Taylor Series

Basic Mathematical Requirements:
- \( f(x) \) is a single valued function and if in the interval \( a<x<b \)
- Its first \((N-1)\) derivatives are continuous
- Its \(N\)th derivative exists, \( f^{(N)}(x) \)

Taylor Series in Cartesian coordinates:

\[
T(x_o + \Delta x, y_o + \Delta y, z_o + \Delta z) = T(x_o, y_o, z_o) \\
+ \sum_{m=1}^{N-1} \frac{1}{m!} \left( \Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y} + \Delta z \frac{\partial}{\partial z} \right)^m \left. T \right|_{x_o, y_o, z_o} \\
+ \frac{1}{N!} \left( \Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y} + \Delta z \frac{\partial}{\partial z} \right)^N \left. T \right|_{x_1, y_1, z_1}
\]

where

\[ x_0 < x_1 < x_0 + \Delta x, \quad y_0 < y_1 < y_0 + \Delta y, \quad z_0 < z_1 < z_0 + \Delta z \]
Example:

Heat conduction two dimensional – steady state with internal heat generation

\[ \frac{\partial^2 T}{\partial x^2} \bigg|_o + \frac{\partial^2 T}{\partial y^2} \bigg|_o + \frac{\dot{q}''}{k} = 0 \]

Sample space on an evenly spaced mesh of points separated by \(\Delta x\) in the x direction and \(\Delta y\) in the y direction, and pick a specific point at which the equation will be evaluated.
Use a Taylor Series to express values of temperature at adjacent points in terms of the values of temperature and its derivatives where the conduction equation is to be evaluated:

\[
T_{x+\Delta x} = T_o + \Delta x \frac{\partial T}{\partial x} \bigg|_o + \frac{\Delta x^2}{2} \frac{\partial^2 T}{\partial x^2} \bigg|_o + \frac{\Delta x^3}{3!} \frac{\partial^3 T}{\partial x^3} \bigg|_o + \cdots \frac{\Delta x^N}{N!} \frac{\partial^N T}{\partial x^N} \bigg|_{x_1}
\]

\[
T_{x-\Delta x} = T_o - \Delta x \frac{\partial T}{\partial x} \bigg|_o + \frac{\Delta x^2}{2} \frac{\partial^2 T}{\partial x^2} \bigg|_o - \frac{\Delta x^3}{3!} \frac{\partial^3 T}{\partial x^3} \bigg|_o + \cdots \frac{(-1)^N \Delta x^N}{N!} \frac{\partial^N T}{\partial x^N} \bigg|_{x_1}
\]

Add and truncate the series

\[
T_{x+\Delta x} + T_{x-\Delta x} = 2T_o + \Delta x^2 \frac{\partial^2 T}{\partial x^2} \bigg|_o + \frac{2\Delta x^4}{4!} \frac{\partial^4 T}{\partial x^4} \bigg|_\xi
\]

thus

\[
\frac{\partial^2 T}{\partial x^2} \bigg|_o = \frac{T_{x+\Delta x} - 2T_o + T_{x-\Delta x}}{\Delta x^2} - \frac{2\Delta x^2}{4!} \frac{\partial^4 T}{\partial x^4} \bigg|_{x_1}
\]
A similar expression is obtained for
\[
\frac{\partial^2 T}{\partial y^2} \bigg|_o = \frac{T_{y+\Delta y} - 2T_o + T_{y-\Delta y}}{\Delta y^2} \quad \frac{2\Delta y^2}{4!} \quad \frac{\partial^4 T}{\partial y^4} \bigg|_{y_1}
\]
\[y - \Delta y < y_1 < y + \Delta y\]

