Proof of Lemma 1

Lemma A.1: Let Assumptions 1-2 be true, let \( \theta \) be a common eigenvector of the matrices \( M_{1,n} \), \( M_{2,n} \), and \( M_2 \) corresponding to a zero eigenvalue, and let \( \theta_\ast \) be any other conformable vector. Moreover, let \( m \to \infty \) at rate \( o(n) \). Then the following hold:

\[
\begin{align*}
n\theta^T \hat{M}_1 \theta & = \theta^T C(1) \left( \int \tilde{W}_k(x) \tilde{W}_k(x)^T dx \right) C(1) \theta \text{ in distr.,} \quad (A.1) \\
n^{-p-1/2} \theta^T \hat{M}_1 \theta_\ast & = \theta^T C(1) \left( \int \tilde{W}_k(x) F(x) \tilde{W}_k(x)^T dx \right) \theta_\ast \text{ in distr.,} \quad (A.2) \\
n^{-2p} \theta_\ast^T \hat{M}_1 \theta_\ast & = \theta_\ast^T \left( \int F(x) F(x)^T dx \right) \theta_\ast + O_p(n^{-p-1/2}), \quad (A.3) \\
m \theta^T \hat{M}_2 \theta & = \theta^T C(1) C(1)^T \theta + O_p(\sqrt{m/n}) + O_p(1/\sqrt{m}). \quad (A.4) \\
n^{-p} \sqrt{m} \theta^T \hat{M}_2 \theta_\ast & = O_p(\sqrt{m/n}) + O_p(1/\sqrt{m}). \quad (A.5) \\
n^{-2p} \theta_\ast^T \hat{M}_2 \theta_\ast & = \theta_\ast^T \left( \int F'(x) F'(x)^T dx \right) \theta_\ast + O_p(n^{-p-1/2}) + O_p(n^{-p/m}). \quad (A.6)
\end{align*}
\]

Before we can prove Lemma A.1, we have to prove (12) and (13) first:

Proof of (12): Let

\[
U_n(x) = (1/\sqrt{n}) \sum_{t=1}^{[nx]} u_t \text{ if } x \in [n^{-1}, 1], \quad U_n(x) = 0 \text{ if } x \in [0, n^{-1})
\]
Then it follows from Assumption 2, the decomposition (11), and the functional central limit theorem, that

\[ U_n(x) \Rightarrow C(1)W_k(x), \tag{A.7} \]

where \( W_k \) is a \( k \)-variate standard Wiener process and "\( \Rightarrow \)" denotes weak convergence. Cf. Billingsley (1968). Next, observe that, with 
\[ g(t) = \beta_{0,n} + \beta_{1,n} t + f_n(t), \]
where \( f_n(t) \) satisfies the conditions in (7), we have
\[ z_t - \hat{\beta}_0 - \hat{\beta}_1 t = u_t + f_n(t) - \left( \hat{\beta}_0 - \beta_{0,n} \right) - \left( \hat{\beta}_1 - \beta_{1,n} \right) t = \hat{u}_t + f_n(t), \]
where \( \hat{u}_t \) is the OLS residual of the regression of \( u_t \) on an intercept and time \( t \), for \( t = 1, \ldots, n \). Denoting the OLS coefficients involved by \( \hat{\delta}_0 \) and \( \hat{\delta}_1 \), respectively, and using the fact that by Lemma 9.6.3 in Bierens (1994, p. 200),
\[ (1/n) \sum_{t=1}^{n} (t/n) u_t = U_n(1) - \int U_n(x) dx, \]
it follows that

\[ \begin{pmatrix} \sqrt{n} \hat{\delta}_0 \\ n \sqrt{n} \hat{\delta}_1 \end{pmatrix} \sim \begin{pmatrix} 1 \\ (1/n) \sum_{t=1}^{n} (t/n) \end{pmatrix}^{-1} \begin{pmatrix} U_n(1) \\ U_n(1) - \int U_n(x) dx \end{pmatrix}. \tag{A.8} \]

This result implies that for \( x \in [0,1] \),
\[ (1/n) \sum_{t=1}^{\lfloor nx \rfloor} \hat{u}_t = U_n(x) - x U_n(1) + 3(x^2 - x) \left( 2 \int U_n(y) dy - U_n(1) \right) + o_p(1), \tag{A.9} \]
where the \( o_p(1) \) term is uniform in \( x \in [0,1] \). Combining (7) and (9), the result (12) follows.

