Outline

1. Interpolating Polynomial Error Bound

2. Example: 2nd Lagrange Interpolating Polynomial Error Bound

3. Example: Interpolating Polynomial Error for Tabulated Data
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Theorem

Suppose \( x_0, x_1, \ldots, x_n \) are distinct numbers in the interval \([a, b]\) and \( f \in C^{n+1}[a, b] \). Then, for each \( x \) in \([a, b]\), a number \( \xi(x) \) (generally unknown) between \( x_0, x_1, \ldots, x_n \), and hence in \((a, b)\), exists with

\[
f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1) \cdots (x - x_n)
\]

where \( P(x) \) is the interpolating polynomial given by

\[
P(x) = f(x_0)L_{n,0}(x) + \cdots + f(x_n)L_{n,n}(x) = \sum_{k=0}^{n} f(x_k)L_{n,k}(x)
\]
Note first that if $x = x_k$, for any $k = 0, 1, \ldots, n$, then $f(x_k) = P(x_k)$, and choosing $\xi(x_k)$ arbitrarily in $(a, b)$ yields the result:

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1) \cdots (x - x_n)$$

If $x \neq x_k$, for all $k = 0, 1, \ldots, n$, define the function $g$ for $t$ in $[a, b]$ by

$$g(t) = f(t) - P(t) - [f(x) - P(x)]\frac{(t - x_0)(t - x_1) \cdots (t - x_n)}{(x - x_0)(x - x_1) \cdots (x - x_n)}$$

$$= f(t) - P(t) - [f(x) - P(x)]\prod_{i=0}^{n} \frac{(t - x_i)}{(x - x_i)}$$
The Lagrange Polynomial: Theoretical Error Bound

\[ g(t) = f(t) - P(t) - [f(x) - P(x)] \prod_{i=0}^{n} \frac{(t - x_i)}{(x - x_i)} \]

Error Bound: Proof (2/6)

Since \( f \in C^{n+1}[a, b] \), and \( P \in C^{\infty}[a, b] \), it follows that \( g \in C^{n+1}[a, b] \).
For \( t = x_k \), we have

\[ g(x_k) = f(x_k) - P(x_k) - [f(x) - P(x)] \prod_{i=0}^{n} \frac{(x_k - x_i)}{(x - x_i)} = 0 - [f(x) - P(x)] \cdot 0 = 0 \]
The Lagrange Polynomial: Theoretical Error Bound

\[ g(t) = f(t) - P(t) - [f(x) - P(x)] \prod_{i=0}^{n} \frac{(t - x_i)}{(x - x_i)} \]

Error Bound: Proof (3/6)

We have seen that \( g(x_k) = 0 \). Furthermore,

\[
g(x) = f(x) - P(x) - [f(x) - P(x)] \prod_{i=0}^{n} \frac{(x - x_i)}{(x - x_i)}
\]

\[
= f(x) - P(x) - [f(x) - P(x)] = 0
\]

Thus \( g \in C^{n+1}[a, b] \), and \( g \) is zero at the \( n + 2 \) distinct numbers \( x, x_0, x_1, \ldots, x_n \).
Error Bound: Proof (4/6)

Since $g \in C^{n+1}[a, b]$, and $g$ is zero at the $n + 2$ distinct numbers $x, x_0, x_1, \ldots, x_n$, by Generalized Rolle’s Theorem there exists a number $\xi$ in $(a, b)$ for which $g^{(n+1)}(\xi) = 0$. So

$$0 = g^{(n+1)}(\xi)$$

$$= f^{(n+1)}(\xi) - P^{(n+1)}(\xi) - [f(x) - P(x)] \frac{d^{n+1}}{dt^{n+1}} \left[ \prod_{i=0}^{n} \frac{(t - x_i)}{(x - x_i)} \right]_{t=\xi}$$

