Outline

1. Unique natural cubic spline interpolant

2. Natural cubic spline approximating $f(x) = e^x$

3. Natural cubic spline approximating $\int_0^3 e^x \, dx$
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1. Unique natural cubic spline interpolant

2. Natural cubic spline approximating $f(x) = e^x$

3. Natural cubic spline approximating $\int_0^3 e^x \, dx$
Existence of a unique natural spline interpolant

Theorem
If $f$ is defined at $a = x_0 < x_1 < \cdots < x_n = b$, then $f$ has a unique natural spline interpolant $S$ on the nodes $x_0, x_1, \ldots, x_n$; that is, a spline interpolant that satisfies the natural boundary conditions

$$S''(a) = 0 \quad \text{and} \quad S''(b) = 0$$
Proof (1/4)

Using the notation

\[ S_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3 \]

the boundary conditions in this case imply that \( c_n = \frac{1}{2} S''_n(x_n) 2 = 0 \)
and that \( 0 = S''(x_0) = 2c_0 + 6d_0(x_0 - x_0) \) so \( c_0 = 0 \).

The two equations \( c_0 = 0 \) and \( c_n = 0 \) together with the equations

\[ h_{j-1}c_{j-1} + 2(h_{j-1} + h_j)c_j + h_jc_{j+1} = \frac{3}{h_j} (a_{j+1} - a_j) - \frac{3}{h_{j-1}} (a_j - a_{j-1}) \]

produce a linear system described by the vector equation \( Ax = b \):
Existence of a unique natural spline interpolant

Proof (2/4)

A is the \((n + 1) \times (n + 1)\) matrix:

\[
A = \begin{bmatrix}
1 & 0 & 0 & \cdots & \cdots & 0 \\
0 & 2(h_0 + h_1) & h_1 & \ddots & \ddots & \vdots \\
0 & h_1 & 2(h_1 + h_2) & h_2 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & h_{n-2} & 2(h_{n-2} + h_{n-1}) & h_{n-1} \\
0 & \cdots & \cdots & 0 & 0 & 1
\end{bmatrix}
\]
Existence of a unique natural spline interpolant

Proof (3/4)

\[ \mathbf{b} \text{ and } \mathbf{x} \text{ are the vectors} \]

\[ \mathbf{b} = \begin{bmatrix} 0 \\ \frac{3}{h_1} (a_2 - a_1) - \frac{3}{h_0} (a_1 - a_0) \\ \vdots \\ \frac{3}{h_{n-1}} (a_n - a_{n-1}) - \frac{3}{h_{n-2}} (a_{n-1} - a_{n-2}) \\ 0 \end{bmatrix} \]

and

\[ \mathbf{x} = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} \]
Proof (4/4)

The matrix $A$ is strictly diagonally dominant, that is, in each row the magnitude of the diagonal entry exceeds the sum of the magnitudes of all the other entries in the row.
Natural Cubic Spline Algorithm

To construct the cubic spline interpolant $S$ for the function $f$, defined at the numbers $x_0 < x_1 < \cdots < x_n$, satisfying $S''(x_0) = S''(x_n) = 0$ (Note: $S(x) = S_j(x) = a_j + b_j (x - x_j) + c_j (x - x_j)^2 + d_j (x - x_j)^3$ for $x_j \leq x \leq x_{j+1}$):

**INPUT** $n; x_0, x_1, \ldots, x_n; a_0 = f(x_0), a_1 = f(x_1), \ldots, a_n = f(x_n)$

**OUTPUT** $a_j, b_j, c_j, d_j$ for $j = 0, 1, \ldots, n - 1$

**Step 1** For $i = 0, 1, \ldots, n - 1$ set $h_i = x_{i+1} - x_i$

**Step 2** For $i = 1, 2, \ldots, n - 1$ set

$$
\alpha_i = \frac{3}{h_i} (a_{i+1} - a_i) - \frac{3}{h_{i-1}} (a_i - a_{i-1})
$$

(Note: In what follows, Steps 3, 4, 5 and part of Step 6 solve a tridiagonal linear system using a Crout Factorization algorithm.)
Natural Cubic Spline Algorithm (Cont’d)

Step 3  Set $l_0 = 1$
\[ \mu_0 = 0 \]
\[ z_0 = 0 \]

Step 4  For $i = 1, 2, \ldots, n - 1$
\[ \text{set } l_i = 2(x_{i+1} - x_{i-1}) - h_{i-1}\mu_{i-1} \]
\[ \mu_i = h_i/l_i \]
\[ z_i = (\alpha_i - h_{i-1}z_{i-1})/l_i \]

Step 5  Set $l_n = 1$
\[ z_n = 0 \]
\[ c_n = 0 \]

Step 6  For $j = n - 1, n - 2, \ldots, 0$
\[ \text{set } c_j = z_j - \mu_j c_{j+1} \]
\[ b_j = (a_{j+1} - a_j)/h_j - h_j(c_{j+1} + 2c_j)/3 \]
\[ d_j = (c_{j+1} - c_j)/(3h_j) \]

Step 7  OUTPUT $(a_j, b_j, c_j, d_j$ for $j = 0, 1, \ldots, n - 1) \&$ STOP
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Example: \( f(x) = e^x \)

Use the data points \((0, 1), (1, e), (2, e^2),\) and \((3, e^3)\) to form a natural spline \( S(x) \) that approximates \( f(x) = e^x \).

