Randomness, Martingales and Differentiability

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Outline

Theses

Thesis 1 (Brattka-Miller-Nies, Pathak, others)

$$\text{randomness} = \text{differentiability}$$

Thesis 2

$$\text{randomness} = \text{convergence}$$

The Lebesgue Differentiation Theorem

Let m be the Lebesgue measure on $[0,1]^d$.

Theorem (Lebesgue Differentiation Theorem)

Let $f : [0,1]^d \to \mathbb{R}$ be an integrable (L^1) function. Define the average

$$A_r f(x) = \frac{\int_{B_r(x)} f(x)\,dx}{m(B_r(x))}.$$

Then $A_r f(x) \to f(x)$ as $r \to 0$ for almost every x.

- "Almost every x" intuitively means if x is "random enough" (relative to the parameters of the theorem) then the above theorem is true for x.
- Algorithmic randomness attempts to make rigorous this notion of "random".

Effective Lebesgue Differentiation Theorem

Theorem (Lebesgue Differentiation Theorem)
Let $f : [0,1]^d \to \mathbb{R}$ be an integrable (L^1) function.
Then $A_r f(x) \to f(x)$ as $r \to 0$ for almost every x.

Theorem (Effective Lebesgue Differentiation Theorem, R.)
Let $f : [0,1]^d \to \mathbb{R}$ be an L^1-computable function.
Then $A_r f(x) \to \widehat{f}(x)$ as $r \to 0$ for all Schnorr randoms x.

Here $\widehat{f}(x)$ is a particular representative of the L^1 equivalence class of f.

Two Effective LDT's

Pathak has already shown the following.

Theorem (Effective LDT for Martin-Löf Randoms, Pathak)

Let $f : [0,1]^d \to \mathbb{R}$ be an L^1-computable function.
Then $A_r f(x) \to \widehat{f}(x)$ as $r \to 0$ for all Martin-Löf randoms x.

Theorem (Effective LDT for Schnorr Randoms, R.)

Let $f : [0,1]^d \to \mathbb{R}$ be an L^1-computable function.
Then $A_r f(x) \to \widehat{f}(x)$ as $r \to 0$ for all Schnorr randoms x.

- Since ML randoms $\subset$ Schnorr randoms, the Schnorr result is tighter.
- Is Schnorr randomness the best we can do? Yes.

Characterization of Schnorr Randomness

Theorem ("The Reversal", R.)

Let $x \in [0,1]^d$ be not Schnorr random. Then there is an L^1-computable function f such that $A_r f(x)$ does not converge as $r \to 0$.

Schnorr randomness is characterized by "differentiability".

Related Work

- Pathak proved the Effective LDT for Martin-Löf randoms.
- Brattka, Miller, Nies characterized computable randomness, Martin-Löf randomness, and weak-2 randomness in terms of differentiability of absolutely-continuous computable functions on $[0,1]$. Martin-Löf case was based on the work of Demuth.
- Pathak, Rojas, Simpson have a different proof of the Effective LDT for Schnorr randoms.
- Kenshi Miyabe characterized Kurtz randomness by the LDT, and has done related work with integral tests.
- Freer, Kjos-Hanssen, Nies characterized computable randomness and Schnorr randomness in terms of differentiability of computable Lipschitz functions.
- Bienvenu, Hölzl, Miller, Nies looked at the Denjoy alternative and the Lebesgue density theorem for Π^0_1 sets.
- ...this meeting.

Outline

Schnorr Random

A set $U \subseteq [0,1]^n$ is Σ^0_1 (effectively open) if it is a union of a open cubes $(a_1, b_1) \times \cdots \times (a_n, b_n)$ with rational coordinates (uniformly).

A Solovay test for **Schnorr** randomness is an uniform sequence (U_i) of Σ^0_1 sets where

$$\sum_i m(U_i) \text{ is finite and computable.}$$

(It is sufficient for, say, each $m(U_i)$ to be computable and $\leq 2^{-i}$.)

Say (U_i) covers $x \in [0,1]^n$ if $x \in U_i$ for infinitely many i.

$x \in [0,1]^n$ is Schnorr random if no Solovay test for Schnorr randomness covers x.

L^1-computable Functions

Any integrable function f can be approximated by rational-valued step functions using the L^1-norm $\|f\|_1 := \int |f| \, dx$.

