# RESEARCH STATEMENT

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Contents



## 1. Overview

My work is on the connections between logic and analysis. On the analysis side, I am concerned with probability theory and related areas—ergodic theory, potential theory, stochastic calculus, etc. On the logical side I am interested in computability theory and proof theory. While there are many deep connections between logic and probability theory, I am especially interested in two areas: rates of convergence and algorithmic randomness.

As for rates of convergence in analysis, it is well known that a sequence $(a_n)$ of real numbers converges iff it is Cauchy: for all $\varepsilon > 0$ there is an $n(\varepsilon)$ such that for all $m, m' \geq n(\varepsilon)$ one has $|a_m - a_{m'}| < \varepsilon$. This function $n(\varepsilon)$ is called a rate of convergence. A rate of convergence is, informally, a witness to the fact that this sequence converges. However, many theorems in analysis do not have explicit rates of convergence, such as the monotone convergence principle: every bounded monotone sequence of reals converges.

Instead, there are other equivalent ways to say that a sequence $(a_n)$ converges, and each of these ways comes with its own notion of a "rate of convergence," which provides some quantitative estimate on the convergence. One very weak rate of convergence, called metastable convergence, comes from Gödel's Dialectica interpretation in proof theory. Avigad, Dean, and I [ADR12b] answered a question of T. Tao about the computability of metastable rates of convergence for the dominated convergence theorem. In a work in progress [Rutf], I also show that for certain theorems in analysis, the metastable rates are computable just from the statement of the theorem. What one needs is that part of the theorem can be axiomatized in continuous logic, a variant of first-order logic designed to work with metric spaces. Avigad and I [AR15] also gave a new quantitative bound on the rate of convergence for the mean ergodic theorem in a uniformly convex Banach space.

As for algorithmic randomness, informally, a point in a probability space (for example, an infinite sequence of fair-coin tosses) is said to be *algorithmically random* if it obeys all "computable probability laws." Different definitions of "computable probability law" lead to different definitions of algorithmic randomness, the most well-known being Martin-Löf randomness. However, there are other algorithmic randomness notions, with names such as Schnorr randomness and computable randomness.

Much of the previous work in algorithmic randomness has focused on studying the Martin-Löf randoms in the probability space of fair-coin tosses (viewed as infinite bit sequences). There have been numerous investigations into the computational power and information content of the Martin-Löf random bit sequences. This has led to many important discoveries.

However, there is a growing trend in algorithmic randomness (1) to explore other notions of randomness besides Martin-Löf randomness, (2) to look at more general probability spaces, such as Brownian motion, and (3) to not only ask questions about computational power, but also to study how algorithmic randomness relates to classical probability theory and analysis. These investigations have led to many important insights into randomness, computability, and computable analysis.

Much of my work is focused on these new approaches. I have made contributions to the theory of Schnorr and computable randomness. I have also developed randomness for capacities (which are basically subadditive measures). Last, I have developed tools for connecting randomness to theories in classical analysis. These tools have allowed me and my coauthors to solve a number of questions in algorithmic randomness [ADR12a, KHNR14, MR13, RR, Rutd, Rut13, Ruta, Rutc, Rutb, Rute, Rutg, Ruth].

In the remaining sections I will provide more information about my projects. Section 2 is on rates of convergence and is intended for a broad mathematical audience. Section 3 is on algorithmic randomness and is accessible to those familiar with either logic or analysis. Appendix A is a more advanced look at my work in algorithmic randomness and computability theory, including some of my future projects. It is intended mostly for experts in the field.

## 2. Rates of convergence

2.1. **Metastable rates of convergence.** As mentioned in the overview, there are many convergence theorems in analysis which do not have an explicit rate of convergence—where a "rate of convergence" for a sequence $(a_n)_{n\in\mathbb{N}}$ of reals is a function $n(\varepsilon)$ satisfying

$$\forall\varepsilon > 0\ \forall m, m' \geq n(\varepsilon)\ |a_n - a_{n'}| < \varepsilon.$$

In order to get a quantitative handle on the convergence, one must explore different ways to say $(a_n)$ converges. Each of these ways comes with their own "rate of convergence." One such way to express convergence is called metastable convergence, and it comes from Gödel's Dialectica interpretation. The sequence $(a_n)_{n\in\mathbb{N}}$ of reals converges iff the following expression holds

$$\forall\varepsilon > 0\ \forall F\colon \mathbb{N} \to \mathbb{N}\ \exists m\ \forall n, n' \in [m, F(m)]\ |a_n - a_{n'}| < \varepsilon. \tag{2.1}$$

While this expression looks complicated, the idea is simple. If $(a_n)_{n\in\mathbb{N}}$ does not converge, then there is some $\varepsilon > 0$ such that $(a_n)$ "jumps by $\varepsilon$" infinitely often. More specifically, for each $m$, there is some pair $n, n' \geq m$ such that $|a_n - a_{n'}| \geq \varepsilon$. Then there is a function $F$ which finds an some upper bound $F(m)$ on $n, n'$. The above expression is saying that the opposite is true. Every such function $F$ fails.

A *rate of metastable convergence* is an upper bound $M(\varepsilon, F)$ on the variable $m$ in (2.1). (This idea extends to any complete metric space.)

While, (2.1) is just equivalent to saying that $(a_n)$ converges, metastable rates have some important advantages. For one, rates of metastable convergence are much more computable and uniform that the usual rates of convergence. For example, if $(a_n)$ is a monotone sequence of reals in the interval $[0, 1]$, then for any $\varepsilon$ and $F$, there is an $m \leq F^{\lfloor 1/\varepsilon\rfloor}(0)$ such that $\forall n, n' \in [m, F(m)]\ |a_n - a_{n'}| < \varepsilon$. (Here $\lfloor 1/\varepsilon\rfloor$ is the number of iterations $FF\cdots F(0)$ of the function $F$.)

Moreover, metastability is an important tool in some recent theorems in analysis, such as T. Tao's [Tao08] proof of the ergodic theorem for multiple ergodic averages for commuting transformations. In that paper, Tao asked for an explicit metastable rate of convergence for the dominated convergence theorem in measure theory. Avigad, Dean, and I [ADR12b] answered this question by using logic tools—such as recursion on a well-founded tree—to extract an explicit rate of metastable convergence from the proof of the dominated convergence theorem.