However, $P(x)$ is a polynomial of degree at most $n$, so the $(n + 1)$st derivative, $P^{(n+1)}(x)$, is identically zero.
The Lagrange Polynomial: Theoretical Error Bound

Error Bound: Proof (5/6)

Also, \( \prod_{i=0}^{n} \frac{t - x_i}{x - x_i} \) is a polynomial of degree \((n + 1)\), so

\[
\prod_{i=0}^{n} \frac{(t - x_i)}{(x - x_i)} = \left[ \frac{1}{\prod_{i=0}^{n}(x - x_i)} \right] t^{n+1} + \text{(lower-degree terms in } t),
\]

and

\[
\frac{d^{n+1}}{dt^{n+1}} \prod_{i=0}^{n} \frac{(t - x_i)}{(x - x_i)} = \frac{(n + 1)!}{\prod_{i=0}^{n}(x - x_i)}
\]
The Lagrange Polynomial: Theoretical Error Bound

Error Bound: Proof (6/6)

We therefore have:

\[ 0 = f^{(n+1)}(\xi) - P^{(n+1)}(\xi) - [f(x) - P(x)] \frac{d^{n+1}}{dt^{n+1}} \left[ \prod_{i=0}^{n} \frac{(t - x_i)}{(x - x_i)} \right]_{t=\xi} \]

\[ = f^{(n+1)}(\xi) - 0 - [f(x) - P(x)] \frac{(n+1)!}{\prod_{i=0}^{n}(x - x_i)} \]

and, upon solving for \( f(x) \), we get the desired result:

\[ f(x) = P(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i) \]
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Example: Second Lagrange Polynomial for $f(x) = \frac{1}{x}$

In an earlier example, we found the second Lagrange polynomial for $f(x) = \frac{1}{x}$ on $[2, 4]$ using the nodes $x_0 = 2$, $x_1 = 2.75$, and $x_2 = 4$. 
The Lagrange Polynomial: 2nd Degree Error Bound

Solution (1/3)

Because \( f(x) = x^{-1} \), we have

\[
f'(x) = -\frac{1}{x^2}, \quad f''(x) = \frac{2}{x^3}, \quad \text{and} \quad f'''(x) = -\frac{6}{x^4}
\]

As a consequence, the second Lagrange polynomial has the error form

\[
f'''(\xi(x)) \frac{1}{3!} (x - x_0)(x - x_1)(x - x_2) = -\frac{1}{\xi(x)^4} (x - 2)(x - 2.75)(x - 4)
\]

for \( \xi(x) \) in \((2, 4)\). The maximum value of \( \frac{1}{\xi(x)^4} \) on the interval is \( \frac{1}{2^4} = 1/16 \).
The Lagrange Polynomial: 2nd Degree Error Bound

Solution (2/3)

We now need to determine the maximum value on [2, 4] of the absolute value of the polynomial

\[ g(x) = (x - 2)(x - 2.75)(x - 4) = x^3 - \frac{35}{4}x^2 + \frac{49}{2}x - 22 \]

Because

\[ g'(x) = 3x^2 - \frac{35}{2}x + \frac{49}{2} = \frac{1}{2}(3x - 7)(2x - 7), \]

the critical points occur at

\[ x = \frac{7}{3} \text{ with } g\left(\frac{7}{3}\right) = \frac{25}{108} \quad \text{and} \quad x = \frac{7}{2} \text{ with } g\left(\frac{7}{2}\right) = -\frac{9}{16} \]
The Lagrange Polynomial: 2nd Degree Error Bound

Solution (3/3)

Hence, the maximum error is

\[
\max_{[2,4]} \left| \frac{f'''(\xi(x))}{3!} \right| \cdot \max_{[2,4]} |(x - x_0)(x - x_1)(x - x_2)|
\]

\[
\leq \frac{1}{3!} \cdot \frac{1}{16} \cdot \frac{9}{16}
\]

\[
= \frac{3}{512}
\]

\[
\approx 0.00586
\]
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Example: Tabulated Data
Use of the Interpolating Polynomial Error Bound