Shift to a compass notation:
\[T_E \equiv T(x_o + \Delta x, y_o)\]
\[T_W \equiv T(x_o - \Delta x, y_o)\]
\[T_N \equiv T(x_o, y_o + \Delta y)\]
\[T_S \equiv T(x_o, y_o - \Delta y)\]
The second order finite difference conduction equation:

\[
\left( \frac{T_E - 2T_O + T_W}{\Delta x^2} \right) + \left( \frac{T_N - 2T_O + T_S}{\Delta y^2} \right) + \frac{\dot{q}'''}{k} = 0
\]

with an error

\[
\varepsilon = \frac{2\Delta x^2}{4!} \left. \frac{\partial^4 T}{\partial x^4} \right|_{x_i} + \frac{2\Delta y^2}{4!} \left. \frac{\partial^4 T}{\partial y^4} \right|_{y_i}
\]

The error is abbreviated as:

\( O(\Delta x^2), O(\Delta y^2) \)
High order approximation

We must consider more points on the mesh when working with the equations. Introduce extended notation:

\[ T_{EE} \equiv T(x_\circ + 2\Delta x, y_\circ) \]
\[ T_{WW} \equiv T(x_\circ - 2\Delta x, y_\circ) \]
\[ T_{NN} \equiv T(x_\circ, y_\circ + 2\Delta y) \]
\[ T_{SS} \equiv T(x_\circ, y_\circ - 2\Delta y) \]

We want an approximation to the second derivative that only involves \( T_\circ, T_E, T_W, T_{EE}, \) and \( T_{WW} \). Look at the Taylor expansions giving the values of \( T \) at these 4 points.
Taylor Expansions in the x direction

\[ T_E = T_0 + \Delta x \frac{\partial T}{\partial x} \bigg|_0 + \frac{\Delta x^2}{2} \frac{\partial^2 T}{\partial x^2} \bigg|_0 + \frac{\Delta x^3}{3!} \frac{\partial^3 T}{\partial x^3} \bigg|_0 + \frac{\Delta x^4}{4!} \frac{\partial^4 T}{\partial x^4} + \cdots \]

\[ T_W = T_0 - \Delta x \frac{\partial T}{\partial x} \bigg|_0 + \frac{\Delta x^2}{2} \frac{\partial^2 T}{\partial x^2} \bigg|_0 - \frac{\Delta x^3}{3!} \frac{\partial^3 T}{\partial x^3} \bigg|_0 + \frac{\Delta x^4}{4!} \frac{\partial^4 T}{\partial x^4} + \cdots \]

\[ T_{EE} = T_0 + 2\Delta x \frac{\partial T}{\partial x} \bigg|_0 + \frac{(2\Delta x)^2}{2} \frac{\partial^2 T}{\partial x^2} \bigg|_0 + \frac{(2\Delta x)^3}{3!} \frac{\partial^3 T}{\partial x^3} \bigg|_0 + \frac{(2\Delta x)^4}{4!} \frac{\partial^4 T}{\partial x^4} + \cdots \]

\[ T_{WW} = T_0 - 2\Delta x \frac{\partial T}{\partial x} \bigg|_0 + \frac{(2\Delta x)^2}{2} \frac{\partial^2 T}{\partial x^2} \bigg|_0 - \frac{(2\Delta x)^3}{3!} \frac{\partial^3 T}{\partial x^3} \bigg|_0 + \frac{(2\Delta x)^4}{4!} \frac{\partial^4 T}{\partial x^4} + \cdots \]
Method of Undetermined Coefficients

Now multiply the equation for $T_E$ by the unknown coefficient “$A$”, multiply the equation for $T_W$ by the unknown “$B$”, multiply the equation for $T_{EE}$ by the unknown “$C$”, and multiply the equation for $T_{WW}$ by the unknown “$D$”. Sum these modified equations.

