Algorithmic randomness for Doob's martingale convergence theorem in continuous time

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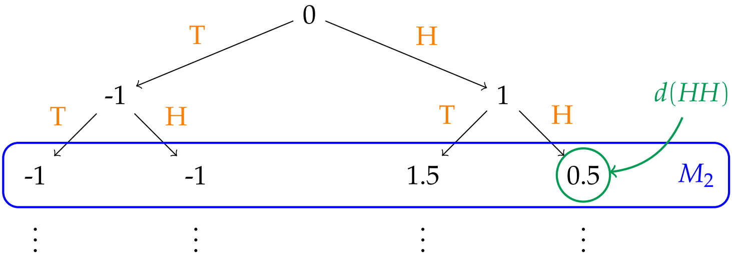
I will use the whiteboard a lot. You may want to sit near the front!

Gambling on coin-flips

In algorithmic randomness, a **martingale** is a gambling strategy on coin-flips.



Note for algorithmic random-ists

If $d(\sigma)$ is a martingale in the computability theory sense and $x \in 2^{\mathbb{N}}$, then we are considering

$$M_n(x) = d(x \upharpoonright n).$$

Computable randomness

Definition

A sequence $\alpha \in 2^{\mathbb{N}}$ is **computably random** if there is no computable, nonnegative martingale such that

$$\limsup_{n\to\infty} M_n(\alpha) = \infty.$$

- Any nonnegative betting strategy will only succeed with probability 0.
- There are countably many computable strategies.
- Therefore, there are measure one many computable randoms.

Randomness hierarchy

Schnorr randoms $\supseteq$ computable randoms $\supseteq$ Martin-Löf randoms

Theorem (R.)

Computable randomness can be generalized to other computable probability measures.

Gambling on Brownian motion (aka continuous random walks)

Suppose we want to *continuously gamble* on a Brownian motion path with a *computable strategy*.

- Q: What the does this even mean?!
  - A: Coming shortly...
- Q: What is this resulting notion of randomness called?
  - A: **Doob randomness**.
- Q: Is a Doob random Brownian motion path the same as a computably randomness Brownian motion path?
  - A: No, not in general. Computable randomness implies Doob randomness, but not conversely.
  - A: However, Doob randomness has a nice characterization in terms of computable randomness.
- Q: How does Doob randomness relate to Schnorr randomness?
  - A: Doob randomness is incomparable with Schnorr randomness.

Gambling on bit-arrays

In order to better understand gambling on Brownian motion W, we **discretize** $W \in C([0,\infty))$ into a sequence of sequences $\omega \in (2^{\mathbb{N}})^{\mathbb{N}}$.

(See white board.)

Then we can gamble on this array by betting on one sequence at a time.

Definition

Some $\omega \in (2^{\mathbb{N}})^{\mathbb{N}}$ is **Doob random** if there are no nonnegative, computable betting strategy which succeeds on ω (win arbitrarily large amounts of money).

Definition

The same results as before hold for Doob randomness on $(2^{\mathbb{N}})^{\mathbb{N}}$.

Martingales informally

- Choose a notion of time, e.g. $\mathbb{N}$ or $[0, \infty)$.
- Decide what information we can see at time t. Once you see information, you cannot forget it. (This can be made formal with a **filtration**.)
- Each martingale $(M_t)_t$ represents a betting strategy.
- M_t is the value of the gambler's capital at time t. It is a function of the sample space.

Defining martingales on $2^{\mathbb{N}}$ (fair-coin measure)

- Let $\alpha_{<n} = (a_0, \dots, a_{n-1})$ for $\alpha \in 2^{\mathbb{N}}$.

Definition (Martingales)

A sequence of integrable functions $(M_n)_{n\in\mathbb{N}}$ is a **martingale** if

1. (adapted) $M_n(\alpha)$ depends only on $\alpha_{<n}$ and n, and
2. (martingale property) For $m \leqslant n$,

$$\mathbb{E}_m(M_n) = M_m.$$

A martingale is **computable** if

1. (computably adapted) $M_n(\alpha)$ is computable from $\alpha_{<n}$ and n.

