Randomness and the Lebesgue Differentiation Theorem

Jason Rute

Department of Mathematical Science
Carnegie Mellon University

Graduate Student Conference in Logic, 2011
University of Illinois - Chicago

Outline

The Lebesgue Differentiation Theorem

Theorem (Lebesgue Differentiation Theorem)

Let m be the Lebesgue measure on $[0,1]^n$. Let $f : [0,1]^n \to \mathbb{R}$ be an integrable (L^1) function. Define the average

$$A_r f(x) = \frac{\int_{B_r(x)} f(x)\,dx}{m(B_r(x))}.$$

Then $A_r f(x) \to f(x)$ as $r \to 0$ for almost every x.

- "Almost every x" intuitively means if x is "random enough" (relative to the parameters of the theorem) then the above theorem is true for x.
- Algorithmic randomness attempts to make rigorous this notion of "random".

An Effective Lebesgue Differentiation Theorem

Theorem (Lebesgue Differentiation Theorem)

Let m be the Lebesgue measure on $[0,1]^n$. Let $f : [0,1]^n \to \mathbb{R}$ be an integrable (L^1) function. Define the average

$$A_r f(x) = \frac{\int_{B_r(x)} f(x)\, dx}{m(B_r(x))}.$$

Then $A_r f(x) \to f(x)$ as $r \to 0$ for almost every x.

Theorem (Effective Lebesgue Differentiation Theorem, R.)

Let $f : [0,1]^n \to \mathbb{R}$ be an L^1-computable function. Then $A_r f(x)$ converges as $r \to 0$ for all Schnorr randoms x.

Two Effective LDT's

Pathak has already shown the following.

Theorem (Effective LDT for Martin-Löf Randoms, Pathak)

Let $f : [0,1]^n \to \mathbb{R}$ be an L^1-computable function. Then $A_r f(x)$ converges as $r \to 0$ for all Martin-Löf randoms x.

Theorem (Effective LDT for Schnorr Randoms, R.)

Let $f : [0,1]^n \to \mathbb{R}$ be an L^1-computable function. Then $A_r f(x)$ converges as $r \to 0$ for all Schnorr randoms x.

- Since ML randoms $\subset$ Schnorr randoms, the Schnorr result is better.
- Is Schnorr randomness the best we can do? Yes.

Characterization of Schnorr Randomness

Theorem ("The Reversal", R.)

Let $x \in [0,1]^n$ be not Schnorr random. Then there is an L^1-computable function f such that $A_r f(x)$ does not converge as $r \to 0$.

Therefore the Effective LDT and the reversal, together characterize Schnorr randomness by "differentiability". This is closely related to the field of Reverse Mathematics.

Related Work

- Pathak proved the Effective LDT for Martin-Löf randoms.
- Brattka, Miller, Nies characterize computable randomness, Martin-Löf randomness, and Π^0_2-randomness in terms of differentiability of absolutely-continuous computable functions on $[0,1]$ (actually, functions of bounded variation). Martin-Löf case based on work of Demuth.
- Pathak, Rojas, and Simpson have a different proof of the Effective LDT for Schnorr randoms.
- Kenshi Miyabe also has a similar result.
- Freer, Kjos-Hanssen, Nies characterizes computable randomness and Schnorr randomness in terms of computable Lipschitz functions.
- The proof in this talk is based the tools of the Brattka, Miller, Nies result.

L^1-computable Functions

Any integrable function f can be approximated by a polynomial p_i with rational coefficients such that $\|f - p_i\|_1 = \int |f - p_i| \leq 2^{-i}$. In other words, rational polynomials are dense in L^1.

Definition

For any $f \in L^1$, a code for f is a sequence of rational polynomials (p_i) converging in the L^1-norm to f that are fast Cauchy, i.e. $\|p_{i+1} - p_i\|_1 \leq 2^{-i}$. We say f is L^1-computable if there exists such a computable code.

The following are all computable from (the codes for) $f, g \in L^1$ and (the codes for) any other parameters:

$$f + g, af, \max(f, g), \min(f, g), f^+, f^-, |f|, \|f\|_1, \int f\,dx, \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f\,dx.$$

Note that on $[0, 1]$, if f is L^1-computable then $F(x) := \int_0^x f\,dx$ is computable, but the converse does not hold.

Functions of Absolute Continuity

Definitions

Take an a.e. differentiable, continuous function $F : [0,1] \to \mathbb{R}$. We say F is absolutely continuous if $F(x) = \int_0^x f(t)\,dt + F(0)$ where f is the derivative of F, i.e. F "satisfies the Fundamental Theorem of Calculus".