2.2. **Variational bounds for the mean ergodic theorem in a uniformly convex Banach space.** A *Banach space* $(\mathbb{B}, \|\cdot\|)$ is a complete normed vector space. The space $\mathbb{B}$ is *uniformly convex* if for all $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that whenever $\|x\|, \|y\| \leq 1$ and $\|x - y\| \geq \varepsilon$, then $\|(x + y)/2\| \leq 1 - \delta(\varepsilon)$. (Informally, this says that the unit ball is more like a round ball than a cube with flat faces.) Moreover, $\mathbb{B}$ is $p$-uniformly convex (for $p \geq 2$) if $\delta(\varepsilon) = K\varepsilon^p$ for some constant $K > 0$. Every uniformly convex Banach space is isomorphic to some $p$-uniformly convex Banach space. The well-known spaces $L^p$ and $\ell^p$ are examples of $p$-uniformly convex spaces (for $p \geq 1$). The following version of the mean ergodic theorem is due to Birkhoff.

**Fact 1** (Mean ergodic theorem)**.** *If* $(\mathbb{B}, \|\cdot\|)$ *is a uniformly convex Banach space, and* $T$ *is a linear operator on* $X$ *such that* $\|T(x)\| = \|x\|$*, then the ergodic averages* $A_n x := \frac{1}{n}\sum_{k=0}^{n-1} T^k(x)$ *converge as* $n \to \infty$*.*

Kohlenbach and Leuştean [KL09], using logic, gave metastable rates of convergence for this theorem, depending only on $\|x\|$ and the modulus of uniform convexity $\delta(\varepsilon)$. Motivated by their result, Avigad and I [AR15] gave the following even stronger "variational bound" on the rate of convergence.

**Theorem 2** (Avigad and Rute)**.** *If* $(\mathbb{B}, \|\cdot\|)$ *is a* $p$*-uniformly convex Banach space, and* $T$ *is a linear operator on* $\mathbb{B}$ *such that is* $\|T(x)\| = \|x\|$*, then for any* $x$ *in* $\mathbb{B}$ *and any nondecreasing sequence* $(t_k)_{k\in\mathbb{N}}$ *of natural numbers,*

$$\sum_k \|A_{t_{k+1}}x - A_{t_k}x\|^p \leq C\|x\|^p$$

*for some constant* $C$*. This result is uniform, in that, if* $\delta(\varepsilon) = K\varepsilon^p$ *is the modulus of uniform-convexity, then* $C$ *depends only on* $K$ *and* $p$*.*

This result generalizes similar results for $\mathbb{B} = L^p$ given by Jones, Kaufman, Rosenblatt, and Wierdl, but uses new ideas such as Pisier's work on "Banach space-valued martingales" and a novel version of the "Calderón transfer principle."

Avigad and I, in the same paper, also showed that there are no uniform variational bounds nor metastable rates for Banach spaces which are not isomorphic to uniformly convex spaces.

2.3. **Metastable rates and continuous logic.** Consider again the mean ergodic theorem for a uniformly convex Banach space. This theorem is about a *structure* $(\mathbb{B}, \|\cdot\|, T, x_n)$ made up of a Banach space $(\mathbb{B}, \|\cdot\|)$, a map $T$, and a sequence of points $x_n \in \mathbb{B}$ (namely, $x_n$ is the ergodic average $A_n x_0$). It is also about a *property* $P$ of this structure, namely that $(\mathbb{B}, \|\cdot\|)$ is uniformly convex, $T$ is a linear isometry, and $x_n = \frac{1}{n}\sum_{k=0}^{n-1} T^k(x_0)$. Then the mean ergodic theorem says that $(x_n)$ converges. Recall that Kohlenbach and Leuştean [KL09] gave a computable metastable rate of convergence for the mean ergodic theorem which is uniform, in that it only depends on $\|x\|$ and the modulus of uniform convexity $\delta(\varepsilon)$.

Given another theorem $(T)$ of the above form, one may ask if the rate of metastable convergence in $(T)$ is also both uniform and computable. Kohlenbach [Koh05], using proof theory, showed that if $(T)$ is provable in some particular formal system $A^\omega[X, d]$, then the rate of metastable convergence is computable from the proof. His result technically requires checking that $(T)$ is provable in $A^\omega[X, d]$. Avigad and Iovino [AI13], using model theory, showed that if the theorem $(T)$ is true and the property $P$ is preserved by ultraproducts (a powerful, but complex tool used in both logic and analysis), then the rate of metastable convergence is uniform. (Their result says nothing about the computability of this rate.) Using two areas of logic—computable analysis and computable continuous model theory—I [Rutf] showed that if the theorem $(T)$ is true and the property $P$ is axiomitizable in continuous logic (an extension of first-order logic used to work with metric spaces) then the corresponding metastable bounds are both uniform and computable from $P$. This result covers the mean ergodic theorem as well as a number of other convergence theorems in analysis.

## 3. Algorithmic randomness

3.1. **Background on computable analysis.** Computability theory is the study of computable objects, but also the study of the computable connections between non-computable objects. A function on the natural numbers $f\colon \mathbb{N}^k \to \mathbb{N}$ is *computable* is there is an algorithm (that is, a computer program) which takes in input $n_1, \dots, n_k$ and returns the output $f(n_1, \dots, n_k)$. Not all functions $f\colon \mathbb{N}^k \to \mathbb{N}$ are computable since there are only countably many computer programs, but uncountably many such functions.

