Asset Prices

Econ 497F Lecture

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Basic Framework

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With concave utility we have diminishing marginal utility which gives the desire to smooth consumption.

- We have not just impatience ($\beta$), but risk aversion
- Investors prefer a consumption stream that is steady over time and across states of nature.
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Definition

Define $x_{t+1}$ as the payoff of an investment.

- E.G., equity, the payoff tomorrow: $x_{t+1} = p_{t+1} + d_{t+1}$; (not rate of return; but value of investment in $t+1$)
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- Then the consumer’s problem is to

$$\max_{\phi} u(c_t) + E_t[\beta u(c_{t+1})] \quad s.t.$$  

$$c_t = e_t - p_t \phi,$$

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- Future consumption depends on the payoff in $t+1$, which you do not know today.
The solution to this problem yields the familiar first-order condition:

\[ p_t u'(c_t) = E_t[\beta u'(c_{t+1})x_{t+1}] \]  \hspace{1cm} (12)
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Expression (13) is *the* central asset pricing formula.

- Notice that consumption is on the RHS and that is endogenous.
We can see the problem graphically in figure 3.
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- Key difference, we don’t know \(c_{t+1}\) or \(x_{t+1}\), just have expectations
Consumer’s Problem

Figure: Consumer’s problem
Stochastic Discount Factor

**Definition**

We can define the stochastic discount factor, \( m_{t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t)} \).

- Clearly \( m \) is the marginal rate of substitution.
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$$p_t = E_t [m_{t+1} x_{t+1}] .$$ (14)
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- With uncertainty for each asset \( i \)
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p_t^i = \frac{1}{R^i} E_t(x^i_{t+1}).
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p^i_t = \frac{1}{R^i} E_t(x^i_{t+1}).
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- \( R^i \), is used to discount the uncertain payoff, \( x^i_{t+1} \).
The sdf thus generalizes the normal approach.
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- One factor, $m_{t+1}$, the same one for each asset, can be used to discount any asset.

What does $p_t = E_t [m_t + 1 x_t + 1]$ imply?

What matters for asset prices is how payoffs are correlated with the value of a dollar in a given state. An asset that pays off when a dollar is valuable is worth more.

When is a dollar more valuable?

$m$ is high when when consumption is expected to be low (so the marginal utility of consumption is high).
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Example

Two-State derivation of SDF.

- Recall relation between $p_i$ and A-D state price:

$$p_i = q_1 x_{1i} + q_2 x_{2i}$$

where the $q$'s are the A-D state prices.
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- Then, clearly,
  \[ p_i = E_j[m_j x_{ji}] \]
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  - Equivalently, the price of any asset is the expected product of the SDF and the payoff.
  - This seems only notational. But involved here is a deep and useful separation. All asset pricing models amount to alternative ways of connecting the stochastic discount factor to the data.
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    \[ R_f = \frac{1}{E(m)} \] (15)
  - notice that we can use (15) to define a shadow risk-free rate even if such a security did not actually exist
Real interest rates

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- Suppose no uncertainty. Then,

$$R_f = \frac{1}{\beta} \left( \frac{c_{t+1}}{c_t} \right)^\gamma$$

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  3. real interest rates are more sensitive to consumption growth when \( \gamma \) is high
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  3. real interest rates are more sensitive to consumption growth when $\gamma$ is high

  1. when utility is highly curved, investor cares more about a smooth consumption profile
Marginal utility and gamma

Graph showing the relationship between marginal utility ($u'$) and gamma ($\gamma$) for different values of gamma: $\gamma = 2$, $\gamma = 5$, $\gamma = 1.1$, and $\gamma = 0.5$. The graph illustrates how the marginal utility decreases as the gamma increases, indicating diminishing marginal utility. The x-axis represents different levels of gamma, and the y-axis shows the corresponding marginal utility values. The graph helps in understanding the impact of risk aversion on asset prices within the context of the Capital Asset Pricing Model (CAPM) and the Arbitrage Pricing Theory (APT).
Thus, when $\gamma$ is higher it takes a larger $\Delta r_f$ to get people to change their consumption path.
Risk-free rate

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Then

$$\ln R_f = \delta + \gamma E_t(\Delta \ln c_{t+1}) - \frac{\gamma^2}{2} \sigma_t^2(\Delta \ln c_{t+1})$$  \hspace{1cm} (17)$$

where $\beta = e^{-\delta}$, $\Delta \ln c_{t+1}$ is the growth rate of consumption, and $\sigma_t^2(\Delta \ln c_{t+1})$ is the variance of the growth rate of consumption.
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- Expression (17) can be decomposed into three terms:
  \[
  \ln R_f = \text{impatience term} + \text{growth of } c \text{ term} - \text{variance of growth of } c \text{ term}
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Implications