Informally

Conditional expectation $\mathbb{E}_n(f)$ is the expected future value of f only using information up to time n (the first n bits of α).

Defining martingales on $2^{\mathbb{N}}$ (fair-coin measure)

- $\alpha_{<n} = (a_0, \ldots, a_{n-1})$ for $\alpha \in 2^{\mathbb{N}}$,
- $\alpha_{<n} \frown \beta$ means the concatenation of $\alpha_{<n}$ and β.

Definition (Martingales)

A sequence of integrable functions $(M_n)_{n \in \mathbb{N}}$ is a **martingale** if

1. (adapted) $M_n(\alpha)$ depends only on $\alpha_{<n}$ and n, and
2. (martingale property) For $m \leqslant n$,

$$\mathbb{E}_m(M_n) = M_m.$$

Definition (Conditional expectation)

The conditional expectation (at time n), is the operator $\mathbb{E}_n : L^1 \to L^1$ defined by

$$\mathbb{E}_n(f)(\alpha) = \int f(\alpha_{<n} \frown \beta)\, d\beta.$$

Defining martingales on $(2^{\mathbb{N}})^{\mathbb{N}}$ (uniform measure)

- $\omega_{<n} = (\omega_0, \ldots, \omega_{n-1})$ for $\omega \in (2^{\mathbb{N}})^{\mathbb{N}}$,
- $\omega_{<n}{}^\frown\beta$ means the concatenation of $\omega_{<n}$ and β.

Definition (Martingales)

A sequence of integrable functions $(M_n)_{n\in\mathbb{N}}$ is a **martingale** if

1. (adapted) $M_n(\omega)$ depends only on $\omega_{<n}$ and n, and
2. (martingale property) For $m \leqslant n$,

$$\mathbb{E}_m(M_n) = M_m.$$

Definition (Conditional expectation)

The conditional expectation (at time n), is the operator $\mathbb{E}_n : L^1 \to L^1$ defined by

$$\mathbb{E}_n(f)(\omega) = \int f(\omega_{<n}{}^\frown\xi)\, d\xi.$$

Defining martingales on Brownian motion

- $W_{\leqslant t} = W \upharpoonright [0,t]$ for $W \in C([0,\infty))$,
- $W_{\leqslant t} \frown X$ means the concatenation of $W_{\leqslant t}$ and X.

Definition (Martingales)

A sequence of integrable functions $(M_t)_{t\in[0,\infty)}$ is a **martingale** if

1. (adapted) $M_t(W)$ depends only on $W_{\leqslant t}$ and t, and
2. (martingale property) For $s \leqslant t$,

$$\mathbb{E}_s(M_t) = M_s.$$

Definition (Conditional expectation)

The conditional expectation (at time t), is the operator $\mathbb{E}_t : L^1 \to L^1$ defined by

$$\mathbb{E}_t(f)(W) = \int f(W_{\leqslant t} \frown X)\, dX.$$

Defining martingales on Brownian motion

- $W_{\leqslant t} = W \restriction [0,t]$ for $W \in C([0,\infty))$,

Definition (Martingales)

A sequence of integrable functions $(M_t)_{t\in[0,\infty)}$ is a **martingale** if

1. (adapted) $M_t(W)$ depends only on $W_{\leqslant t}$ and t, and
2. (martingale property) For $s \leqslant t$,

$$\mathbb{E}_s(M_t) = M_s.$$

A martingale $(M_t)_{t\in[0,\infty)}$ is **computable** if

1. (computably adapted) $M_t(W)$ is computable from $W_{\leqslant t}$ and t.

Doob's martingale convergence theorem

Consider $(\Omega, \mathcal{B}, \mathbb{P})$ to be $C([0,\infty))$ with the Wiener measure.

Theorem (Doob's martingale convergence theorem[1])

- *Let $(M_t)_{t\in[0,\infty)}$ be a nonnegative martingale on Brownian motion.*
- *Then $M_t(W)$ converges as $t \to \infty$ for almost all Brownian motion paths W.*

We can then define **Doob randomness** so that this theorem holds:

Theorem (Kjos-Hanssen, Nguyen, R.)