However, in computable analysis, we are interested in working with uncountable topological spaces, such as the unit interval $[0, 1]$ (with the standard topology). For example, a *computable function* $f\colon [0, 1] \to [0, 1]$ is the effective (that is computable) analogue of a (uniformly) continuous function. To compute $f(x)$ the algorithm performs the following steps: (1) The computer asks the user for some rational $\varepsilon > 0$ which specifies how close the computer's output needs to be to $f(x)$. (2) Given this $\varepsilon$, the computer gives the user a small rational $\delta > 0$ and asks for a rational approximation $r \in \mathbb{Q}$ such that $|x - r| \leq \delta$. (3) Given this $r$, the computer outputs a rational $q \in \mathbb{Q}$ such that $|f(x) - q| \leq \varepsilon$. Notice that the computer never knows $x$, just an approximation $r \approx x$, and never computes $f(x)$ exactly, just an approximation $q \approx f(x)$. Nonetheless, this algorithm can provide an output of arbitrarily good precision. It is clear from this description that such a computable $f$ is (uniformly) continuous. Conversely, every computable function in a standard calculus book—$e^x$, $\sin x$, and even the standard example of a nowhere-differentiable continuous function—are all computable. Most objects in analysis have such a computable analogue—computable real numbers, computable metric spaces, computable open and closed sets, computable (and hence continuous) functions $f : A \to B$ (where $A$ and $B$ are computable metric spaces), and computable Borel probability measures.

3.2. **Background on algorithmic randomness.** Algorithmic randomness arises from the concept of a "computable null set." One works on a *computable probability space* $(X, \mu)$, that is a computable complete separable metric space $X$ with a computable Borel probability measure $\mu$. For example, the space $X = \{0, 1\}^{\mathbb{N}}$ of infinite binary sequences, and the *fair-coin* measure $\mu$ given as the product measure on $\{0, 1\}^{\mathbb{N}}$ where 0 and 1 are given equal weight. Recall that a set $N \subseteq X$ is $\mu$-null if it can be covered by an open set of arbitrary small $\mu$-measure. This leads to the following definition.

**Definition 3.** A *Martin-Löf test* is a computable sequence $(U_n)$ of computable open sets such that $\mu(U_n) \leq 2^{-n}$. A set $N \subseteq X$ is a *Martin-Löf null set* if $N \subseteq \bigcap_n U_n$ for some Martin-Löf test $(U_n)$, and $x \in X$ is *Martin-Löf random* if $x$ is not in any Martin-Löf null set. Since there are only countably many Martin-Löf null sets, the set of Martin-Löf randoms has $\mu$-measure one.

I am also interested in two other randomness notions, *computable randomness* and *Schnorr randomness*. The definition of Schnorr randomness is the same as Martin-Löf randomness except that except that a *Schnorr test* is a Martin-Löf test $(U_n)$ where the function $n \mapsto \mu(U_n)$ is also computable. (Computable randomness takes more work to define. See Definition 15.) We have that the following hierarchy of randomness notions.

$$\text{Martin-Löf randoms} \subset \text{computably random} \subset \text{Schnorr random}$$

A major portion of my work has been to develop tools to work with all three randomness notions. This is especially true of Schnorr and computable randomness. For example, computable randomness had only been defined on the space $\{0,1\}^{\mathbb{N}}$ via a betting game (called a martingale) which bets successively on each bit of the sequence $x \in \{0,1\}^{\mathbb{N}}$. I extended this definition to other computable metric spaces [Rutd]. This work allowed Kjos-Hannsen, Nyugen, and I to study computably random Brownian motion [KHNR14].

Moreover, I have developed Schnorr randomness for noncomputable measures [Rutg]. This new definition extends work by Miyabe and me characterizing Schnorr and computable randomness on product measures [MR13]. This work has lead to new ideas in the computability theory of uniform computations (see Subsection A.1). I have also developed a number of other tools for working with Schnorr and computable randomness. (See Subsection A.2 for more details.)

3.3. **Algorithmic randomness and classical analysis.** One thing remarkable about algorithmic randomness is that we can take an "almost-everywhere" theorem and ask for which randomness notions does this theorem hold when "almost-every $x$" is replaced with "every random $x$." For example, consider this simple version of the strong law of large numbers:

**Fact 4.** *For almost-every sequence $x = (x_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ (w.r.t. the fair-coin measure), we have that $\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} x_k = \frac{1}{2}$. (That is, the density of 1s in the string is $1/2$.)*

It is well-known that for Schnorr random $x$, $\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} x_k = \frac{1}{2}$. Hence one can replace "for almost-every $x$" with "for every Schnorr random $x$." However, the situation is even more interesting when there are parameters in the theorem. For example, consider a version of the Lebesgue differentiation theorem.

**Fact 5.** *Let $f\colon [0,1] \to \mathbb{R}$ be an integrable function. For almost-every real $x \in [0,1]$ (w.r.t. the Lebesgue measure), we have that $\frac{1}{2r}\int_{x-r}^{x+r} f(y)\,dy$ converges as $r \to 0$.*

For all $x \in [0,1]$, it is easy to construct some integrable $f$ such that $\frac{1}{2r}\int_{x-r}^{x+r} f(y)\,dy$ diverges as $r \to 0$. On the other hand, we have something remarkable if we restrict ourselves to only "computably integrable functions $f$" (which are the computable analogues of integrable functions). This next result was proved independently by me [Rut13, Rutc] and by Pathak, Rojas, and Simpson [PRS14].

**Theorem 6** (Rute; Pathak, Rojas, and Simpson)**.** *The following are equivalent for a real $x \in [0,1]$.*

1. *$x$ is Schnorr random.*
2. *For all computably integrable $f\colon [0,1] \to \mathbb{R}$, we have that $\frac{1}{2r}\int_{x-r}^{x+r} f(y)\,dy$ converges as $r \to 0$.*

In this way, the Lebesgue differentiation theorem characterizes Schnorr randomness. There are many results which show that various almost-everywhere theorems in analysis characterize either Schnorr randomness, computable randomness, or Martin-Löf randomness. I have contributed to this research program by finding a number of other characterizations. For example, I [Rut13, Rutc], also Franklin and I [FR], showed that Schnorr and Martin-Löf randomness can be characterized by different versions of the martingale convergence theorem.