Definition

An L^1-code is a sequence of rational valued step functions (p_i) such that

$$\|p_{i+1} - p_i\|_1 \leq 2^{-2i}.$$

We say f is L^1-computable if there is an L^1-code (p_i) such that $p_i \to f$ in the L^1-norm and (p_i) is uniformly computable.

The following are computable from (the codes for) $f, g \in L^1$ and (the codes for) any other parameters:

$$f + g, af, |f|, \|f\|_1, \int f \, dx, \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f \, dx.$$

Representatives for L^1-computable functions

Pathak proved this theorem for ML randomness. It can be extended to Schnorr randomness.

Definition

Let f be an L^1-computable function with L^1-code (p_i).
Define $\widehat{f} = \lim_i p_i$ (where the limit exists).

Theorem

Let (p_i) be the L^1-code for an L^1-computable function f.

1. *(Existence) $p_i(z)$ converges for Schnorr randoms z.*
2. *(Uniqueness) If (q_i) is a another L^1-code for f, then $(p_i(z) - q_i(z)) \to 0$ on Schnorr randoms z.*

Here is where we need that $\|p_{i+1} - p_i\| < 2^{-2i}$, else (p_i) may not converge pointwise.

L^1-computable functions v.s. canonical representations

For every L^1-computable function f there are **two** related but different structures:

1. The L^1-equivalence class of f.
2. The canonical L^1-representative $\hat{f}$.

These canonical representatives are the same layerwise computable functions of Hoyrup and Rojas, except those are only defined up to ML randomness.

$\hat{f}$ can also be defined for measurable (not necessary integrable) functions using the metric $d(f,g) = \int \min(1, |f-g|)$.

Pointwise Convergence

- Almost uniform convergence:
  Uniform rate of convergence (up to small set).
- Almost everywhere convergence:
  Rate of convergence unique to each point.

Egorov's theorem

In a probability space,

$$\text{almost everywhere} \Leftrightarrow \text{almost uniform.}$$

Ergorov's theorem is very non-computable!

- Reverse-math strength is 2-WWKL.
- There are effective notions between a.e. and a.u.

Effective Convergence

Definitions

$f_n \to f$ almost uniformly if there is a modulus $n(\varepsilon, \delta)$ such that

- In words: For every $\delta > 0$, all but δ-measure of points obey the rate of convergence $n(\varepsilon, \delta)$,
  i.e. $|f_k(x) - f(x)| \leq \varepsilon$ for all $k \geq n(\varepsilon, \delta)$.
- In symbols:

$$m(\{x \mid \exists k \geq n(\varepsilon, \delta)\ |f_k(x) - f(x)| > \varepsilon\}) \leq \delta.$$

$f_n \to f$ effectively a.e. if there is a computable modulus $n(\varepsilon, \delta)$.

"Effectively a.e." may more accurately be called "effectively (a.u.) almost uniform".

L^1-codes and a.e. (a.u.) convergence

A.e. (a.u.) convergence can be expressed in terms of L^1-codes.
(Hence it is invariant under a.e. equivalence.)

Theorem

Let (f_i) and f all be L^1-functions. Let p_i be rational step functions such that $\|p_i - f_i\| \leq 2^{-2i}$ for each i. Let (q_i) be an L^1-code for f. The following are equivalent.

1. *$f_i \to f$ effectively a.e. with some modulus $n(\varepsilon, \delta)$.*
2. *$(p_i - q_i) \to 0$ effectively a.e. with some modulus $m(\varepsilon, \delta)$.*

The result is uniform: $n(\varepsilon, \delta)$ and $m(\varepsilon, \delta)$ are computable from each other, independent of (f_i), f, (p_i), and (q_i).

“Finitary” form of a.e. (a.u.) convergence

For L^1-functions, a.e. (a.u.) convergence has a “finitary” version.

Theorem

Let (f_i) and f all be L^1-functions. TFAE:

1. *$f_i \to f$ effectively a.e. with some modulus $n(\varepsilon, \delta)$, i.e.*

$$\forall \varepsilon, \delta \;\; m(\{x \mid \exists i \geq n(\varepsilon, \delta) \; |f_i(x) - f(x)| > \varepsilon\}) \leq \delta.$$

2. *There is an sequence (i_k) such that*

$$\forall k \;\; m\left(\left\{x \mid \exists i \in [i_k, i_{k+1}] \; |f_i(x) - f(x)| > 2^{-k}\right\}\right) \leq 2^{-k}.$$

The result is uniform: $n(\varepsilon, \delta)$ and (i_k) are computable from each other, independent of (f_i) and f.