- *Let $(M_t)_{t\in[0,\infty)}$ be a nonnegative* ***computable*** *martingale on Brownian motion.*
- *Then $M_t(W)$ converges as $t \to \infty$ for all* ***Doob randoms*** *$W \in C([0,\infty))$.*

[1] Doob's martingale convergence theorem can be stated much more generally than this.

Doob's martingale convergence theorem

Consider $(\Omega, \mathcal{B}, \mathbb{P})$ to be $(2^{\mathbb{N}})^{\mathbb{N}}$ with the uniform probability measure.

Theorem (Doob's martingale convergence theorem[2])

- *Let $(M_n)_n$ be a nonnegative martingale on sequences of sequences.*
- *Then $M_n(\omega)$ converges as $n \to \infty$ for almost all $\omega = (\omega_n) \in (2^{\mathbb{N}})^{\mathbb{N}}$.*

We can then define **Doob randomness** so that this theorem holds:

Theorem (Kjos-Hanssen, Nguyen, R.)

- *Let $(M_n)_{n\in\mathbb{N}}$ be a nonnegative* ***computable*** *martingale on sequences of sequences.*
- *Then $M_n(\omega)$ converges as $n \to \infty$ for all* ***Doob randoms*** *$\omega = (\omega_n) \in (2^{\mathbb{N}})^{\mathbb{N}}$.*

[2] Doob's martingale convergence theorem can be stated much more generally than this.

Our goal

We want to characterize the points for which computable martingales converge on Brownian motion.

Three notions of martingale convergence on $2^{\mathbb{N}}$

Equivalent definitions of Doob randomness on $(2^{\mathbb{N}})^{\mathbb{N}}$.

Theorem (Kjos-Hanssen, Nguyen, R.)

The following are equivalent for $\omega = (\omega_n) \in (2^{\mathbb{N}})^{\mathbb{N}}$.

1. $M_n(\omega)$ *converges for all computable martingales* $(M_n)_n$.
2. $\limsup_{n\to\infty} M_n(\omega) < \infty$ *for all computable martingales* $(M_n)_n$.
3. $\liminf_{n\to\infty} M_n(\omega) < \infty$ *for all computable martingales* $(M_n)_n$.

Proof.

(1) $\Rightarrow$ (2): Buy-low-sell-high.
(2) $\Rightarrow$ (3): Save money.

□

Six notions of convergence on Brownian motion

Equivalent definitions of Doob randomness on Brownian motion.

Theorem (Kjos-Hanssen, Nguyen, R.)

The following are equivalent for a ***Schnorr random*** $W \in C([0,\infty))$.

1. $M_t(\omega)$ *converges for all* ***continuous-time*** *computable martingales* $(M_t)_{t\in[0,\infty)}$.
2. $\limsup_{t\to\infty} M_n(\omega) < \infty$ *for all continuous-time comp. marting.* $(M_t)_{t\in[0,\infty)}$.
3. $\liminf_{t\to\infty} M_n(\omega) < \infty$ *for all continuous-time martingales* $(M_t)_{t\in[0,\infty)}$.
4. $M_n(\omega)$ *converges for all* ***discrete-time*** *computable martingales* $(M_n)_{n\in\mathbb{N}}$.
5. $\limsup_{n\to\infty} M_n(\omega) < \infty$ *for all discrete-time comput. martingales* $(M_n)_{n\in\mathbb{N}}$.
6. $\liminf_{n\to\infty} M_n(\omega) < \infty$ *for all discrete-time computable martingales* $(M_n)_{n\in\mathbb{N}}$.

Mapping Brownian motion to $(2^{\mathbb{N}})^{\mathbb{N}}$.

