Schnorr randomness for noncomputable measures

Jason Rute

University of Hawaii–Manoa

(Soon to be at Penn State)

CiE
July 2013

Why Schnorr randomness
is just as good as Martin-Löf randomness
and possibly even better!

Main Ideas

- Algorithmic randomness lets us talk about points that behave randomly.
- There are a few well-behaved randomness notions, including
  - Martin-Löf randomness
  - Schnorr randomness
- Martin-Löf randomness is robustly defined on noncomp. prob. measures.
- Schnorr randomness has not (until now) been **robustly defined** on noncomputable probability measures.
- In this talk, I will ...
  1. Discuss Schnorr randomness relative to noncomputable oracles.
  2. Define Schnorr randomness on noncomp. probability measures.
  3. Present theorems showing this definition is the correct one.
  4. Discuss other possible definitions and why they do not work.

Schnorr randomness and Martin-Löf randomness

Definition

A **lower semicomputable function** is

$$t = \sup_n f_n$$

for a computable sequence of computable functions (f_n).

Definition

For a computable probability measure μ on $2^{\mathbb{N}}$,

1. $x \in \mathsf{ML}_\mu$ iff $t(x) < \infty$ for all **ML (integral) tests** t:
   - t is a nonnegative lower semicomputable function
   - $\int t \, d\mu < 1$.
2. $x \in \mathsf{SR}_\mu$ iff $t(x) < \infty$ for all **Schnorr (integral) tests** t:
   - t is a nonnegative lower semicomputable function
   - $\int t \, d\mu = 1$.

Good news / Bad news

Paraphrase from Downey and Hirschfeldt's book

- **Good news:** Most results for ML randomness can be extended to Schnorr randomness **if one makes the objects effective enough**.
- **Bad news:**
  1. Schnorr randomness has no universal test.
  2. Van Lambalgen's thm for SR fails (really badly).

Theorem (van Lambalgen)

$$x \in \mathsf{ML} \wedge y \in \mathsf{ML}^x \quad \Leftrightarrow \quad x \oplus y \in \mathsf{ML}.$$

Observation (Kjos-Hanssen)

There is $x \oplus y \in \mathsf{SR}$ such that $x \equiv_T y$!

Good news / Good news

Paraphrase from Downey and Hirschfeldt's book

- **Good news:** Most results for ML randomness can be extended to Schnorr randomness **if one makes the objects effective enough**.
- **Good news:**
  1. Schnorr randomness has no universal test.
     Hence we can compute points that behave randomly!
     (Schnorr randomness is the randomness notion for RCA_0.)
  2. Van Lambalgen's thm **works fine** for SR (after natural adjustments).
     And it is a quite useful theorem!
  3. **Schnorr randomness is very important in the computable analysis of measure theory.**

History of Schnorr randomness (SR) and van Lambalgen's Theorem (vLT) $x \in \mathsf{ML} \wedge y \in \mathsf{ML}^x \Leftrightarrow x \oplus y \in \mathsf{ML}$

Merkle - Miller - Nies - Reimann - Stephan, 2006; Yu, 2007; Kjos-Hanssen

"Easy ($\Leftarrow$) direction" of vLT does not hold for SR (with usual relativization).

"Folklore"

"Hard ($\Rightarrow$) direction" of vLT "obviously" holds for SR. (No proof!)

Franklin - Stephan, 2010

Introduced "truth-table SR" (to characterize Schnorr-triviality).

Miyabe, 2011

vLT holds for "truth-table SR".

Franklin - Stephan, 2011

"Hard ($\Rightarrow$) direction" of vLT holds for SR. (Proof given. Not obvious!)

History of Schnorr randomness (SR) and van Lambalgen's Theorem (vLT) $x \in \mathsf{ML} \wedge y \in \mathsf{ML}^x \Leftrightarrow x \oplus y \in \mathsf{ML}$

Miyabe - Rute, 2013

Renamed "truth-table SR" $\mapsto$ "uniformly relativized SR".

Miyabe - Rute, 2013

vLT holds for "uniformly relativized SR" (correcting the proof of Miyabe).

Rute, 201x

vLT holds for "uniformly relativized SR" for **noncomputable measures**.

Uniformly relativized Schnorr randomness

Uniformly relativized Schnorr randomness

- A **uniform Schnorr test** is a class $\{t^x\}_{x\in 2^{\mathbb{N}}}$ of tests,
  - t^x is nonnegative and lower semicomputable uniformly from $x \in 2^{\mathbb{N}}$
  - $\int t^x \, d\mu = 1$ for all $x \in 2^{\mathbb{N}}$.
- $y_0 \in \mathsf{SR}_\mu^{x_0}$ iff $t^{x_0}(y_0) < \infty$ for all uniform Schnorr tests $\{t^x\}_{x\in 2^{\mathbb{N}}}$.

ML_μ^x can be defined the same way. It is equivalent to the usual definition.