Such uniform, finitary results are of recent interest to both Proof Theorists and Analysts/Number Theorists (e.g. Tao).

Effective a.e. (a.u.) convergence implies convergence on Schnorrs

Theorem (also implicit in Pathak, Rojas, Simpson)

Assume (f_i) and f are uniformly L^1-computable functions.
If $f_i \to f$ effectively a.e. (a.u.),
then $\widehat{f_i}(z) \to \widehat{f}(z)$ for Schnorr randoms z.

Main Idea of Proof:

Use the finitary version of effective a.e. (a.u.) convergence and use that the set in red is a finite union of rational cubes.

$$m\left(\left\{x \mid \exists i \in [i_k, i_{k+1}]\ |p_i| \geq 2^{-k}\right\}\right) \leq 2^{-k}.$$

Hence we have a Solovay test for Schnorr randomness.

Convergence on Schnorr randoms

Theorem

Assume (f_i) and f are uniformly L^1-computable functions.
If $f_i \to f$ effectively a.e., then $\widehat{f_i}(z) \to \widehat{f}(z)$ for Schnorr randoms z.

Similar results:

Theorem

Assume (f_i) and f are uniformly computable functions.
If $f_i \to f$ effectively a.e., then $f_i(z) \to f(z)$ for Schnorr randoms z.

Theorem

Assume (f_i) and f are uniformly computably measurable functions.
If $f_i \to f$ effectively a.e., then $\widehat{f_i}(z) \to \widehat{f}(z)$ for Schnorr randoms z.

For measurable functions use the metric, $d(f,g) = \int \min(1, |f-g|)$.

Applications

If we have effective a.e. convergence (computable analysis),
then we have convergence on Schnorr randoms (randomness).

Theorem

Let (U_i) is a Solovay test for Schnorr randomness.
There is a computable sequence (p_i) of rational step functions such that

- $p_i \to 0$ *effectively a.e.*
- $\|p_{i+1} - p_i\|_1 \leq 2^{-2i}$ *for each* i
- *If* x *is covered by* (U_i)*, then* $(p_i(x))$ *diverges.*

This reversal characterizes Schnorr randomness.

It also shows there is not an L^1-representative defined on, say, Kurtz randoms.

Convergence Tests

Convergence characterizes Schnorr randomness.

This gives a notion of a "convergence test".

Work in Progress

Can replace "effectively a.e. (a.u.)" to get convergence tests for weak-2 randomness, ML randomness, computable randomness, and Kurtz randomness.

Outline

The Main Theorems to Prove

Stronger versions:

Theorem (Effective LDT for Schnorr Randoms)

Let $f : [0,1]^d \to \mathbb{R}$ be an L^1-computable function. Then

$$A_r \left| f - \widehat{f}(z) \right| (z) := \frac{\int_{B_r(z)} \left| f(x) - \widehat{f}(z) \right| dx}{m(B_r(z))} \to 0$$

as $r \to 0$ for all Schnorr randoms z.

Theorem (LDT Reversal)

Let (U_i) be a Solovay test for Schnorr test. There is an L^1-computable function f such that $A_r f(z)$ does not converge as $r \to 0$ for any z covered by (U_i).

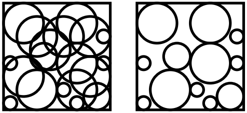
Structure of the Proof

Geometric part of Proof. Reduce the geometric complexity of the problem.

Martingale part of Proof. Use theorems about martingales to analyze the convergence.

Geometric Part

- The LDT is a very geometric theorem.
- For example, if the balls were replaced with arbitrary rectangles or ellipse, it would not be true. (But cubes are OK.)
- One approach is to replace overlapping balls with disjoint balls/cubes.



- Classical proofs use the Vitali Covering Theorem to do this.
- That is what Pathak, Rojas, and Simpson used.
- Brattka, Miller, Nies used a different approach.
- This proof uses yet another approach, based off a proof of Morayne and Solecki.

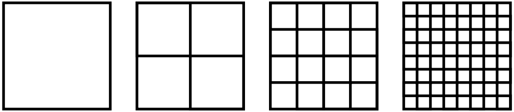
Work with Dyadic Cubes Instead

Work with dyadic cubes instead of balls.