Another connection between randomness and analysis is reverse mathematics. While mathematics is about proving theorems from axioms, reverse mathematics goes farther by figuring out exactly which axioms (in this case axioms of second-order arithmetic) are needed to prove each theorem of mathematics. Avigad, Dean, and I [ADR12a] answered an old question of Simpson about which axioms are needed to prove the dominated convergence theorem. (Specifically, we showed it is equivalent to $2\text{-}\mathsf{RAN} + \mathsf{B}\Sigma_2$. Here $2\text{-}\mathsf{RAN}$ is the existence of a particular type of randomness, called 2-randomness, and $\mathsf{B}\Sigma_2$ is the proof-theoretic principle of $\Sigma_2$ collection.)

I am working on a survey [Rute] of computable and constructive measure theory starting with the work of Brouwer in 1919. This survey carefully shows how many different mathematical traditions (Brouwer's intuitionism, Russian constructivism, reverse mathematics, etc.) have repeatedly developed the same ideas about measure theory and its computable content. Moreover, I explain how algorithmic randomness, especially Schnorr randomness, is implicitly connected to all this work.

A computable understanding of measure theory goes beyond results in randomness. For example, D. Roy (a statistician) and C. Freer study probabilistic programming languages. Avigad and I worked with them to figure out how to best represent "graphons" in a probabilistic programming language [AFRR]. (A *graphon* or *graph limit* is a measure-theoretic structure arising from the Aldous-Hoover theorem concerning exchangeable arrays of random variables.)

Moreover, I have developed a theory of computable conditional probability [Ruta], which besides being useful for randomness (see Subsection A.2), has independent interest in computable analysis and statistics.

3.4. **Algorithmic randomness and ergodic theory.** One of the most successful connections between algorithmic randomness and classical analysis has been in ergodic theory. Ergodic theory is the study of measure preserving flows. It is closely linked to information theory, which, in turn, is closely linked to algorithmic randomness. Despite the attention this subject has received, there are still many open questions.

A measure $\mu$ on $\{0,1\}^{\mathbb{N}}$ is *shift-invariant* if it is invariant under the left shift map—that is $\mu(T^{-1}(A)) = \mu(A)$ for all measurable sets $A$ where $T(x_0x_1\ldots) = x_1x_2\ldots$. A shift-invariant measure $\mu$ is *ergodic* if $T^{-1}(A) = A$ implies that $\mu(A) = 0$ or 1.

I have investigated computable versions of the ergodic decomposition theorem, which says that each shift-invariant measure is a convex combination of ergodic measures. Specially, consider the frequency of occurrences of a finite bit-string $\sigma \in \{0,1\}^*$ in an infinite bit-sequence $x \in \{0,1\}^{\mathbb{N}}$,

$$\operatorname{freq}(\sigma, x) := \lim_{n\to\infty} \frac{\#\{k < n : \sigma = x_k x_{k+1} \ldots x_{k+|\sigma|-1}\}}{n}.$$

For $\mu$-almost-every $x$, we have that (1) $\operatorname{freq}(\sigma, x)$ exists for every $\sigma$, and (2) $\operatorname{freq}(\sigma, x)$ defines a measure $\mu_x$ given by

$$\mu_x\{x\colon \sigma = x_0 x_1 \ldots x_{|\sigma|-1}\} = \operatorname{freq}(\sigma, x).$$

Moreover, (3) $\mu$ is a convex combination of the $\mu_x$ in the sense that for measurable sets $A$, $\mu(A) = \int \mu_x(A)\, d\mu(x)$, and (4) $\mu_x$ is an ergodic measure for $\mu$-almost-every $x$. A special case of a shift-invariant measure is an *exchangeable measure*, that is, $\mu$ is invariant under any map $T(x_0 x_1 \ldots) = x_{f(0)} x_{f(1)} \ldots$ where $f\colon \mathbb{N} \to \mathbb{N}$ is injective. In this case, $\mu_x$ is almost surely a Bernoulli measure (a measure corresponding to i.i.d. tosses of a weighted coin).

In the case of exchangeable measures, I have shown the following computable version of (1) through (4) above.

**Theorem 7.** *If $\mu$ is a computable exchangeable measure and $x$ is $\mu$-Schnorr random then (1) $\operatorname{freq}(\sigma, x)$ exists for all $\sigma$, (2) $\operatorname{freq}(\sigma, x)$ defines a measure $\mu_x$, (3) $x$ is $\mu_x$-Schnorr random, and (4) $\mu_x$ is a Bernoulli measure.*

If $\mu$ is merely a computable shift-invariant measure, then Schnorr randomness is not sufficient, but by results of V'yugin [V'y98] and Hoyrup and Rojas [HR09], if $x$ is Martin-Löf random, then (1) $\operatorname{freq}(\sigma, x)$ exists for all $\sigma$ and (2) $\operatorname{freq}(\sigma, x)$ defines a measure $\mu_x$. However, for (3), Reimann and I [RR] used the ergodic theory technique of "cutting and stacking" to show the following negative result.

**Theorem 8** (Reimann and Rute)**.** *There exists a computable exchangeable measure $\mu$ and a $\mu$-Martin-Löf random $x$ such that $x$ is not $\mu_x$-Martin-Löf random.*

This answers a question of Hoyrup [Hoy13]. However, for (4), Reimann and I are working on the following question which is still open.

**Question 9.** *Is the measure $\mu_x$ an ergodic measure for all $\mu$-Martin-Löf random $x$.*

This is an interesting question since most natural almost-everywhere properties, when made computable, hold for Martin-Löf randoms. However, this result seems to resist being true for Martin-Löf randoms. If that is the case, then there is something special about this theorem compared to the other a.e. theorems in ergodic theory.

3.5. **Capacities, random closed sets, and Brownian motion.** A fruitful area that I have been working on is randomness for capacities. A *capacity* $C$ is much like a measure $\mu$ except that it is subadditive, that is for disjoint sets $A$, $B$ one has $C(A \cup B) \leq C(A) + C(B)$ instead of the usual $\mu(A \cup B) = \mu(A) + \mu(B)$. As with other analytic objects, there is a suitable definition of *computable capacity*.

In analogy to Martin-Löf randomness on measures, say that $x$ is *$C$-Martin-Löf random* for a computable capacity $C$ if $x \notin \bigcap U_n$ for any computable sequence of opens sets $U_n$ such that $C(U_n) \leq 2^{-n}$. Randomness for capacities is far from generalization for generalization's sake. I have been able to use the concept to unify a number of concepts and results in randomness. I also was able to use it to solve some open problems in the randomness of Brownian motion.