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- Notice that with power utility $\gamma$ governs intertemporal substitution, risk aversion and precautionary savings. More general utility functions allow this to be separated
Risk Corrections

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- the first term on RHS is discounted present value, second term is risk adjustment
Risk Corrections

- It is useful to substitute for $m$ in (18)

\[
p = \frac{E(x)}{R_f} + \frac{cov[\beta u'(c_{t+1}), x_{t+1}]}{u'(c_t)}
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(19)

Since $u_0$ declines as $c_r$ rises, an asset's price is lower if the payo¤ positively covaries with consumption. Conversely, an asset that pays o¤ when consumption is low has a higher price. Insurance riskier securities must o¤er higher returns to get investors to hold them.
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- It is not risk itself that is compensated with return, only risk correlated with $m$ matters.
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- thus the rate of return on this asset equals the risk-free rate, no matter how large is the variance of its payoff, $\sigma^2(x)$.

- We could let $\sigma^2(x) \longrightarrow$ a billion, but if $cov(m, x) = 0$, it will still yield only the risk-free rate. Even if people are totally risk averse.
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  - Only *systematic* risk generates a risk correction.
  - Systematic risk is that part of the variance that is correlated with $m$. Idiosyncratic risk is the residual variance.
  - This is the risk that can be diversified away. Since it is not systematically related to $m$, we can avoid this risk. Idiosyncratic risk is like the residual in a regression equation.
Systematic risk

\[ \sigma_p^2 = \sigma^2(e)/n \]

**Figure**: Systematic Risk
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Equity Premium

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  - The maximum Sharpe ratio is thus given by the tangent to the market portfolio: \[
  \frac{E(R_P) - R_f}{\sigma(R_P)},
  \]
  as in figure 5.
Sharpe Ratio

**Figure:** Sharpe Ratio Efficient Portfolios
Equity Premium

To derive the maximum Sharpe ratio, start from the basic equation

\[ 1 = E(mR_i) \]
\[ = E(m)E(R_i) + \rho_{m,R_i}\sigma(R_i)\sigma(m) \]
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\[ E(R_i) = R_f - \rho_{m,R_i}\sigma(R_i)\frac{\sigma(m)}{E(m)} \] (21)

Notice that a correlation coefficient cannot be greater than unity, so this gives us a bound on asset prices:
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\[ |E(R_i) - R_f| \leq \sigma(R_i) \frac{\sigma(m)}{E(m)} \] (22)
From (22) we derive the maximum Sharpe ratio, since 
\( \rho_{m,R_p} = 1 \):

\[
\frac{E(R_P) - R_f}{\sigma(R_P)} = \frac{\sigma(m)}{E(m)} = \sigma(m)R_f
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this is an important result: the maximum Sharpe ratio – the slope of the capital allocation line – is governed by the volatility of \( m \), the stochastic discount factor.
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What does \( \sigma(m) \), depend on? We know that 

\[ m = \beta \frac{u'(c_{t+1})}{u'(c_t)} \]

so it depends on the volatility of consumption, and on how curved is the utility function – that is how little we like irregular consumption.
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If consumption growth is volatile we have a higher equity premium – a larger Sharpe ratio.
Equity Premium

if we had power utility then

\[
\frac{E(R_P) - R_f}{\sigma(R_P)} = \frac{\sigma \left[ \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} \right]}{E \left[ \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} \right]}
\]

Over the last 50 years in the US, real stock returns have averaged \( \sim 9\% \) with \( \sigma = 16\% \), while the real return on T-bills is about 1\%. 

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- and if consumption growth was lognormal

\[
\frac{E(R_P) - R_f}{\sigma(R_P)} \approx \gamma \sigma (\Delta \ln c)
\]  \hspace{1cm} (24)

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Equity Premium

Using this in (24) we have

\[
\frac{E(R_P) - R_f}{\sigma(R_P)} = \frac{.09 - .01}{.16} = .5
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But consumption growth is rather smooth, with \( \sigma \approx 1\% \). This implies that \( \gamma = 50 \) to reconcile (24).
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This is the equity premium puzzle. Why are the returns to equity so high?

- Consumption does not seem volatile enough to explain it, and people do not seem so risk averse ($\gamma = 4$ would seem pretty high).
- Very hard to explain, many have tried.