Solution (1/3)

The step size is $h$, so $x_j = jh$, $x_{j+1} = (j + 1)h$, and

$$|f(x) - P(x)| \leq \frac{|f^{(2)}(\xi)|}{2!} |(x - jh)(x - (j + 1)h)|.$$

Hence

$$|f(x) - P(x)| \leq \max_{\xi \in [0,1]} e^\xi \max_{x_j \leq x \leq x_{j+1}} |(x - jh)(x - (j + 1)h)|,$$

$$\leq \frac{e}{2} \max_{x_j \leq x \leq x_{j+1}} |(x - jh)(x - (j + 1)h)|.$$
Consider the function \( g(x) = (x - jh)(x - (j + 1)h) \), for \( jh \leq x \leq (j + 1)h \). Because

\[
g'(x) = (x - (j + 1)h) + (x - jh) = 2 \left( x - jh - \frac{h}{2} \right),
\]

the only critical point for \( g \) is at \( x = jh + \frac{h}{2} \), with

\[
g \left( jh + \frac{h}{2} \right) = \left( \frac{h}{2} \right)^2 = \frac{h^2}{4}
\]

Since \( g(jh) = 0 \) and \( g((j + 1)h) = 0 \), the maximum value of \( |g'(x)| \) in \([jh, (j + 1)h]\) must occur at the critical point.
Use of the Interpolating Polynomial Error Bound

Solution (3/3)

This implies that

\[ |f(x) - P(x)| \leq \frac{e}{2} \max_{x_j \leq x \leq x_{j+1}} |g(x)| \leq \frac{e}{2} \cdot \frac{h^2}{4} = \frac{eh^2}{8}. \]

Consequently, to ensure that the error in linear interpolation is bounded by \(10^{-6}\), it is sufficient for \(h\) to be chosen so that

\[ \frac{eh^2}{8} \leq 10^{-6}. \]

This implies that \(h < 1.72 \times 10^{-3}\).

Because \(n = \frac{(1-0)}{h}\) must be an integer, a reasonable choice for the step size is \(h = 0.001\).
Questions?
Reference Material
Suppose $f \in C[a, b]$ is $n$ times differentiable on $(a, b)$. If $f(x) = 0$ at the $n + 1$ distinct numbers $a \leq x_0 < x_1 < \ldots < x_n \leq b$, then a number $c$ in $(x_0, x_n)$, and hence in $(a, b)$, exists with $f^{(n)}(c) = 0$.
The Lagrange Polynomial: Theoretical Error Bound

Suppose $x_0, x_1, \ldots, x_n$ are distinct numbers in the interval $[a, b]$ and $f \in C^{n+1}[a, b]$. Then, for each $x$ in $[a, b]$, a number $\xi(x)$ (generally unknown) between $x_0, x_1, \ldots, x_n$, and hence in $(a, b)$, exists with

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n + 1)!}(x - x_0)(x - x_1) \cdots (x - x_n)$$

where $P(x)$ is the interpolating polynomial given by

$$P(x) = f(x_0)L_{n,0}(x) + \cdots + f(x_n)L_{n,n}(x) = \sum_{k=0}^{n} f(x_k)L_{n,k}(x)$$
Example: $f(x) = \frac{1}{x}$

Use the numbers (called nodes) $x_0 = 2$, $x_1 = 2.75$ and $x_2 = 4$ to find the second Lagrange interpolating polynomial for $f(x) = \frac{1}{x}$.

Solution (Summary)

$$P(x) = \sum_{k=0}^{2} f(x_k)L_k(x)$$
$$= \frac{1}{3}(x - 2.75)(x - 4) - \frac{64}{165}(x - 2)(x - 4) + \frac{1}{10}(x - 2)(x - 2.75)$$
$$= \frac{1}{22}x^2 - \frac{35}{88}x + \frac{49}{44}$$