$$AT_E + BT_W + CT_{EE} + DT_{WW} = (A + B + C + D)T_O$$

$$+ (A - B + 2C - 2D) \Delta x \left. \frac{\partial T}{\partial x} \right|_O$$

$$+ (A + B + 4C + 4D) \frac{\Delta x^2}{2} \left. \frac{\partial^2 T}{\partial x^2} \right|_O$$

$$+ (A - B + 8C - 8D) \frac{\Delta x^3}{3!} \left. \frac{\partial^3 T}{\partial x^3} \right|_O$$

$$+ (A + B + 16C + 16D) \frac{\Delta x^4}{4!} \left. \frac{\partial^4 T}{\partial x^4} \right|_O + \cdots$$
We want the summed equations to provide an approximation to the second derivative of $T$, so we require that the coefficient of the second derivative in the summed equation be equal to 1. We further require that the coefficients of the first, third, and fourth derivatives be zero. This gives us four equations that can be solved for values of $A$, $B$, $C$, and $D$. They are:

\[
(A - B + 2C - 2D) = 0
\]
\[
(A + B + 4C + 4D) \frac{\Delta x^2}{2} = 1
\]
\[
(A - B + 8C - 8D) = 0
\]
\[
(A + B + 16C + 16D) = 0
\]

The solution is:

\[
A = B = \frac{4}{3} \frac{1}{\Delta x^2}
\]
\[
C = D = -\frac{1}{12\Delta x^2}
\]
Replacing these values in the summed equation and rearranging gives:

\[
\left. \frac{\partial^2 T}{\partial x^2} \right|_O = \frac{4}{3\Delta x^2} [T_E - T_O] + \frac{4}{3\Delta x^2} [T_W - T_O] - \frac{1}{12\Delta x^2} [T_{EE} - T_O] - \frac{1}{12\Delta x^2} [T_{WW} - T_O] + \varepsilon
\]

\[
= \frac{1}{12\Delta x^2} \left[ -T_{EE} + 16T_E - 30T_O + 16T_W - T_{WW} \right] + \varepsilon
\]

What is the expression for the error, \( \varepsilon \)?
Cylindrical coordinate system (DEM)

Example: Steady State Heat Conduction

\[ \nabla^2 T = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial (r\theta)^2} + \frac{\partial^2 T}{\partial z^2} = 0 \]

Assume \( \Delta z, \Delta \theta \) and, \( \Delta r \) are uniform

\[ \left. \frac{\partial^2 T}{\partial z^2} \right|_0 = \frac{T_B - 2T_0 + T_A}{\Delta z^2} \]

Define

\[ T_N \equiv T(r_o + \Delta r, \theta_o) \]
\[ T_S \equiv T(r_o - \Delta r, \theta_o) \]
\[ T_E \equiv T(r_o, \theta_o + \Delta \theta) \]
\[ T_W \equiv T(r_o, \theta_o - \Delta \theta) \]
Write the Taylor Series expansion for each of these variables

\[ T_E = T_o + \frac{r_o \Delta \theta}{r_o} \frac{\partial T}{\partial \theta} \bigg|_o + \frac{(r_o \Delta \theta)^2}{r_o^2 2!} \frac{\partial^2 T}{\partial \theta^2} \bigg|_o + \cdots \]

\[ T_W = T_o + \frac{r_o \Delta \theta}{r_o} \frac{\partial T}{\partial \theta} \bigg|_o + \frac{(r_o \Delta \theta)^2}{r_o^2 2!} \frac{\partial^2 T}{\partial \theta^2} \bigg|_o + \cdots \]

\[ T_N = T_o + \Delta r \frac{\partial T}{\partial r} \bigg|_o + \frac{\Delta r^2}{2!} \frac{\partial^2 T}{\partial r^2} \bigg|_o + \cdots \]

\[ T_S = T_o - \Delta r \frac{\partial T}{\partial r} \bigg|_o + \frac{\Delta r^2}{2!} \frac{\partial^2 T}{\partial r^2} \bigg|_o + \cdots \]
Pattern of Solution

