ULTRAFILTERS AND ERGODIC THEORY (EXTENDED ABSTRACT)

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This talk is a survey on how ultralimits (or p-limits) can be used to prove recurrence results in Ergodic theory, which in turn have combinatorial consequences such as the following.

Theorem 0.1. *Partition* $\mathbb{N}$ *into* $A_1 \cup \ldots \cup A_k$. *Then there is some* i *(*$1 \leq i \leq k$*) and some* $x, y, z \in A_i$ *such that* $x - y = z^2$.

The main idea is that recurrence results can be strengthened by replacing averaging limits with ultralimits, while the proofs remain similar.

1. Background on ultrafilters

Recall the following definitions and facts.

Definition 1.1. An *ultrafilter* p on $\mathbb{N}$ is a collection of subsets of $\mathbb{N}$ such that the following hold: (1) For all $A, B \subseteq \mathbb{N}$, if $A \in p$ and $A \subseteq B$, then $B \in p$. (2) For all $A, B \subseteq \mathbb{N}$, if $A \in p$ and $B \in p$, then $A \cap B \in p$. (3) $\varnothing \notin p$. (4) If $\mathbb{N} = A_1 \cup \ldots \cup A_k$, then $A_i \in p$ for some i ($1 \leq i \leq k$).

The collection of ultrafilters on $\mathbb{N}$ is denoted $\beta\mathbb{N}$, and is homeomorphic (under an appropriate topology) to the Stone-Čech compactification of $\mathbb{N}$. Informally, an ultrafilter is a measure of largeness. Some set A is p-large, or has p-measure one if $A \in p$.

Definition 1.2. Addition on $\mathbb{N}$ is extended to $\beta\mathbb{N}$ via

$$\forall A \subseteq \mathbb{N} \quad A \in (p + q) \Leftrightarrow \{n \in \mathbb{N} \mid (A - n) \in p\} \in q$$

where $A - n = \{m \in \mathbb{N} \mid m + n \in A\}$.

Definition 1.3 (See [BD08].)**.** Given $p \in \beta\mathbb{N}$, say that

(1) p is *idempotent* if $p + p = p$.
(2) p is *essential idempotent* if p is idempotent and for all $A \in p$, A has positive upper Banach density, i.e.

$$\limsup_{|N-M|\to\infty} \frac{1}{N - M} |A \cap [M, \ldots, N - 1]| > 0.$$

(3) p is *minimal idempotent* if p is idempotent and p belongs to a minimal right ideal of $(\beta\mathbb{N}, +)$.

Let $A \subseteq \mathbb{N}$. Say that A is an "IP", D, C set if, respectively, $A \in p$ for *some* idempotent, essential idempotent, minimal idempotent $p \in \mathbb{N}$. Say that A is a IP^*, D^*, C^* set if, respectively, $A \in p$ for *all* idempotent, essential idempotent, minimal idempotent $p \in \mathbb{N}$. (C sets are also called *central sets*. What I refer to as "IP" sets can also be characterized combinatorially via Hindman's theorem.)

The previous definitions are measures of "largeness" and they satisfy the following implications.

$$IP^* \Rightarrow D^* \Rightarrow C^* \Rightarrow C \Rightarrow D \Rightarrow \text{“}IP\text{”}$$

2. Ultralimits

In this section let $(x_n)_{n\in\mathbb{N}}$ be a sequence from a compact Hausdorff space X. Let $p \in \beta\mathbb{N}$.

Definition 2.1. Say that $p\text{-}\lim_{n\in\mathbb{N}} x_n = a$ if and only if for all neighborhoods U of a, we have $\{n \in \mathbb{N} \mid x_n \in U\} \in p$. This is the *ultralimit* of (x_n) under p. (An alternate notation is $\lim_{n\to p} x_n$; here $n \to p$ in the topology of $\beta\mathbb{N}$.)

Proposition 2.2. *The ultralimit $p\text{-}\lim_{n\in\mathbb{N}} x_n$ exists (because X is compact) and is unique (because X is Hausdorff).*

This next proposition is a consequence of ultrafilter addition.

Proposition 2.3. *Given $p, q \in \beta\mathbb{N}$,*

$$\underset{n\in\mathbb{N}}{p\text{-}\lim}\, \underset{m\in\mathbb{N}}{q\text{-}\lim}\, x_{n+m} = \underset{k\in\mathbb{N}}{q{+}p\text{-}\lim}\, x_k.$$

Hence if $p \in \beta\mathbb{N}$ is idempotent, then

$$\underset{n\in\mathbb{N}}{p\text{-}\lim}\, \underset{m\in\mathbb{N}}{p\text{-}\lim}\, x_{n+m} = \underset{k\in\mathbb{N}}{p\text{-}\lim}\, x_k.$$

3. A refinement of a theorem of Sárközy and Furstenberg.

Using ultralimits, one can state and prove the following refinement of a theorem of Sárközy [Sár78] and Furstenberg [Fur81].