While there are many notions of capacity in the analysis literature, one of the most important is Riesz capacity, which is defined via potential theory. The *$s$-energy* of a measure $\mu$ on $\mathbb{R}^n$ (or $\{0,1\}^{\mathbb{N}}$) is defined via the double integral

$$\mathcal{E}_s(\mu) = \iint |x-y|^{-s}\, d\mu(x) d\mu(y).$$

Then the *$s$-dimensional Riesz capacity* of a Borel set $A \subseteq \mathbb{R}^n$ (or $\{0,1\}^{\mathbb{N}}$) is defined by

$$C_s(A) = \sup \mu(A)$$

where the supremum is over all Borel measures $\mu$ on $\mathbb{R}^n$ (or $\{0,1\}^{\mathbb{N}}$) such that $\mathcal{E}_s(\mu) \leq 1$. As with measures, sets $A$ with $C_s(A) = 0$ are small in some sense. Also, the $s$ relates to the Hausdorff dimension (fractal dimension) of the set $A$. Specifically, the Hausdorff dimension of a set $A$ is the infimum of all $s$ such that $C_s(A) = 0$.

Surprisingly, randomness for $s$-dimensional Reisz capacity has already been studied under a different name and a different definition. Kjos-Hanssen and Diamondstone defined a sequence $x \in \{0,1\}^{\mathbb{N}}$ to be *$s$-energy random* if it is Martin-Löf random for some measure $\mu$ such that the $s$-energy of $\mu$ is finite. They created the concept in an attempt to characterize a certain class of sequences (namely, the sequences in the Martin-Löf random closed sets of Barmpalias, Brodhead, Cenzer, Dashi, and Weber). I showed these two concepts, $s$-energy randomness and Martin-Löf randomness for $s$-dimensional Riesz capacity, are equivalent. I [Rutb] also completed the characterization that Kjos-Hanssen and Diamondstone started.

Recently, J. Miller and I [MR] gave a characterization of $s$-energy randomness in terms of *a priori complexity $KM$*. Given a finite bit string $\sigma \in \{0,1\}^*$, $KM(\sigma)$ is a measure of the computational complexity of $\sigma$. (It is similar to, but different than, prefix-free Kolmogorov complexity $K(x)$, which is the length of the shortest computer program, written in binary, that can compute $x$—under suitable assumptions on the programming language used.) In this next theorem, $x \upharpoonright n$ is the string given by the first $n$ bits of $x$.

**Theorem 10** (Miller and Rute [MR]). *The following are equivalent for $x \in \{0,1\}^{\mathbb{N}}$ and computable $0 < s < 1$.*

(1) $\sum_n 2^{sn - KM(x\upharpoonright n)} < \infty$.
(2) *$x$ is $s$-energy random (that is, $x$ is Martin-Löf random for $s$-dimensional Riesz capacity).*

By relativizing this last result to a noncomputable oracle, we get a new characterization of sets with null $s$-dimensional Riesz capacity.

**Corollary 11.** *A set $A \subseteq \{0,1\}^{\mathbb{N}}$ has null $s$-dimensional Riesz capacity iff there is some oracle $b \in \{0,1\}^{\mathbb{N}}$ such that for all $x \in A$ one has $\sum_n 2^{sn - KM^b(x \upharpoonright n)} = \infty$.*

Another application of $s$-energy randomness is the characterization of special points associated with Martin-Löf random Brownian motion. A Brownian motion is roughly a continuous-time random walk. Specifically, it can be thought of as a computable probability measure (called the Wiener measure) on the space $C[0,1]$ of continuous functions $\omega\colon [0,1] \to \mathbb{R}$. Almost-surely, a Brownian motion path $\omega$ starts with value 0 at time $t = 0$ and oscillates up and down randomly like a stock price. Allen, Bienvenu, and Slaman studied the times $t$ such that $\omega(t) = 0$ for Martin-Löf random Brownian motion paths $\omega$, but they did not give a full characterization. Using energy randomness, I [Rutb] gave the following characterization.

**Theorem 12** (Rute)**.** *The following are equivalent for $t \in (0,1]$.*

1. *$\omega(t) = 0$ for some Martin-Löf random Brownian motion path $\omega$.*
2. *$t$ is $\frac{1}{2}$-energy random.*

I also gave similar characterizations for the points visited by an $n$-dimensional Martin-Löf random Brownian motion path and the points where two planar (that is 2-dimensional) Martin-Löf random Brownian motion paths intersect [Rutb].

## Appendix A. Further work in algorithmic randomness (for experts)

This appendix is mostly for experts in computability theory.

### A.1. (Future project) Uniform reducibility.

Recall $x \in \{0,1\}^{\mathbb{N}}$ is *Turing reducible* to $y \in \{0,1\}^{\mathbb{N}}$ ($x \leq_T y$) if there is a partial computable map $f\colon \subseteq\{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ such that $f(y) = x$. If the map $f$ above is total, then $x$ is *truth-table reducible* to $y$ ($x \leq_{tt} y$); this is the uniform counterpart to Turing reducibility.

My work shows that uniform reductions are quite important when working with Schnorr randomness. Moreover, uniform reductions also very important in constructive mathematics. However, in many cases it is necessary to use oracles in more exotic spaces than $\{0,1\}^{\mathbb{N}}$. For example, in algorithmic randomness, it is common to treat a measure $\mu$ as an oracle.

Miller [Mil04] extended Turing reducibility to other computable metric spaces, for example the space $\mathcal{M}_1(\{0,1\}^{\mathbb{N}})$ of Borel probability measures on $\{0,1\}^{\mathbb{N}}$. This new reducibility led to a new degree structure, called the *continuous degrees*, which contains the Turing degrees as a proper subset. I have found it necessary to do something similar for truth-table reduction.