In this instance we are free to either deal with all four expansions as a single sum, or group the radial and theta equations separately. I will follow the pattern in the previous higher order example. Multiply the equation for $T_E$ by the unknown coefficient “A”, multiply the equation for $T_W$ by the unknown “B”, multiply the equation for $T_N$ by the unknown “C”, and multiply the equation for $T_S$ by the unknown “D”. Sum these modified equations. We want the summed equations to provide an approximation to the radial and azimuthal derivatives in the conduction equations, so we require that the coefficient of the first derivative of $T$ with respect to $r$ in the summed equation be equal to $1/r_o$. The coefficient of the second radial derivative must be one. We require that the coefficient of the first derivative with respect to theta be zero. The coefficient of the second derivative must be $1/r_o^2$. This again gives us four equations that can be solved for values of A, B, C, and D. The solution is:

\[
A = B = \frac{1}{(r_o \Delta \theta)^2}
\]

\[
C = \frac{2r_o + \Delta r}{2r_o \Delta r^2}
\]

\[
D = \frac{2r_o - \Delta r}{2r_o \Delta r^2}
\]
The finite difference equation is

\[
\frac{T_E}{r_o^2 \Delta \theta^2} + \frac{T_W}{r_o^2 \Delta \theta^2} + \left[ \frac{2r_o + \Delta r}{2r_o \Delta r^2} \right] T_N + \left[ \frac{2r_o - \Delta r}{2r_o \Delta r^2} \right] T_s + \frac{T_B}{\Delta z^2} + \frac{T_A}{\Delta z^2} \\
-2 \left[ \frac{1}{r_o^2 \Delta \theta^2} + \frac{1}{\Delta r^2} + \frac{1}{\Delta z^2} \right] T_o \equiv 0
\]
What is the error?

\[ \varepsilon = -\frac{Ar_0^3 \Delta \theta^3}{r_0^3 3!} \frac{\partial ^3 T}{\partial \theta^3} \bigg|_0 - \frac{Ar_0^4 \Delta \theta^4}{r_0^4 4!} \frac{\partial ^4 T}{\partial \theta^4} \bigg|_0 - \ldots \]

\[ + \frac{Br_0^3 \Delta \theta^3}{r_0^3 3!} \frac{\partial ^3 T}{\partial \theta^3} \bigg|_0 - \frac{Br_0^4 \Delta \theta^4}{r_0^4 4!} \frac{\partial ^4 T}{\partial \theta^4} \bigg|_0 - \ldots \]

\[ - C \frac{\Delta r^3}{3!} \frac{\partial ^3 T}{\partial \theta^3} \bigg|_0 - \ldots \]

\[ + D \frac{\Delta r^3}{3!} \frac{\partial ^3 T}{\partial r^3} \bigg|_0 - \ldots \]

\[ - \frac{2 \Delta z^2}{4!} \frac{\partial ^4 T}{\partial z^4} \bigg|_{\xi_3} - \ldots \]
Or

\[ \varepsilon = - \frac{2\Delta \theta^2}{4! r_0^2} \left( \frac{\partial^4 T}{\partial \theta^4} \right)_{\theta_1} - \frac{\Delta r^2}{3! r_0} \left( \frac{\partial^3 T}{\partial r^3} \right)_{r_2} - \frac{2\Delta z^2}{4!} \left( \frac{\partial^4 T}{\partial z^4} \right)_{z_3} \]

\[ \theta_0 < \theta_1 < \theta_0 + \Delta \theta \]

\[ r_0 < r_1 < r_0 + \Delta r \]

\[ z_0 < z_1 < z_0 + \Delta z \]
Special treatment near origin

\[ \nabla^2 T = \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \]

When symmetry exists at the origin (Geometrical & Thermal)

\[ \left. \frac{\partial^2 T}{\partial \theta^2} \right|_{r=0} = 0 \]

\[ \left. \frac{\partial T}{\partial r} \right|_{r=0} = 0 \quad \text{so} \quad \left. \frac{1}{r} \frac{\partial T}{\partial r} \right|_{r=0} = 0 \]

Indeterminate

\[ \lim_{r \to 0} \left. \frac{1}{r} \frac{\partial T}{\partial r} \right|_{r=0} = 2 \frac{\partial^2 T}{\partial r^2} \]
Non-Symmetric Cases

when no symmetry at \( r = 0 \), use Cartesian coordinates for the node at the origin:

\[
\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}
\]

Everywhere else use cylindrical coordinate system