Solution (1/7)

With \( n = 3, h_0 = h_1 = h_2 = 1 \) and the notation

\[
S(x) = S_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3
\]

for \( x_j \leq x \leq x_{j+1} \), we have

- \( a_0 = 1, \quad a_1 = e \)
- \( a_2 = e^2, \quad a_3 = e^3 \)
Natural Spline Interpolant

Solution (2/7)

So the matrix $A$ and the vectors $\mathbf{b}$ and $\mathbf{x}$ given in the Natural Spline Theorem have the forms

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 0 \\ 3(e^2 - 2e + 1) \\ 3(e^3 - 2e^2 + e) \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

The vector-matrix equation $A\mathbf{x} = \mathbf{b}$ is equivalent to the system:

$$c_0 = 0$$
$$c_0 + 4c_1 + c_2 = 3(e^2 - 2e + 1)$$
$$c_1 + 4c_2 + c_3 = 3(e^3 - 2e^2 + e)$$
$$c_3 = 0$$
Natural Spline Interpolant

Solution (3/7)

This system has the solution $c_0 = c_3 = 0$ and, to 5 decimal places,

\[ c_1 = \frac{1}{5}(-e^3 + 6e^2 - 9e + 4) \approx 0.75685 \]
\[ c_2 = \frac{1}{5}(4e^3 - 9e^2 + 6e - 1) \approx 5.83007 \]
Natural Spline Interpolant

Solution (4/7)

Solving for the remaining constants gives

\[ b_0 = \frac{1}{h_0} (a_1 - a_0) - \frac{h_0}{3} (c_1 + 2c_0) \]
\[ = (e - 1) - \frac{1}{15} (-e^3 + 6e^2 - 9e + 4) \approx 1.46600 \]

\[ b_1 = \frac{1}{h_1} (a_2 - a_1) - \frac{h_1}{3} (c_2 + 2c_1) \]
\[ = (e^2 - e) - \frac{1}{15} (2e^3 + 3e^2 - 12e + 7) \approx 2.22285 \]

\[ b_2 = \frac{1}{h_2} (a_3 - a_2) - \frac{h_2}{3} (c_3 + 2c_2) \]
\[ = (e^3 - e^2) - \frac{1}{15} (8e^3 - 18e^2 + 12e - 2) \approx 8.80977 \]
Natural Spline Interpolant

**Solution (5/7)**

\[ d_0 = \frac{1}{3h_0}(c_1 - c_0) = \frac{1}{15}(-e^3 + 6e^2 - 9e + 4) \approx 0.25228 \]

\[ d_1 = \frac{1}{3h_1}(c_2 - c_1) = \frac{1}{3}(e^3 - 3e^2 + 3e - 1) \approx 1.69107 \]

and

\[ d_2 = \frac{1}{3h_2}(c_3 - c_1) = \frac{1}{15}(-4e^3 + 9e^2 - 6e + 1) \approx -1.94336 \]
Natural Spline Interpolant

Solution (6/7)

The natural cubic spline is described piecewise by

\[
S(x) = \begin{cases} 
1 + 1.46600x + 0.25228x^3 & \text{for } x \in [0, 1] \\
2.71828 + 2.22285(x - 1) + 0.75685(x - 1)^2 + 1.69107(x - 1)^3 & \text{for } x \in [1, 2] \\
7.38906 + 8.80977(x - 2) + 5.83007(x - 2)^2 - 1.94336(x - 2)^3 & \text{for } x \in [2, 3]
\end{cases}
\]

The spline and its agreement with \( f(x) = e^x \) are as shown in the following diagram.
Solution (7/7): Natural spline and its agreement with $f(x) = e^x$
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Example: The Integral of a Spline

Approximate the integral of \( f(x) = e^x \) on \([0, 3]\), which has the value

\[
\int_0^3 e^x \, dx = e^3 - 1 \approx 20.08553692 - 1 = 19.08553692,
\]

by piecewise integrating the spline that approximates \( f \) on this integral.