**Definition 13** (Rute)**.** Given two computable metric spaces $X$ and $Y$, say that $x \in X$ is *uniformly reducible to* $y \in Y$ ($x \leq_u y$) if there is an effectively closed ($\Pi^0_1$) subset of $C \subseteq Y$ and a computable map $f\colon C \to X$ such that $f(y) = x$.

This definition gives rise to a new degree structure extending the truth-table degrees. I plan to explore the properties of this new degree structure. It seems to be a more natural setting in which to investigate many of the topics related to randomness and computable analysis. For example, McNicholl and I can use uniform degrees to characterize the points found on some computable Jordan curves in $\mathbb{R}^n$. (Specifically, $x \in \mathbb{R}^n$ is on some computable Jordan curve iff there is some $r \in \mathbb{R}$ such that $x \equiv_u r$. Moreover, this implies that whenever $x$ is $\mu$-Schnorr random for some computable measure $\mu$ on $\mathbb{R}^n$, then $x$ is on some computable Jordan curve.)

A.2. **Developing tools for working with randomness.** To efficiently work with randomness and analysis, one needs the proper tools, including the right definitions, lemmas, and techniques. A large part of my work has been in developing such tools, especially for Schnorr and computable randomness. One tool, already known to hold for Martin-Löf randomness, is van Lambalgen's theorem.

**Fact 14** (Van Lambalgen)**.** *Let $\mu$ and $\nu$ be computable probability measures. Then, $(x, y)$ is $\mu \otimes \nu$-Martin-Löf random iff both $x$ is $\mu$-Martin-Löf random and $y$ is $\nu$-Martin-Löf independently of $x$.*

Here $\mu \otimes \nu$ is the usual product measure, and one says that $y$ *is $\nu$-Martin-Löf independently of* $x$ if there is no test $(U_n)$ computable from $x$ such that $\mu(U_n) \leq 2^{-n}$ and $y \in \bigcap_n U_n$. (While it looks very different, this is the algorithmic randomness version of "independence" as found in probability theory.)

Miyabe and I showed that this theorem also holds for Schnorr randomness, except using a different definition of "independent" [MR13]. Namely, $x$ is *$\mu$-Schnorr random independently of $y$* if there is a family of tests $\{(U_n^y)\}_{y \in \{0,1\}^{\mathbb{N}}}$, such that for all $y \in \{0,1\}^{\mathbb{N}}$ and all $n \in \mathbb{N}$, $\mu(U_n^y) \leq 2^{-n}$ and such that $y, n \mapsto \mu(U_n^y)$ is computable. (This concept of a new notion of independence for Schnorr randomness was first developed by Franklin and Stephan using a very different definition. Miyabe published a proof of van Lambalgen's for Schnorr randomness using the Franklin-Stephan definition, but Miyabe's paper contained significant errors. Miyabe and I published a new proof and revised the Franklin-Stephan definition to be more clear.)

Using a similar technique, I developed a theory of $\mu$-Schnorr randomness for noncomputable measures $\mu$ [Rutg]. Randomness for noncomputable measures is important for many of the results in Martin-Löf randomness, including those mentioned in Subsections 3.4 and 3.5 of this research statement.

I also extended computable randomness—which was previously only defined on $\{0,1\}^{\mathbb{N}}$—to all computable metric spaces. This definition, given below, while quite different from previous definitions of computable randomness, has many advantages. In particular, it resembles similar definitions for Martin-Löf and Schnorr randomness.

**Definition 15** (Rute)**.** If $(X, \mu)$ is a computable probability space, then $x \in X$ is *$\mu$-computably random* if $t(x) < \infty$ for all lower semicomputable functions $t\colon X \to [0, \infty]$ with a computable measure $\nu$ on $X$ such that for all Borel sets $A \subseteq X$,

$$\int_A t \, d\mu \leq \nu(A).$$

Many results in randomness depend on how randomness behaves under a measure-preserving map.

**Fact 16.** *Let $T\colon X_1 \to X_2$ be a measurable map where $(X_1, \mu)$ is a computable probability space, $X_2$ is a computable metric space, and $\mu_T$ denotes the push-forward measure given by $\mu_T(A) = \mu(T^{-1}(A))$.*

(1) *(Randomness preservation) If $x$ is $\mu$-Martin-Löf random and $T$ is computable, then $T(x)$ is $\mu_T$-Martin-Löf random.*
(2) *(No-randomness-from-nothing) If $y$ is $\mu_T$-Martin-Löf random and $T$ is computable, then there is some $\mu$-Martin-Löf random $x$ such that $T(x) = y$.*

Randomness preservation holds for Schnorr randomness, but not computable randomness. (This last observation was made by me [Rutd] and independently by Bienvenu and Porter.) However, no-randomness-from-nothing for Schnorr and computable randomness was an open question which I solved as follows.

**Theorem 17** (Rute [Ruth])**.** *No-randomness-from-nothing holds for computable randomness but not Schnorr randomness. Moreover, Martin-Löf randomness is the weakest possible randomness notion satisfying both randomness preservation and no-randomness-from-nothing.*

Nonetheless, I showed that both randomness preservation and no-randomness-from-nothing hold for Schnorr randomness and computable randomness under a natural additional condition: the conditional probability map $x \mapsto \mu(\cdot \mid T = x)$ is computable [Ruta]. Moreover, for Schnorr and Martin-Löf randomness, one can combine van Lambalgen's theorem, randomness preservation, and no-randomness-from-nothing into one theorem as follows.