Definition

A dyadic cube on $[0,1]^d$ is a cube of the form.

$$\left[\frac{i_0}{2^j}, \frac{i_0+1}{2^j}\right) \times \cdots \times \left[\frac{i_{d-1}}{2^j}, \frac{i_{d-1}+1}{2^j}\right).$$



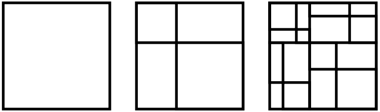
Call the corresponding σ-algebra's $\mathscr{B}_0, \mathscr{B}_1, \mathscr{B}_2, \ldots$, i.e. $\mathscr{B}_k$ is the (σ-algebra formed by the) set of dyadic cubes of size 2^{-k}.

Use $[x \upharpoonright k]$ for the dyadic interval of size 2^{-k} containing x.

Filtrations

Technically, a filtration is an increasing sequence of σ-algebras

$$\mathscr{F}_0 \subseteq \mathscr{F}_1 \subseteq \mathscr{F}_2 \subseteq \dots$$



The only specific filtration we will use is $(\mathscr{B}_k)$.

Say $\mathscr{F}_k \nearrow \mathscr{F}_\infty$ if $\mathscr{F}_\infty$ is the minimal σ-algebra containing $\bigcup_i \mathscr{F}_i$.

$\mathscr{B}_k \nearrow \mathscr{B}$, where $\mathscr{B}$ is the Borel σ-algebra.

Conditional Expectation

Let f be an L^1-function. Let $(\mathscr{B}_k)$ be the filtration of dyadic cubes. Then the conditional expectation $E[f \mid \mathscr{B}_k]$ is a function from $[0,1]^d \to \mathbb{R}$ such

$$E[f \mid \mathscr{B}_k](x) := \frac{\int_{[x \restriction k]} f\, dx}{m([x \restriction k])}.$$

Further, $E[f \mid \mathscr{B}_k]$ is L^1-computable from (codes for) f, k, and $E\widehat{[f \mid \mathscr{B}_k]} = E\left[\widehat{f} \mid \mathscr{B}_k\right]$.

Also, $E[f \mid \mathscr{B}] = f$.

Replace Balls with Dyadic Cubes.

Let $t+Q=\{t+x \mid x\in Q\}$ and $\mathscr{B}_k^t=\{t+Q \mid Q\in\mathscr{B}_k^t\}$.

Theorem (Morayne and Solecki)

Let f be an integrable function. Let $z\in[0,1]^d$. The following are equivalent.

1. $A_r\left|f-f(z)\right|(z)\to 0$ as $r\to 0$.
2. $E[|f-f(z)| \mid \mathscr{B}_k^t](z)\to 0$ as $k\to\infty$ for all $t\in\{0,\frac{1}{3}\}^d$.

Proof Sketch.

Consider one dimension. Take a ball $B_r(x)$. It is in a dyadic interval (or $\frac{1}{3}$-shift) that is at most 3 times longer. The rest is approximation. □

Necessary Criteria

Notice we are using $|f - f(z)|$ and that the average goes to 0. This is necessary for the approximation to work. And will come up later.

Outline

Levy 0-1 Law

By last result, it is enough to show $E\left[\left|f-\hat{f}(z)\right| \mid \mathscr{B}_k^t\right](z)$ converges when z is Schnorr random and f is L^1-computable.

Theorem (Levy 0-1 Law)

Given a filtration $(\mathscr{F}_k)$ and an L^1-function f, then

$$E[f \mid \mathscr{F}_k] \to E[f \mid \mathscr{F}_\infty]$$

both in L^1-norm and pointwise almost-everywhere.
Therefore, if f is $\mathscr{F}_\infty$-measurable, then

$$E[f \mid \mathscr{F}_k] \to f$$

both in L^1-norm and pointwise almost-everywhere.

Effective Levy 0-1 Law

Theorem (Effective Levy 0-1 Law, R.)

If $E[f \mid \mathscr{F}_k]$ is L^1-computable (uniformly in k) and $E[f \mid \mathscr{F}_\infty]$ is L^1-computable, then

$$E[f \mid \mathscr{F}_k] \to E[f \mid \mathscr{F}_\infty]$$

effectively in the L^1-norm and effectively a.e.
Hence on Schnorr randoms z,

$$E\widehat{[f \mid \mathscr{F}_k]}(z) \to E\widehat{[f \mid \mathscr{F}_\infty]}(z).$$

Corollary.
If f is L^1-computable and then on Schnorr randoms z,

$$E\left[\widehat{f} \mid \mathscr{B}_k\right](z) \to \widehat{f}(z).$$

Outline of Proof

Step 1: Notice $M_k := E[f \mid \mathscr{F}_k]$ is a martingale.