We say F is effectively absolutely continuous if its derivative f is L^1-computable.

effectively absolutely continuous

$\Rightarrow$ absolutely continuous and computable

But the converse doesn't hold.

Theorem (R.)

$x \in [0,1]$ is Schnorr random $\Leftrightarrow$
$A_r f(x)$ converges for all L^1-computable functions f.

Corollary

$x \in [0,1]$ is Schnorr random $\Leftrightarrow$
$F'(x)$ exists for all effectively absolutely continuous functions F.

Proof Sketch.

Let f be the derivative of F, hence $F(x) = \int_0^x f(x)\,dx$. Then

$$F'(x) = \lim_{r \to 0} \frac{F(x+r) - F(x-r)}{2r} = \lim_{r \to 0} \frac{\int_{x-r}^{x+r} f(x)\,dx}{2r} = A_r f(x). \quad \square$$

Translation Between L^1 and Abs. Cont.

Theorem (Brattka, Miller, Nies and Freer, Kjos-Hanssen, Nies)

$x \in [0,1]$ is Martin-Löf random (resp. computably random) $\Leftrightarrow$ $F'(x)$ exists for all absolutely continuous functions F s.t

- *F is computable*
- *(and resp. nondecreasing or Lipschitz).*

Corollary

$x \in [0,1]$ is Martin-Löf random (resp. computably random) $\Leftrightarrow$ $A_r f(x)$ converges for all integrable functions f s.t.

- *$\int_0^x f(t)\, dt$ is computable*
- *(and resp. nonnegative or bounded).*

Schnorr Random

Definitions

A set $U \subseteq [0,1]^n$ is Σ^0_1 (effectively open) if it is a union of a computable sequence of open cells $(a_1, b_1) \times \cdots \times (a_n, b_n)$ with rational endpoints.

A Martin-Löf test is a uniformly computable sequence (U_i) of Σ^0_1 sets such that

$$m(U_i) \leq 2^{-i}.$$

A Schnorr test is a uniformly computable sequence (U_i) of Σ^0_1 sets such that

$$m(U_i) = 2^{-i}.$$

$x \in [0,1]^n$ is Martin-Löf random if $x \notin \bigcap U_i$ for any ML test (U_i).
$x \in [0,1]^n$ is Schnorr random if $x \notin \bigcap U_i$ for any Schnorr test (U_i).

Borel-Cantelli-Solovay Test

Theorem (Borel-Cantelli Lemma)

Let (A_k) be a sequence of sets such that $\sum m(A_k) < \infty$. Then for almost every x, x is in at most finitely many A_k.

Theorem (Solovay Lemma for Martin-Löf Randoms)

Let (U_k) be a uniformly computable sequence of Σ^0_1 sets such that $\sum m(U_k) < \infty$. (Call (U_k) a Solovay test for Martin-Löf randomness.) Then for all Martin-Löf randoms x, x is in at most finitely many U_k. Further, if x is not Martin-Löf random, there is a Solovay test (U_k) such that $x \in U_k$ for infinitely many k.

Borel-Cantelli-Solovay Test

for Schnorr Randomness

Theorem (Borel-Cantelli Lemma)

Let (A_k) be a sequence of sets such that $\sum m(A_k) < \infty$. Then for almost every x, x is in at most finitely many A_k.

Theorem (Solovay Lemma for Schnorr Randoms, Hoyrup-Rojas)

Let (U_k) be a uniformly computable sequence of Σ^0_1 sets such that $\sum m(U_k)$ is finite and computable. (Call (U_k) a Solovay test for Schnorr randomness.) Then for all Schnorr randoms x, x is in at most finitely many U_k. Further, if x is not Schnorr random, there is a Solovay test (U_k) such that $x \in U_k$ for infinitely many k.

The Main Theorems to Prove

Theorem (Effective LDT for Schnorr Randoms)
Let $f : [0,1]^n \to \mathbb{R}$ be an L^1-computable function. Then

$$A_r f(x) := \frac{\int_{B_r(x)} f(x)\,dx}{m(B_r(x))}$$

converges as $r \to 0$ for all Schnorr randoms x.

Theorem (LDT Reversal)
Let $x \in [0,1]^n$ be not Schnorr random. Then there is an L^1-computable function f such that $A_r f(x)$ does not converge as $r \to 0$.

Structure of the Proof

Geometric part of Proof. Reduce the geometric complexity of the problem.

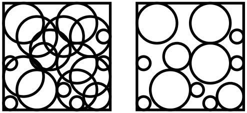
Martingale part of Proof. Use theorems about martingales to analyze the convergence.