Note: From the previous example, the natural cubic spline \( S(x) \) that approximates \( f(x) = e^x \) on \([0, 3]\) is described piecewise by

\[
S(x) = \begin{cases} 
1 + 1.46600x + 0.25228x^3 & \text{for } x \in [0, 1] \\
2.71828 + 2.22285(x - 1) + 0.75685(x - 1)^2 + 1.69107(x - 1)^3 & \text{for } x \in [1, 2] \\
7.38906 + 8.80977(x - 2) + 5.83007(x - 2)^2 - 1.94336(x - 2)^3 & \text{for } x \in [2, 3]
\end{cases}
\]
Natural Spline Interpolant

Solution (1/4)

We can therefore write

\[
\int_0^3 S(x) = \int_0^1 \left[ 1 + 1.46600x + 0.25228x^3 \right] \, dx \\
+ \int_1^2 \left[ 2.71828 + 2.22285(x - 1) + 0.75685(x - 1)^2 \\
+ 1.69107(x - 1)^3 \right] \, dx \\
+ \int_2^3 \left[ 7.38906 + 8.80977(x - 2) + 5.83007(x - 2)^2 \\
- 1.94336(x - 2)^3 \right] \, dx
\]
Natural Spline Interpolant

Solution (2/4)

Integrating and collecting values from like powers gives

\[
\int_0^3 S(x) = \left[ x + 1.46600 \frac{x^2}{2} + 0.25228 \frac{x^4}{4} \right]_0^1 \\
+ \left[ 2.71828(x - 1) + 2.22285 \frac{(x - 1)^2}{2} \right]_0^2 \\
+ \left[ 0.75685 \frac{(x - 1)^3}{3} + 1.69107 \frac{(x - 1)^4}{4} \right]_0^3 \\
+ \left[ 7.38906(x - 2) + 8.80977 \frac{(x - 2)^2}{2} \right]_0^1 \\
+ \left[ 5.83007 \frac{(x - 2)^3}{3} - 1.94336 \frac{(x - 2)^4}{4} \right]_0^2
\]
Natural Spline Interpolant

Solution (3/4)

Therefore:

\[
\int_0^3 S(x) = (1 + 2.71828 + 7.38906) \\
+ \frac{1}{2} (1.46600 + 2.22285 + 8.80977) \\
+ \frac{1}{3} (0.75685 + 5.83007) \\
+ \frac{1}{4} (0.25228 + 1.69107 - 1.94336) \\
= 19.55229
\]
Natural Spline Interpolant

Solution (4/4)

Because the nodes are equally spaced in this example the integral approximation is simply

\[
\int_0^3 S(x) \, dx = (a_0 + a_1 + a_2) + \frac{1}{2} (b_0 + b_1 + b_2) + \frac{1}{3} (c_0 + c_1 + c_2) + \frac{1}{4} (d_0 + d_1 + d_2)
\]
Questions?
Reference Material
Cubic Spline Interpolant

Definition

Given a function \( f \) defined on \([a, b]\) and a set of nodes \( a = x_0 < x_1 < \cdots < x_n = b \), a cubic spline interpolant \( S \) for \( f \) is a function that satisfies the following conditions:

(a) \( S(x) \) is a cubic polynomial, denoted \( S_j(x) \), on the subinterval \([x_j, x_{j+1}]\) for each \( j = 0, 1, \ldots, n - 1 \);

(b) \( S_j(x_j) = f(x_j) \) and \( S_j(x_{j+1}) = f(x_{j+1}) \) for each \( j = 0, 1, \ldots, n - 1 \);

(c) \( S_{j+1}(x_{j+1}) = S_j(x_{j+1}) \) for each \( j = 0, 1, \ldots, n - 2 \); (Implied by (b).)

(d) \( S'_{j+1}(x_{j+1}) = S'_j(x_{j+1}) \) for each \( j = 0, 1, \ldots, n - 2 \);

(e) \( S''_{j+1}(x_{j+1}) = S''_j(x_{j+1}) \) for each \( j = 0, 1, \ldots, n - 2 \);

(f) One of the following sets of boundary conditions is satisfied:

(i) \( S''(x_0) = S''(x_n) = 0 \) (natural (or free) boundary);

(ii) \( S'(x_0) = f'(x_0) \) and \( S'(x_n) = f'(x_n) \) (clamped boundary).
Theorem

A strictly diagonally dominant matrix $A$ is nonsingular.

Moreover, in this case, Gaussian elimination can be performed on any linear system of the form $Ax = b$ to obtain its unique solution without row or column interchanges, and the computations will be stable with respect to the growth of round-off errors.
Natural Spline Interpolant: Linear System $A\mathbf{x} = \mathbf{b}$

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ h_0 & 2(h_0 + h_1) & h_1 & \cdots & \cdots & \vdots \\ 0 & h_1 & 2(h_1 + h_2) & h_2 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & h_{n-2} & 2(h_{n-2} + h_{n-1}) & h_{n-1} \\ 0 & \cdots & \cdots & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} 3 \frac{a_2 - a_1}{h_1} - 3 \frac{a_1 - a_0}{h_0} \\ \vdots \\ 3 \frac{a_n - a_{n-1}}{h_{n-1}} - 3 \frac{a_{n-1} - a_{n-2}}{h_{n-2}} \\ 0 \end{bmatrix}$$

and

$$\mathbf{x} = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix}$$