Step 2: Extract fast converging subsequence from M_k.

Step 3: Show effective L^1-convergence and effective a.e. convergence on the subsequence.

Step 4: Show effective L^1-convergence and effective a.e. convergence on the whole sequence using martingale properties.

Step 1: Notice M_k is a Martingale

A martingale on a filtration $(\mathscr{F}_k)$ is a sequence of functions (M_k) such that M_k is $\mathscr{F}_k$-measurable and

$$E\left[M_{k+1} \mid \mathscr{F}_k\right] = M_k.$$

Let

$$M_k := E\left[f \mid \mathscr{F}_k\right].$$

It is well known that M_k is a martingale, by

$$E\left[M_{k+1} \mid \mathscr{F}_k\right] = E\left[E\left[f \mid \mathscr{F}_{k+1}\right] \mid \mathscr{F}_k\right] = E\left[f \mid \mathscr{F}_k\right] = M_k.$$

Martingales have very nice convergence properties!

Step 2: Extract Fast Converging Subsequence

Since $M_k \to f$ in the L^1-norm (Levy's 0-1 Law) and f is L^1-computable we can find a subsequence $\left(M_{k_j}\right)$ such that

$$\left\|M_{k_j} - M_\infty\right\|_1 \leq 2^{-(2j+1)}.$$

Therefore

$$\left\|M_{k_{j+1}} - M_{k_j}\right\|_1 \leq 2^{-2j}.$$

Step 3: Show convergence on subsequence

Effectively Cauchy in L^1:

$$\left\| M_{k_{j+1}} - M_{k_j} \right\|_1 \leq 2^{-2j}.$$

Effectively a.e. Cauchy (use Markov's inequality):

$$m\left(\left\{x \mid \left|M_{k_{j+1}}(x) - M_{k_j}(x)\right| \geq 2^{-k}\right\}\right) \leq \frac{\left\| M_{k_{j+1}} - M_{k_j} \right\|_1}{2^{-k}} \leq \frac{2^{-2k}}{2^{-k}} = 2^{-k}.$$

Hence, we have effective L^1 and a.e. convergence on the subsequence.

Step 4: Show convergence on martingale

Consider the interval $[k_j, k_{j+1}]$.
$M_k - M_{k_j}$ is a martingale for $k \geq k_j$.

The L^1-norms of martingales are nondecreasing, so

$$\max_{k \in [k_j, k_{j+1}]} \left\| M_k - M_{k_j} \right\|_1 \leq \left\| M_{k_{j+1}} - M_{k_j} \right\|_1 \leq 2^{-2j}.$$

Can use submartingale inequality in place of Markov's inequality, so

$$m\left(\left\{x \mid \max_{k \in [k_j, k_{j+1}]} \left| M_k(x) - M_{k_j}(x) \right| \geq 2^{-j} \right\}\right) \leq \frac{\left\| M_{k_{j+1}} - M_{k_j} \right\|_1}{2^{-j}} \leq \frac{2^{-2j}}{2^{-j}} = 2^{-j}$$

These are sufficient for effective L^1 and a.e. convergence.

An aside: Backward's martingales

- This same proof works for "backwards martingales" (which always converge in the L^1-norm).
- This has applications to the strong law of large numbers and to parts of de Finetti's theorem.
- Backwards martingales are very similar to Ergodic averages.

Finish LDT Proof

This ends the proof of the effective Levy 0-1 law.

For LDT...

- We need to show $E\left[\left|f-\widehat{f}(z)\right| \mid \mathscr{B}_k^t\right](z) \to 0$.
- If the real number $\widehat{f}(z)$ is computable,
  then $\left|f-\widehat{f}(z)\right|$ is L^1-computable
  and $E\left[\left|f-\widehat{f}(z)\right| \mid \mathscr{B}_k^t\right]$ is L^1-computable.
  Then $E\left[\left|f-\widehat{f}(z)\right| \mid \mathscr{B}_k^t\right](z) \to \hat{f}(z)-\hat{f}(z)=0$.
  We are done.
- But $\widehat{f}(z)$ need not be computable...
  Instead, can use the L^1-code of f to approximate.