Reversal. Construct a martingale that doesn't converge on a Schnorr test and read off an L^1-computable function.

Outline

Geometric Part

- The LDT is a very geometric theorem.
- For example, if the balls were replaced with arbitrary rectangles or ellipse, it would not be true. (But cubes are OK.)
- One approach is to replace overlapping balls with disjoint balls/cubes.



- Classical proofs use the Vitali Covering Theorem to do this.
- Brattka, Miller, Nies use a different approach, which this talk follows.

Work with Cubic Cells Instead

Work with cells instead of balls.

Definition

A closed cell Q on $[0,1]^n$ is a set of the form $Q = [a_1, b_1] \times \cdots \times [a_n, b_n]$. A cubic cell is a cell with equal lengths in all dimensions.

It is enough to approximate the balls with cubic cells. We wish to prove this equivalent version of the LDT.

Theorem

Let $f : [0,1]^n \to \mathbb{R}$ be an L^1-computable function. Let x be a Schnorr random. Let $(Q_k)_{k \in \mathbb{N}}$ be a decreasing sequence of cubes which converge to x. (x need not be center of Q_k's.) Then

$$\frac{\int_{Q_k} f(x)\,dx}{m(Q_k)}$$

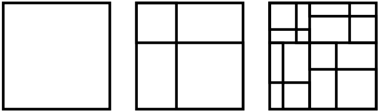
converges as $k \to \infty$.

Filtrations

Technically, a filtration is an increasing sequence of σ-algebras

$$\mathscr{F}_0 \subseteq \mathscr{F}_1 \subseteq \mathscr{F}_2 \subseteq \dots$$

For our purposes, we care only about filtrations made-up of finite partitions of $[0,1]^n$ into cells, e.g.



Call such a filtration of cells computable if it is "uniformly codable". Define $\mathscr{F}_\infty$ to be the minimal σ-algebra containing $\bigcup_i \mathscr{F}_i$. So $(\mathscr{F}_i)$ "converges to" $\mathscr{F}_\infty$.

Conditional Expectation

Let f be an L^1-function. Let $\mathscr{F}$ be a finite partition of $[0,1]^n$ into cells $Q_1, \ldots, Q_k$. Then the conditional expectation $E[f \mid \mathscr{F}]$ is a function from $[0,1]^n \to \mathbb{R}$ such that for $x \in Q_i$

$$E[f \mid \mathscr{F}](x) := \frac{\int_{Q_i} f\, dx}{m(Q_i)}.$$

Further, $E[f \mid \mathscr{F}]$ is L^1-computable from (codes for) $f, \mathscr{F}$, and $\frac{\int_{Q_i} f\, dx}{m(Q_i)}$ is computable from $f, \mathscr{F}, i$.

The LDT talks about simultaneous convergence over a "net of filtrations". We extract one filtration to work with.

Theorem (Insipred by Brattka, Miller, Nies)

Let $f : [0,1]^n \to \mathbb{R}$ *be a non-negative* L^1*-computable function. For any* x*, if for some decreasing sequence of cubic cells* (Q_i) *(not necessarily "computable") converging to* x*,*

$$\frac{\int_{Q_i} f(x)\,dx}{m(Q_i)}$$

diverges, then there is some computable filtration $(\mathscr{F}_k)$ *(made up of finite partitions of cells) such that* $\mathscr{F}_\infty = \mathscr{B}$ *(Borel* σ*-algebra) and*

$$E[f \mid \mathscr{F}_k](x)$$

diverges as $k \to \infty$*.*

f^+, f^- are L^1-computable

The requirement that f be non-negative in the previous lemma is OK, since the decomposition

$$f = f^+ - f^-$$

is computable from (the L^1 code for) f.

Outline

Levy 0-1 Law

By the Key Lemma, it is enough to show $E[f \mid \mathscr{F}_k](x)$ converges when x is Schnorr random and f is L^1-computable.

Theorem (Levy 0-1 Law)

Given a filtration $(\mathscr{F}_k)$ and an L^1-function f, then

$$E[f \mid \mathscr{F}_k] \to E[f \mid \mathscr{F}_\infty]$$

both in L^1-norm and pointwise almost-everywhere.
Therefore, if f is $\mathscr{F}_\infty$-measurable, then

$$E[f \mid \mathscr{F}_k] \to f$$

both in L^1-norm and pointwise almost-everywhere.

Effective Levy 0-1 Law

Theorem (Effective Levy 0-1 Law, R.)