"Truth-table Schnorr randomness" is a misnomer

- Not obvious what it means for a **test** to be "truth-table reducible" to x.
- Not obvious how to handle oracles in other spaces, e.g., $\mathbb{N}^{\mathbb{N}}$ or $\mathsf{Prob}(2^{\mathbb{N}})$.
- Different answers to above can lead to different randomness notions.

Van Lambalgen's theorem for Schnorr randomness

Notation reminder

SR^x_μ will mean Schnorr random **uniformly relative** to the oracle x.

Theorem (Miyabe, Miyabe - Rute)

Let λ *be fair coin measure. Then*

$$x \in \mathsf{SR}_\lambda \wedge y \in \mathsf{SR}^x_\lambda \quad \Leftrightarrow \quad x \oplus y \in \mathsf{SR}_\lambda.$$

Proof of hard ($\Rightarrow$) direction.

Uses ideas from computable analysis (similar to layerwise computability). □

Randomness for noncomputable measures

Definition (Levin)

- A **uniform ML test** is a class $\{t_\mu\}_{\mu \in \mathsf{Prob}(2^\mathbb{N})}$ such that
  - t_μ is lower semicomputable uniformly from μ
  - $\int t_\mu \, d\mu < 1$ for all μ.
- $x_0 \in \mathsf{ML}_{\mu_0}$ iff $t_{\mu_0}(x_0) < \infty$ for all uniform ML tests $\{t_\mu\}_{\mu \in \mathsf{Prob}(2^\mathbb{N})}$.

Definition (Rute)

- A **uniform Schnorr test** is a class $\{t_\mu\}_{\mu \in \mathsf{Prob}(2^\mathbb{N})}$ such that
  - t_μ is lower semicomputable uniformly from μ
  - $\int t_\mu \, d\mu = 1$ for all μ.
- $x_0 \in \mathsf{SR}_{\mu_0}$ iff $t_{\mu_0}(x_0) < \infty$ for all uniform Schnorr tests $\{t_\mu\}_{\mu \in \mathsf{Prob}(2^\mathbb{N})}$.

Randomness for noncomputable oracles and measures

Let X and Y be computable metric spaces (e.g. $\mathbb{R}$, $C([0,1])$, L^1, $\mathsf{Prob}(2^{\mathbb{N}})$).

Definition (Rute)

- A **uniform Schnorr test** is a class $\{t^a_\mu\}_{\mu \in \mathsf{Prob}(X),\, a \in Y}$ such that
  - t^a_μ is lower semicomputable uniformly from μ and a
  - $\int t^a_\mu \, d\mu = 1$ for all μ and a.
- $x_0 \in \mathsf{SR}^{a_0}_{\mu_0}$ iff $t^{a_0}_{\mu_0}(x_0) < \infty$ for all uniform Schnorr tests $\{t^a_\mu\}_{\mu \in \mathsf{Prob}(X),\, a \in Y}$.

Notation: SR^a_μ

- The oracle a is in the exponent.
- Measure μ is in the subscript.
- The relativization is always uniform unless stated otherwise.

Van Lambalgen for noncomputable measures

Let μ and ν be (noncomputable) measures.

Theorem (Levin)

$$x \in \mathsf{ML}_{\mu}^{\nu} \wedge y \in \mathsf{ML}_{\nu}^{x,\mu} \quad \Leftrightarrow \quad (x,y) \in \mathsf{ML}_{\mu\otimes\nu}.$$

Theorem (Rute)

$$x \in \mathsf{SR}_{\mu}^{\nu} \wedge y \in \mathsf{SR}_{\nu}^{x,\mu} \quad \Leftrightarrow \quad (x,y) \in \mathsf{SR}_{\mu\otimes\nu}.$$

Van Lambalgen for kernels

Let μ be a (noncomputable) measure.
Let $\kappa : 2^{\mathbb{N}} \to \mathsf{Prob}(2^{\mathbb{N}})$ be a (noncomputable) kernel.
Define

$$(\mu * \kappa)(A) = \int \left(\int \mathbf{1}_A(x,y)\, d\kappa_x(y) \right) d\mu(x).$$

Theorem (Levin, Takahashi, Bienvenu - Hoyrup - Shen)

$$x \in \mathsf{ML}^{\kappa}_{\mu} \wedge y \in \mathsf{ML}^{x,\mu,\kappa}_{\kappa(x)} \quad \Leftrightarrow \quad (x,y) \in \mathsf{ML}^{\kappa}_{\mu*\kappa}.$$

Theorem (Rute)

$$x \in \mathsf{SR}^{\kappa}_{\mu} \wedge y \in \mathsf{SR}^{x,\mu,\kappa}_{\kappa(x)} \quad \Leftrightarrow \quad (x,y) \in \mathsf{SR}^{\kappa}_{\mu*\kappa}.$$