Outline

Theorem (LDT Reversal)

Let (U_i) be a Solovay test for Schnorr test. There is an L^1-computable function f such that $A_r f(z)$ does not converge as $r \to 0$ for any z covered by (U_i).

Enough to show $E[f \mid \mathscr{B}_k](z)$ does not converge.

The Reversal

There are many reversals. This is the simplest.

- Let (U_i) be a Solovay test for Schnorr randomness.
- WLOG, each U_i is a dyadic cube.
- Let $f = \sum_i \mathbf{1}_{U_i}$.
- f is L^1-computable, with L^1-code $p_k = \sum_{i=0}^{i_k - 1} m(U_i)$, where i_k is such that

$$\|f - p_k\|_1 = \sum_{i=0}^{\infty} m(U_i) - \sum_{i=0}^{i_k - 1} m(U_i) = \sum_{i=i_k}^{\infty} m(U_i) < 2^{-2k}$$

- If z is covered by U_i then $f(z) = \infty$. Further,

$$E[f \mid \mathscr{B}_k](z) > \#\{U_i \ni z \text{ of size } \geq 2^{-k}\} \to \infty.$$

This is the integral test construction of Kenshi Miyabe.

Outline

Absolutely Continuous Functions

A function $F : [0,1] \to \mathbb{R}$ is absolutely continuous if $F(x) = \int_0^x f(x)\,dx + F(0)$.

Absolutely continuous functions are the ones "satisfying the fundamental theorem of calculus."

Theorem (Lebesgue)

If $F(x) = \int_0^x f(x)\,dx + F(0)$ and f is integrable, then F is differentiable a.e. and $F' = f$ a.e.

This is a corollary to (a version of the) LDT. Since

$$F'(x) = \lim_{\substack{b \to x^+ \\ a \to x^-}} \frac{F(b) - F(a)}{b - a} = \frac{\int_a^b f(x)\,dx}{b - a} \to f \quad \text{a.e.}$$

Absolutely Continuous Functions

Theorem (Lebesgue)

If $F(x) = \int_0^x f(x)\,dx + F(0)$ and f is integrable, then F is differentiable a.e. and $F' = f$ a.e.

Theorem (Effective Version)

If $F(x) = \int_0^x f(x)\,dx + F(0)$ and f is L^1-computable, then F is differentiable on Schnorr randoms and $F' = \hat{f}$ on Schnorr randoms.

Paradigm

Theorem: bounded $\Rightarrow$ converges

Effective Theorem: bound is computable and limit is L^1-computable $\Rightarrow$ effective a.e. convergence and convergence on Schnorr randoms.

Martingale convergence

A martingale (M_k) is L^1-bounded if $\sup_k \|M_k\|_1 < \infty$. Call $\sup_k \|M_k\|_1$ the L^1-bound.

- Nonnegative martingales are L^1-bounded since $\|M_0\|_1 = \|M_1\|_1 = \cdots$.
- Martingales of the form $M_k = E[f \mid \mathscr{F}_k]$ are L^1-bounded with an L^1-bound $\|f\|_1$.

Theorem (Doob)

If (M_k) is a martingale and is L^1-bounded, then (M_k) converges a.e. to a limit M_∞.

Martingale convergence

Theorem (Doob)

If (M_k) is a martingale and is L^1-bounded,
then (M_k) converges a.e. to a limit M_∞.

(M_k) is an L^1-computable martingale if (M_k) is a uniform sequence of L^1-computable functions.

Theorem (Effective Martingale Convergence, R.)

If (M_k) is an L^1-computable martingale, the L^1-bound is computable, and the limit M_∞ is L^1-computable,
then $M_k \to M_\infty$ effectively a.e. (so $\widehat{M}_k \to \widehat{M}_\infty$ on Schnorr randoms).

Effective Martingale Convergence Theorem

Theorem (Effective Martingale Convergence, R.)

If (M_k) is a martingale, the L^1-bound is computable, and the limit M_∞ is computable, then $M_k \to M_\infty$ effectively a.e. (so it converges on Schnorr randoms to $\widehat{M}_\infty$).

The proof is very similar to the effective Levy 0-1 law:

- Find subsequence which converges fast in measure (not in L^1-norm) to M_∞.
- We have effective a.e. convergence on the subsequence.
- Ignore points which don't converge fast enough.
- Use the submartingale inequality to show the martingale does not deviate much in between the subsequence.