Theorem 3.1 (See [Ber96])**.** *Let $q \in \mathbb{Q}[n]$ such that $q(\mathbb{Z}) \subseteq \mathbb{Z}$, and $q(0) = 0$. Let $(X, \mathcal{B}, \mu, T)$ be an invertible measure preserving system, with $A \in \mathcal{B}$. For any idempotent $p \in \beta\mathbb{N}$,*

$$\underset{n\in\mathbb{N}}{p\text{-}\lim}\, \mu(A \cap T^{q(n)}A) \geq \mu(A)^2.$$

Also, since p is an arbitrary idempotent, for all $\varepsilon > 0$,

$$\{n \in \mathbb{N} \mid \mu(A \cap T^{q(n)}A) > \mu(A)^2 - \varepsilon\} \text{ is } IP^*. \tag{1}$$

Theorem 0.1 follows from Theorem 3.1 by the Furstenberg correspondence principle and the fact that one of the A_i in the partition is both "IP" and has positive upper Banach density (see [Ber03]). This is stronger than the Sárközy and Furstenberg result which only shows that there is a monochromatic x, y such that $x - y$ is a perfect square. The added strength comes from (1) which lets us choose some z in the same part as x and y

The proof of Theorem 3.1, however, is similar to more classical versions using averaging limits: The space of L^2 functions is decomposed into two orthogonal subspaces. One decomposition is handled using an idempotent ultralimit version of the van der Corput trick, while the other makes heavy use of Proposition 2.3.

4. Ultralimits and Factors

Using stronger ultrafilters, we can characterize the Kronecker factor and weak mixing.

Proposition 4.1 (See [Ber03, BD08]). *Let $(X, \mathcal{B}, \mu, T)$ be an invertible measure preserving system. Let $p \in \beta\mathbb{N}$ be an essential (or minimal) idempotent. Then the Kronecker factor is*

$$\left\{ f \in L^2 \;\middle|\; p\text{-}\lim_{n\in\mathbb{N}} T^n f = f \right\}$$

(Here the ultralimit is in the weak-topology. Hence the unit ball is compact and the ultralimit exits.)

Idempotent ultrafilters can be used to characterize mild mixing as well (see [Ber03]).

Using essential idempotents and Proposition 4.1, one can prove a version of Szemerédi's theorem for generalized polynomials [BM10].

All the ultralimit definitions and facts on $\mathbb{N}$ also extend to any countable group G. Hence, one can use minimal idempotents in βG with a version of Proposition 4.1 to prove the following nonamenable, noncommutative version of Roth's theorem for groups.

Theorem 4.2 ([BM07]). *Let G be a countable group. Let $(X, \mathcal{B}, \mu)$ be a probability space. Let $(T_g)_{g\in G}$ and $(S_g)_{g\in G}$ be measure preserving actions of G such that $T_g S_h = S_h T_g$ for all $g, h \in G$. Then all $A \in \mathcal{B}$ and for all $\varepsilon > 0$,*

$$\{g \in \mathbb{N} \mid \mu(A \cap T_g^{-1} A \cap S_g^{-1} T_g^{-1} A) > \varepsilon\} \text{ is } C^*.$$

For further information on ultralimits in ergodic theory and additive combinatorics, see the surveys [Ber96, Ber03, Ber10].

References

[BD08] Vitaly Bergelson and Tomasz Downarowicz. Large sets of integers and hierarchy of mixing properties of measure preserving systems. *Colloq. Math.*, 110(1):117–150, 2008.

[Ber96] Vitaly Bergelson. Ergodic Ramsey theory—an update. In *Ergodic theory of $\mathbf{Z}^d$ actions (Warwick, 1993–1994)*, volume 228 of *London Math. Soc. Lecture Note Ser.*, pages 1–61. Cambridge Univ. Press, Cambridge, 1996.

[Ber03] Vitaly Bergelson. Minimal idempotents and ergodic Ramsey theory. In *Topics in dynamics and ergodic theory*, volume 310 of *London Math. Soc. Lecture Note Ser.*, pages 8–39. Cambridge Univ. Press, Cambridge, 2003.

[Ber10] Vitaly Bergelson. Ultrafilters, IP sets, dynamics, and combinatorial number theory. In *Ultrafilters across mathematics*, volume 530 of *Contemp. Math.*, pages 23–47. Amer. Math. Soc., Providence, RI, 2010.

[BM07] V. Bergelson and R. McCutcheon. Central sets and a non-commutative Roth theorem. *Amer. J. Math.*, 129(5):1251–1275, 2007.

[BM10] V. Bergelson and R. McCutcheon. Idempotent ultrafilters, multiple weak mixing and Szemerédi's theorem for generalized polynomials. *J. Anal. Math.*, 111:77–130, 2010.

[Fur81] H. Furstenberg. *Recurrence in ergodic theory and combinatorial number theory*. Princeton University Press, Princeton, N.J., 1981. M. B. Porter Lectures.

[Sár78] A. Sárközy. On difference sets of sequences of integers. III. *Acta Math. Acad. Sci. Hungar.*, 31(3-4):355–386, 1978.