Given a computable filtration $(\mathscr{F}_k)$ such that $\mathscr{F}_\infty = \mathscr{B}$ and an L^1-computable f, then

$$E[f \mid \mathscr{F}_k]$$

converges effectively in the L^1-norm, and converges pointwise on Schnorr randoms.

Step 1: Notice $M_k := E[f \mid \mathscr{F}_k]$ is a martingale.

Step 2: Approximate M_k to be rational-valued.

Step 3: Extract fast converging subsequence from M_k.

Step 4: Create Solovay test of points that don't converge "fast enough".

Step 1: Notice M_k is a Martingale

A martingale on a filtration $(\mathscr{F}_k)$ is a sequence of functions (M_k) such that M_k is $\mathscr{F}_k$-measurable and

$$E\left[M_{k+1} \mid \mathscr{F}_k\right] = M_k .$$

Let

$$M_k := E\left[f \mid \mathscr{F}_k\right].$$

It is well known that M_k is a martingale, by

$$E\left[M_{k+1} \mid \mathscr{F}_k\right] = E\left[E\left[f \mid \mathscr{F}_{k+1}\right] \mid \mathscr{F}_k\right] = E\left[f \mid \mathscr{F}_k\right] = M_k .$$

Martingales have very nice convergence properties!

Step 2: Approximate M_k to Be Rational Valued

By a straight-forward approximation argument, M_k can be assumed to be rational-valued. (Not entirely accurate, but close enough.)

Step 3: Extract Fast Converging Subsequence

Since $M_k \to f$ in the L^1-norm (Levy's 0-1 Law) and f is L^1-computable we can find a subsequence $\left(M_{k_j}\right)$ such that

$$\left\|M_{k_j} - f\right\|_1 \leq 2^{-(2j+1)}.$$

Therefore

$$\left\|M_{k_{j+1}} - M_{k_j}\right\|_1 \leq 2^{-2j}.$$

Actually, we can show $M_k \to f$ effectively in L^1-norm.

Step 4: Create Solovay Test

First notice that by Markov's Inequality we have

$$m\left\{x : \left|M_{k_{j+1}}(x) - M_{k_j}(x)\right| \geq 2^{-j}\right\} \leq \frac{\left\|M_{k_{j+1}} - M_{k_j}\right\|_1}{2^{-j}} \leq \frac{2^{-2j}}{2^{-j}} = 2^{-j}.$$

But we need something stronger. Since (M_k) is a martingale, we have this stronger inequality (Ville's Inequality/Doob's Submartingale Inequality).

$$m\left\{x : \max_{k_j \leq k \leq k_{j+1}} \left|M_k(x) - M_{k_j}(x)\right| \geq 2^{-j}\right\} \leq \frac{\left\|M_{k_{j+1}} - M_{k_j}\right\|_1}{2^{-j}} \leq 2^{-j}.$$

Step 4: Create Solovay Test

Let

$$U_j = \left\{ x : \max_{k_j \le k \le k_{j+1}} \left| M_k(x) - M_{k_j}(x) \right| \ge 2^{-j} \right\}.$$

Then

- (U_j) is uniformly Σ^0_1 (ignoring the boundaries of the cells).
- $m(U_j) \le 2^{-j}$ so $\sum m(U_j) \le 2$.
- $m(U_j)$ is computable (since U_j is a finite union of cells from the filtration and (M_k) is rational valued).

It follows that (U_j) is a Solovay test for Schnorr randomness.

If x is a Schnorr random, then x is in only finitely-many U_k. So, $M_k(x)$ is Cauchy.

Further, the rate of convergence is computable from x's "randomness deficiency", i.e. the highest k such that $x \in U_k$.

End of Proofs

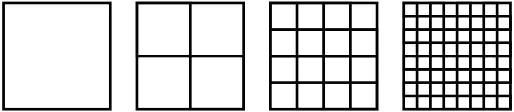
This ends the proof of the Effective Levy 0-1 Law,
and hence the proof of the Effective Lebesgue Differentiation Theorem.

Outline

Construction of L^1 Function

Remains to prove the reversal.

Let $\mathscr{F}_k$ be the partition of dyadic cubes on $[0,1]^n$.



It is enough for each non-Schnorr x to find an L^1-computable f such that $E[f \mid \mathscr{F}_k](x)$ diverges.

Uniformly Integrable Martingales

Theorem

A martingale M_k is of the form $E[f \mid \mathscr{F}_k]$ if and only if it is uniformly integrable, i.e.

$$\sup_n \int_{\{x:|M_k(x)|>C\}} |M_k| \, dx \to 0 \quad \text{as } C \to \infty.$$

This is the property that assures M_k converges in the L^1-norm.