- $C([0,1])$ with the Wiener measure and $2^{\mathbb{N}}$ with the fair-coin measure are computable, atomless probability spaces. Therefore by Hoyrup and Rojas there is an almost-everywhere computable isomorphism

$$B : C([0,1]) \quad \leftrightarrow \quad 2^{\mathbb{N}}.$$

- This extends to an a.e. computable isomorphism

$$B' : C([0,\infty)) \quad \leftrightarrow \quad (2^{\mathbb{N}})^{\mathbb{N}}.$$

- B' (and its inverse) preserves Schnorr randomness, computable randomness, and Doob+Schnorr randomness.
- B' maps discrete-time martingales (M_n) on Brownian motion to martingales (N_n) on $(2^{\mathbb{N}})^{\mathbb{N}}$.

Characterizations of Doob randomness on $(2^{\mathbb{N}})^{\mathbb{N}}$

Theorem (Kjos-Hanssen, Nguyen, R.)

The following are equivalent for $\omega \in (2^{\mathbb{N}})^{\mathbb{N}}$. ***(See the whiteboard for a proper explanation.)***

1. *ω is Doob random.*
2. *There is no $\mathbb{N} \times \mathbb{N}$-indexed martingale which succeeds on ω.*
3. *There is no martingale which bets on finitely-many bits of $\omega_0, \omega_1, \ldots$ which succeeds on ω.*
4. *For all computable functions $f : \mathbb{N} \to \mathbb{N}$, the part of ω below the graph of f is computably random uniformly relative to the part of ω above the graph of f.*

Doob randomness vs other randomness notions

Theorem (Kjos-Hanssen, Nguyen, R.)

Consider the following for $\omega \in (2^{\mathbb{N}})^{\mathbb{N}}$.

1. *ω is computably random.*
2. *$(\omega_n, \omega_{n+1}, \ldots)$ is computably random uniformly relative to $(\omega_0, \ldots, \omega_{n-1})$.*
3. *ω is Doob random.*

(1) $\Rightarrow$ (2) $\Rightarrow$ (3) and none of the implications reverse.

Theorem (Kjos-Hanssen, Nguyen, R.)

Consider the following for $\omega \in (2^{\mathbb{N}})^{\mathbb{N}}$.

1. *ω is computably random.*
2. *$(\omega_n, \omega_{n+1}, \ldots)$ is CR uniformly relative to $(\omega_0, \omega_1, \ldots)$ and ω is Schnorr rand.*
3. *ω is Doob random and Schnorr random.*
4. *ω is Schnorr random.*

(1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4)
and none of the implications reverse ***(except possibly (2) $\Rightarrow$ (3))****.*

Doob randomness vs Schnorr randomness

Theorem (Kjos-Hanssen, Nguyen, R.)

For $(2^{\mathbb{N}})^{\mathbb{N}}$*, Doob randomness is incomparable with Schnorr randomness.*

Doob randomness vs other randomness notions

Theorem (Kjos-Hanssen, Nguyen, R.)

Consider the following for $W \in C([0,\infty))$.

1. *W is computably random.*
2. *$W_{\geqslant n}$ is CR uniformly relative to $W_{\leqslant n}$ for all n, and W is Schnorr rand.*
3. *W is Doob random and Schnorr random.*
4. *W is Schnorr random.*

(1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) and none of the implications reverse ***(except possibly (2) $\Rightarrow$ (3))****.*

Theorem (Kjos-Hanssen, Nguyen, R.)

For Brownian motion, Doob randomness is incomparable with Schnorr randomness.

Final Remarks

- Doob randomness has some bad properties (it is not invariant under permutations on $2^{\mathbb{N}\times\mathbb{N}}$).
- These bad properties suggest Doob randomness should not be studied on an arbitrary computable probability measure.
- Instead it should be studied on a **structured probability measure**: in this case, a probability measure with a notion of time—a **filtered probability space**.

- The martingales used here are not the most general. We could consider:
  - L^1-computable martingales (layerwise-computable and integrable)
  - Nonnegative martingales with a computable L^1-bound: $\sup_t \|M_t\|_{L^1}$.
- The results should basically be the same (except Doob randomness would only make sense on Schnorr random points).

Thank You!

These slides will be available on my webpage:

`math.cmu.edu/~jrute`

Or just Google™ me, "Jason Rute".