Van Lambalgen for infinite product spaces

Let μ be a computable measure.
Let μ^∞ be its infinite-dimensional product measure.
Let $x = (x_n)$ and $y = (y_n)$. Define

$$x \sim y \quad \Leftrightarrow \quad \forall n\, x_n =^* y_n.$$

Theorem (Miyabe)

$$x_0 \in \mathsf{ML}_\mu \wedge x_1 \in \mathsf{ML}_\mu^{x_0} \wedge \ldots \wedge x_n \in \mathsf{ML}_\mu^{x_0,\ldots,x_{n-1}} \wedge \ldots$$
$$\Leftrightarrow \quad x \sim y \quad \textit{for some} \quad y \in \mathsf{ML}_{\mu^\infty}.$$

Theorem (Kjos-Hanssen - Nyugen - Rute)

$$x_0 \in \mathsf{SR}_\mu \wedge x_1 \in \mathsf{SR}_\mu^{x_0} \wedge \ldots \wedge x_n \in \mathsf{SR}_\mu^{x_0,\ldots,x_{n-1}} \wedge \ldots$$
$$\Leftrightarrow \quad x \sim y \quad \textit{for some} \quad y \in \mathsf{SR}_{\mu^\infty}.$$

Blind (Hippocratic, oracle-free) randomness for a biased coin

Let μ_p be a p-Bernoulli measure (for a noncomputable $p \in [0,1]$).

Theorem (Kjos-Hanssen)

$x \in \mathsf{ML}_{\mu_p}$ iff $t(x) < \infty$ for all ***blind tests*** *t,*

- *t is a lower semicomputable function (with no knowledge of p or μ_p)*
- *$\int t\, d\mu_p < 1$.*

Theorem (Rute)

$x \in \mathsf{SR}_{\mu_p}$ iff $t(x) < \infty$ for all ***blind tests*** *t,*

- *t is a lower semicomputable function (with no knowledge of p or μ_p)*
- *$q \mapsto \int t\, d\mu_q$ is computable.*

Uniform Schnorr set tests do not exist

We could try to define Schnorr randomness on μ using uniform "set tests".

Definition

A **uniform Schnorr set test** is a sequence $\{(U_n^\mu)\}_{\mu \in \mathsf{Prob}(2^{\mathbb{N}})}$ where

- U_n^μ is Σ_1^0 uniformly in μ and n,
- $\mu(U_n^\mu)$ is computable uniformly in μ and n, and
- $\mu(U_n^\mu) \to 0$.

Theorem (Hoyrup)

The only uniform Schnorr set test is the trivial one:

$$U_n^\mu = \varnothing \quad \textit{(for all nand } \mu\textit{)!}$$

Schnorr's approach

Definition (Schnorr)

x is SR_μ (Schnorr) if there is no test of this form:

- a computable measure ν and
- a computable order function f such that

$$\exists^\infty n \quad \frac{\nu(x \upharpoonright n)}{\mu(x \upharpoonright n)} \geqslant f(n).$$

Observation

This is similar to the blind (Hippocratic) martingale tests used by Kjos-Hanssen - Taveneaux - Thapen for computable randomness.

My guess is that

$$\mathsf{SR}_\mu(\mathit{Schnorr}) \quad \supsetneq \quad \mathsf{SR}_\mu(\mathit{uniform}).$$

Reimann - Slaman approach

Definition (Reimann - Slaman)

x is ML_μ (non-uniform / Reimann-Slaman) if there is some name $a \in 2^{\mathbb{N}}$ of μ such that $t(x) < \infty$ for all ML tests (non-uniformly) computable from a.

Theorem (Day - Miller)

ML_μ *(uniform / Levin)* $=$ ML_μ *(non-uniform / Reimann-Slaman).*

However, for Schnorr randomness...

Theorem (Rute)

SR_μ *(uniform)* $\supsetneq$ SR_μ *(non-uniform). (Too strong!)*

Proof idea.

Use that van Lambalgen's theorem fails for non-uniformly relativized SR. □

Question

Definition

$x \in \mathsf{SR}_{\mu_0}$ **(semi-uniform)** iff
for all **semi-uniform Schnorr tests** $\{t_a\}_{a \in 2^{\mathbb{N}}}$ as below,
$t_{a_0}(x) < \infty$ for some name a_0 of μ_0,

- t_a is a lower semicomputable function uniformly in a,
- $\int t_a \, d\mu_a = 1$ where μ_a is the measure with name a.

Question

Is SR_μ (semi-uniform) equivalent to SR_μ (uniform)?

Summary

- Schnorr rand. can be relativized to noncomputable oracles and measures.
- Theorems for ML randomness hold for Schnorr randomness.
- I believe these results will be useful in computable analysis.
  (Schnorr randomness is very important in computable analysis.)
- Schnorr randomness is just as good as ML randomness!
  ...and possibly even better?

Thank You!

These slides will be available on my webpage:

math.cmu.edu/~jrute

Or just Google™ me, "Jason Rute".