Sketch of Construction

- Construct martingale M_k.
- Show M_k is uniformly integrable, hence $M_k = E[f \mid \mathscr{F}_k]$ for some L^1 function f.
- Show f is L^1-computable.

The Martingale

Martingales correspond to betting strategies. Bet as follows:

- Take Solovay test (U_i) for Schnorr randomness.
- Separate U_i into disjoint union of dyadic cubes.
- For every dyadic cube $Q \in \mathscr{F}_k$, ask if Q is in disjoint union of some U_i. (Pick disjoint unions such that this is decidable.)
  - If so, bet 1 (doesn't matter if win or lose)
  - If not, bet nothing.

Easy to see that the following are equivalent

- $M_k(x)$ does not converge
- $M_k(x)$ bets infinitely often
- x is in infinitely many U_i
- x passes the Schnorr test (U_i) (so x is not Schnorr random).

Uniformly Integrable

The martingale is uniformly integrable since $\sum m(U_k) < \infty$. (Hence same construction works for Martin-Löf randomness.)

The martingale converges to an L^1-computable function f since $\sum m(U_k)$ is computable.

Let f be the L^1 limit of the martingale. If in $[0,1]$ let F be the corresponding absolutely continuous function. F is a "saw-tooth function. (Hence this construction is similar to the one of Brattka, Miller, Nies.)

Outline

Measures v.s. L^1 functions

An extension of Lebesgue Differentiation theorem is as follows.

Theorem

Let ν be a finite signed measure on $[0,1]^n$. Then

$$\frac{\nu(B_r(x))}{m(B_r(x))} \to \frac{d\nu}{dm}$$

where $d\nu/dm$ is the Radon-Nikodym derivative.

On $[0,1]$,

- $F(x) = \int_0^x f\,dx$ is an absolutely continuous function.
- $G(x) = \nu([0,x])$ is a function of bounded variation.

Hence this theorem is equivalent to the fact that a function of bounded variation is differentiable almost-everywhere.

Measures v.s. L^1 functions

An effective version is as follows.

Theorem (R.)

Let ν be a computable finite signed measure (resp. computable measure) on $[0,1]^n$. Then

$$\frac{\nu(B_r(x))}{m(B_r(x))}$$

converges as $r \to 0$ for Martin-Löf randoms x (resp. computable randoms x).

This implies a result of Brattka, Miller, Nies (about functions of bounded variation), but is a bit stronger.

What is the derivative?

Take f L^1-computable. Let

$$\hat{f} := \limsup_k \frac{\int_{B_r(x)} f\,dx}{m(B_r(x))}.$$

This is a unique (up to Schnorr randoms) representative for f.

Similarly, take a computable measure ν. Let

$$\frac{\hat{d\nu}}{dm} := \limsup_{r\to 0} \frac{\nu(B_r(x))}{m(B_r(x))}.$$

Again this is unique (up to randoms).

Can these "effective" representatives of the L^1-class be characterized?

This is related to layerwise computable functions of Hoyrup and Rojas, as well as previous work of Pathak.

Converging to 0

Partial result:

Theorem (R.)

If ν is computable measure (resp. computable signed measure) and

$$\lim_{r \to 0} \frac{\nu(B_r(x))}{m(B_r(x))} = 0$$

for almost every x, then

$$\lim_{r \to 0} \frac{\nu(B_r(x))}{m(B_r(x))} = 0$$

for Schnorr randoms (resp. ML randoms) x.

The reversal also holds.

Convergence of Martingales

Levy's 0-1 law is related to Doob's Martingale Convergence Theorem.

Theorem (Doob's Martingale Convergence Theorem)

If (M_k) is an L^1-bounded martingale, then M_k converges almost everywhere as $k \to \infty$.

Effective versions of this have been given by Takahashi and Dean for Martin-Löf randomness and Brattka, Miller, Nies for computable randomness.

The techniques in this talk can be applied to get more effectivizations of Doob's Martingale Convergence Theorem.

Summary

- The standard randomness notions allow one to prove effective versions of "almost everywhere" convergence theorems in Analysis and Probability.
- These effectivizations can be reversed (similar to Reverse Mathematics) to characterize known (or new?) types of randomness.
- Martingales are a useful tool for the study of randomness, even outside their usual form in the computability community. What other tools from modern probability and analysis can be used to study randomness?
- A better understanding of the computable aspects of Measure Theory, Probability Theory, and Ergodic Theory may lead to a better understanding of randomness.
- And *vice-versa*.

Thank You!

These slides are available on my webpage:

`math.cmu.edu/~jrute`.

Or just Google me, "Jason Rute".