**Theorem 18** (Rute)**.** *Let $T\colon X_1 \to X_2$ be a measurable map where $(X_1, \mu)$ is a computable probability space, $X_2$ is a computable metric space, and $x \mapsto \mu(\cdot \mid T = x)$ is computable (as a function from points to measures), then the following are equivalent.*

(1) *$x$ is $\mu$-random and $y = T(x)$.*
(2) *$y$ is $\mu_T$-random and $x$ is $\mu(\cdot \mid T = x)$-random independently $y$.*

*(Here "random" means either Schnorr random or Martin-Löf random.)*

Last, when working with randomness and analysis, it is necessary to work with measurable sets and measurable functions in a computable manner. To see why this is tricky, consider the set $A = \{x \in \{0,1\}^{\mathbb{N}} \colon \sup_n \frac{1}{n} \sum_{k=0}^{n-1} x_k \geq 3/4\}$ and its characteristic function $\mathbf{1}_A : \{0,1\}^{\mathbb{N}} \to \{0,1\}$. Both $A$ and $\mathbf{1}_A$ are measurable and easy to describe; however, $\mathbf{1}_A$ is not continuous—not even on a measure one set—and therefore not computable. Building on the work of others, I worked out a theory of *computably measurable sets and functions*, and showed that Schnorr randomness behaves nicely with respect to these measurable functions [Rut13, Rutc]. A computably measurable function $f(x)$—such as $\sin \frac{1}{x}$—may not be well-defined at certain values of $x$. Nonetheless, $f(x)$ is well-defined whenever $x$ is Schnorr random. Moreover, if $x$ is Schnorr random then it is "typical" for $f$ in every way that matters. For example, if $f$ is a computably Lebesgue integrable function on $[0,1]^n$, then $x$ is a Lebesgue point of $f$. If $f$ is further $\{0,1\}$-valued, then $x$ is a point of density of the either the set $\{y : f(y) = 1\}$ or the set $\{y : f(y) = 0\}$. Last, Schnorr randomness is preserved under limits of functions as follows.

**Theorem 19** (Rute [Rut13, Rutc]). *Assume $f_n\colon [0,1]^n \to \mathbb{R}$ is a computable sequence of computably measurable functions and $f\colon [0,1] \to \mathbb{R}$ is computably measurable. Assume $f_n$ converges to $f$ almost-everywhere with a* computable rate of convergence*, that is, there is a computable function $n(\varepsilon_1, \varepsilon_2)$ such that for all $\varepsilon_1, \varepsilon_2 > 0$ and for all $m, m' \geq n(\varepsilon_1, \varepsilon_2)$ one has*

$$\mu \{x : |f_m(x) - f_{m'}(x)| > \varepsilon_1\} \leq \varepsilon_2.$$

*Then $f_n(x)$ converges to $f(x)$ for all Schnorr randoms $x$.*

All these above results, while technical, make it much easier to work with algorithmic randomness in an analysis setting. For example, much of the work on Martin-Löf random Brownian motion follows easily from these lemmas (and also holds for Schnorr random Brownian motion).

A.3. **(Future project) Randomness for Bessel capacities and Sobolev functions.** I plan to extend my work on capacities (Subsection 3.5) to nonlinear Bessel capacities $B_{\alpha,p}$. This family of capacities is important to nonlinear potential theory and to the theory of Sobolev functions. I conjecture my result with Miller (Theorem 10) can be generalized as follows.

**Conjecture 20.** *For $x \in \{0,1\}^{\mathbb{N}}$ and computable $\alpha > 0$ and $p > 1$ ($0 < \alpha p < 1$), the following are equivalent.*

(1) $\sum_n \left(2^{(1-\alpha p)n - KM(x\restriction n)}\right)^{q-1} < \infty$ *where $q$ is the conjugate $p/(p-1)$.*
(2) *$x$ is Schnorr random for the capacity $B_{\alpha,p}$.*

I also plan to explore Schnorr randomness for capacities. Most of the methods involved in proving the theorems from Subsection 3.5 on Martin-Löf randomness for capacities do not easily extend to Schnorr randomness; it is not even clear what the correct definition of Schnorr randomness for capacities is. However, I believe that these difficulties can be resolved, and that their resolution will lead to important insights into the structure of Schnorr randomness.

I hope to apply Schnorr randomness for Bessel capacities $B_{k,p}$ to the study of computably Sobolev functions. (Very roughly, a Sobolev function $u$ in the Sobolev space $W^{k,p}(\mathbb{R}^n)$ is a function $u\colon \mathbb{R}^n \to \mathbb{R}$ which is $k$-differentiable and all $k$ derivatives are $p$-integrable. The *computably Sobolev functions* are the computable analogues of the Sobolev functions.)

Recall that for every computably integrable function $f \in L^1(\mathbb{R}^n)$, every Schnorr random point (with respect to the Lebesgue measure) behaves "typically" for $f$. For example, by Theorem 6 a point is Schnorr random iff it is a Lebesgue point of every computably integrable function. I am working on proving that Schnorr randomness for Bessel capacities accurately captures the notion of "typical points" for computably Sobolev functions, as in the following conjecture of mine.

**Conjecture 21.** *For $x \in \mathbb{R}^n$, $k \in \mathbb{N}$, and computable $p > 1$, the following are equivalent.*

1. *$x$ is a Lebesgue point of every computably Sobolev function $u \in W^{k,p}(\mathbb{R}^n)$.*
2. *$x$ is Schnorr random for the capacity $B_{k,p}$.*

A.4. **(Future project) Axioms for randomness.** The evidence in the literature supports the view that there is not one correct randomness notion, but instead a number of important randomness notions, e.g. Schnorr randomness, computable randomness, and Martin-Löf randomness. One of my research goals is to study the space of "natural randomness notions."

The main idea is that one can think of a random point as a triple $\langle x, \mu, a\rangle$ with the interpretation that $x$ is $\mu$-random relative to the oracle $a$. Then one can consider sets $\mathcal{R}$ of such triples. One can then define a set $\mathcal{R}$ of triples to be a *natural randomness notion* if it satisfies a certain basic set of axioms. The axioms are as follows.