Measures and Radon-Nikodym derivative

A signed measure ν is like a measure but the mass could be negative. The norm $\|\nu\|$ is the sum of the positive mass and negative mass.

Theorem

Let ν be a signed measure on $[0,1]^d$ with $\|\nu\| < \infty$. Then

$$\frac{\nu(B_r(x))}{m(B_r(x))} \to \frac{d\nu}{dm} \quad m\text{-a.e.}$$

where $d\nu/dm$ is the Radon-Nikodym derivative.

Measures and Radon-Nikodym derivative

Theorem

Let ν be a signed measure on $[0,1]^d$ with $\|\nu\| < \infty$. Then

$$\frac{\nu(B_r(x))}{m(B_r(x))} \to \frac{d\nu}{dm} \quad \textit{m-a.e.}$$

A computable signed measure is a computable linear transformation on the continuous functions (use the Riesz representation theorem).

Theorem (R.)

Let ν be a computable signed measure on $[0,1]^d$, $\|\nu\|$ is computable, and $d\nu/dm$ is L^1-computable. Then

$$\frac{\nu(B_r(x))}{m(B_r(x))} \to \widehat{\frac{d\nu}{dm}} \quad \textit{on Schnorr randoms.}$$

Bounded Variation Functions

A function $F : [0,1] \to \mathbb{R}$ is of bounded variation if the total variation is finite.

Theorem

If F is a function of bounded variation, then F is differentiable a.e.

Can code bounded variation functions as functions on dense sequences of reals. Can define pseudo-differentiability as well.

Theorem

Let $A = \{a_n\}$ be a dense sequence of reals in $[0,1]$ and let $F : A \to \mathbb{R}$ be computable (i.e. $F(a_n)$ is computable from n). Assume the total variation is computable, and F' is L^1-computable, then F is psuedo-differentiable on Schnorr randoms and $F' = \widehat{(F')}$ on Schnorr randoms.

Paradigm

Theorem: bounded ⇒ converges

Effective Theorem: bound is computable and limit is L^1-computable
⇒ effective a.e. convergence and convergence on Schnorr randoms.

Characterizing Randomness by Theorems

Theorem: bounded $\Rightarrow$ converges

	Bound is comp.	Bound exists	No bound
Limit is comp.	SR	ML	W2R
Limit exists	CR*	ML	W2R

* The computable randomness result holds for signed measures and functions of bounded variation. For martingale convergence it depends on the filtration.

The CR, ML, and W2R results are due to Brattka-Miller-Nies (with some translation between theorems), except for martingales convergence which is due to Takahashi and separately Ed Dean, (with reversals by R.).

Characterizing Randomness by Theorems

Theorem: bounded $\Rightarrow$ converges

Each structure (measures, martingales, and functions of bounded variation) can be decomposed in two ways.

		Bound is comp.	
		Increasing/ Positive	Decreasing/ Negative
Limit is comp.	Singular & Continuous	SR	SR
	Singular & Discrete	SR	SR
	Absolutely Continuous	SR	SR

One example:
$x \in [0,1]$ is Schnorr random $\Leftrightarrow$ $F(x)$ is differentiable for all increasing computable functions with derivative 0 a.e.

Sketch of Constructions

Construct martingales:

- Take Schnorr test.
- When in new level of the test, bet an integer amount.
- For martingale 1, do nothing else.
- For martingale 2, systematically bet all the money away on one point to go back to 0.
- For martingale 3, systematically bet all the money away on a dense set of points to go back to 0.
- All martingales are L^1-bounded with computable bound.
- Martingale 1 is uniformly integrable, Martingale 2 is singular (converges to 0 a.e.) with only "atoms", and Martingale 3 is singular with no "atoms".

Summary

- The standard randomness notions allow one to prove effective versions of "almost everywhere" convergence theorems in Analysis and Probability.
- These effectivizations can be reversed (similar to Reverse Mathematics) to characterize known (or new?) types of randomness.
- Martingales are a useful tool for the study of randomness, even outside their usual form in the computability community. What other tools from modern probability and analysis can be used to study randomness?
- A better understanding of the computable aspects of Measure Theory, Probability Theory, and Ergodic Theory may lead to a better understanding of randomness. (E.g., a.e. convergence.)
- And *vice-versa*.

Thank You!

These slides are available on my webpage:

`math.cmu.edu/~jrute`.

Or just Google me, "Jason Rute".