1. For all $\mu$ and $a$, $\mu\{x : \langle x, \mu, a\rangle \in \mathcal{R}\} = 1$.
2. If $b$ is computable uniformly in $\langle a, \mu\rangle$, then $\langle x, \mu, a\rangle \in \mathcal{R}$ implies $\langle x, \mu, b\rangle \in \mathcal{R}$.
3. If $\langle x, \mu, a\rangle \in \mathcal{R}$ and $T\colon \Omega \to \Omega$ is $\mu$-computably measurable uniformly in $a$ with code $(f_n)_{n\in\mathbb{N}}$ uniformly computable in $a$, then $f_n(x)$ converges as $n \to \infty$.
4. If $\mu$ is computable uniformly in $a$, $T\colon \Omega \to \Omega$ is $\mu$-computably measurable uniformly in $a$, and $y \mapsto \mu(\cdot \mid T = y)$ is $\mu_T$-computably measurable uniformly in $a$, then
$$\left(\begin{array}{c}\langle x, \mu, a\rangle \in \mathcal{R}\\ \text{and } y = T(x)\end{array}\right) \quad \text{iff} \quad \left(\begin{array}{c}\langle y, \mu_T, a\rangle \in \mathcal{R} \text{ and}\\ \langle x, \mu(\cdot \mid T = y), \langle y, a\rangle\rangle \in \mathcal{R}\end{array}\right).$$

Here "uniformly computable," refers to the notion of uniform reduction given in Subsection A.1. The code referenced in the third axiom is a sequence of computable functions $f_n$ which converge to $T$ in a certain metric. This axiom, which is a variation of Theorem 19, is there to guarantee that the term $T(x)$ is well-defined in the fourth axiom. It is also equivalent to saying that if $\langle x, \mu, a\rangle \in \mathcal{R}$ then $x$ is Schnorr $\mu$-random uniformly relative to $a$. The fourth axiom is a version of Theorem 18 and specifies which basic operations preserve randomness. These axioms are quite powerful. For example, they can be used to prove the following.

**Theorem 22** (Rute)**.** *If $\mu$ and $\nu$ are uniformly computable relative to $a$ and $\mu \ll \nu$ where the density $f = \frac{d\nu}{d\mu}$ is $L^1(\mu)$-computable uniformly in $a$, then*
$$\langle x, \mu, a\rangle \in \mathcal{R} \quad \textit{iff} \quad \langle x, \nu, a\rangle \in \mathcal{R}.$$

I would like to characterize these natural randomness notions (as perhaps versions of Schnorr randomness relative to a class of oracles). I also want to consider other variations of this definition either using stronger axioms or other reducibilities, such as polynomial-time, Turing, hyper-arithmetic, or constructible reducibility.

A.5. **(Future project) Degrees of randomness.** There have been many attempts in the literature to characterize "degrees of randomness," that is, to find a reflexive and transitive relation $x \le y$ which captures the idea that $y$ is more random (or less random) than $x$. Some such candidate reducibilities are $\leq_{LR}$, $\leq_{Sch}$, $\leq_K$, and $\leq_{vL}$ [DH10, ch. 10].

I am working on a new approach. By thinking of random points as infinitesimally small point masses, one may attempt to compare the masses of two random points. The axioms in the previous section suggest such a relation. For randoms $\langle x,\mu,a\rangle$ and $\langle y,\nu,b\rangle$, the *randomness quotient* $\frac{\langle x,\mu,a\rangle}{\langle y,\nu,b\rangle}$ is a partially defined nonnegative real. Namely, $x,\mu,a,y,\nu,b \mapsto \frac{\langle x,\mu,a\rangle}{\langle y,\nu,b\rangle}$ is the partial function having the smallest domain which satisfies the following conditions.

(1) If $\langle x,\mu,a\rangle$ is random, and both $a$ and $b$ are computable uniformly in the other, then $\frac{\langle x,\mu,a\rangle}{\langle x,\mu,b\rangle} = 1$.
(2) If $\mu$ is computable uniformly in $a$, $T\colon \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ is $\mu$-effectively measurable uniformly in $a$, $y \mapsto \mu(\cdot \mid T=y)$ is $\mu_T$-effectively measurable uniformly in $a$, and $y = T(x)$, then
$$\frac{\langle x,\mu,a\rangle}{\langle y,\mu_T,a\rangle} = \mu(x \mid T=y) \qquad \text{and} \qquad \frac{\langle x,\mu,a\rangle}{\langle x,\mu(\cdot \mid T=y),\langle y,a\rangle\rangle} = \mu_T(y).$$
(3) If $\frac{\langle x,\mu,a\rangle}{\langle y,\nu,b\rangle}$ and $\frac{\langle y,\nu,b\rangle}{\langle z,\pi,c\rangle}$ are defined, then $\frac{\langle x,\mu,a\rangle}{\langle z,\pi,c\rangle} = \frac{\langle x,\mu,a\rangle}{\langle y,\nu,b\rangle}\cdot\frac{\langle y,\nu,b\rangle}{\langle z,\pi,c\rangle}$.
(4) If $\frac{\langle x,\mu,a\rangle}{\langle y,\nu,b\rangle} > 0$ then $\frac{\langle y,\nu,b\rangle}{\langle x,\mu,a\rangle} = \left(\frac{\langle x,\mu,a\rangle}{\langle y,\nu,b\rangle}\right)^{-1}$.

Note that if $\nu(\{y\}) > 0$, then the randomness quotient $\frac{\langle x,\mu,a\rangle}{\langle y,\nu,b\rangle}$ is equal to the actual quotient of masses $\frac{\mu(\{x\})}{\nu(\{y\})}$, justifying the terminology. I have shown that the randomness quotient is well-defined, even in the case that $\nu(\{y\}) = 0$, and have shown that this quotient has interesting properties such as the following.

**Theorem 23** (Rute)**.** *If $\langle x,\mu,a\rangle$ is random, $\mu \ll \nu$, and the density $f = \frac{d\nu}{d\mu}$ is $L^1(\mu)$-computable uniformly in $\langle \mu,a\rangle$, then $\frac{\langle x,\nu,\langle\mu,a\rangle\rangle}{\langle x,\mu,a\rangle} = f(x)$.*

One can then define two "less random than" relations as follows.

(1) $\langle x,\mu,a\rangle \leq \langle y,\nu,b\rangle$ iff $\frac{\langle x,\mu,a\rangle}{\langle y,\nu,b\rangle} \leq 1$.
(2) $\langle x,\mu,a\rangle \leq^+ \langle y,\nu,b\rangle$ iff $\frac{\langle x,\mu,a\rangle}{\langle y,\nu,b\rangle} < \infty$.

I plan to investigate the properties of these new relations. I also plan to investigate whether the idea that a random is an infinitesimally small point mass can be made formal using tools from nonstandard analysis, for example Loeb measures.

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