**Proof of (13):** It follows from (8) and the decomposition (11) that
\[
\hat{F}_n'(t/n) - n^p F_{p,n}'(t/n) = (1/m) \sum_{j=0}^{m-1} \left( z_{t-1-j} - \hat{\beta}_0 - \hat{\beta}_1(t+1-j) \right) - (1/m) \sum_{j=0}^{m-1} f_n(t+1-j) = (1/m) \sum_{j=0}^{m-1} \hat{u}_{t+1-j}
\]

\[
= (1/m) \sum_{j=0}^{m-1} u_{t+1-j} - \hat{\delta}_0 - \hat{\delta}_1(t+1 - (m+1)/2) = (1/m) \sum_{j=0}^{m-1} u_{t+1-j} + O_p(1/\sqrt{n})
\]

\[
= C(1)(1/m) \sum_{j=0}^{m-1} \epsilon_{t+1-j} + \frac{v_{t+1-m} - v_{t+1-m}}{m} + O_p(1/\sqrt{n}),
\]

where the \( O_p \) term is uniform in \( t \). This proves the result involved.

**Proof of** (1), (2), and (3): These parts of Lemma 1 follow directly from (12) and Assumption 1.

**Proof of** (4): It follows from (13) that

\[
m\theta^T \hat{M}_z \theta = \frac{m}{n} \sum_{t=m}^{n} \left( (1/m) \sum_{j=0}^{m-1} \theta^T (z_{t-j} - \hat{\beta}_0 - \hat{\beta}_1(t-j)) \right)^2
\]

\[
= \frac{1}{n} \sum_{t=m}^{n} \left( \theta^T C(1)(1/\sqrt{m}) \sum_{j=0}^{m-1} \epsilon_{t-j} + \frac{\theta^T (v_{t+1-m} - v_{t+1-m})}{\sqrt{m}} + O_p(\sqrt{m/n}) \right)^2
\]

\[
= \theta^T C(1) C(1)^T \theta + O_p(1/\sqrt{m}) + O_p(\sqrt{m/n}).
\]

The last equality follows from the fact that under Assumption 3,

\[
(1/n) \sum_{t=m}^{n} \left( (1/\sqrt{m}) \sum_{j=0}^{m-1} \epsilon_{t-j} \right) \left( (1/\sqrt{m}) \sum_{j=0}^{m-1} \epsilon_{t-j} \right)^T = I_k + O_p(\sqrt{m/n}),
\]

which is proved in Lemma A.2 below, and that

\[
(1/n) \sum_{t=m}^{n} (v_{t+1-m} - v_{t+1-m}) (v_{t+1-m})^T = O_p(1).
\]
Since the first $O_p$ term is dominated by the last one, (4) follows from (10).

**Proof of (5) and (6):** It follows from (13) that

$$n^{-p} \sqrt{m} \theta^T \hat{M}_2 \theta_* = \frac{1}{n} \sum_{t=m-1}^{n-1} \theta^T C(1) \left( \frac{1}{\sqrt{m}} \right) \sum_{j=0}^{m-1} \frac{v_{t+1}-v_{t+1-m}}{\sqrt{m}} F_n^j(t/n)^T \theta_* + \frac{1}{n} \sum_{t=m-1}^{n-1} \frac{v_{t+1}-v_{t+1-m}}{\sqrt{m}} F_n^j(t/n)^T \theta_* + O_p\left( \frac{\sqrt{m/n}}{n} \right)$$

$$+ \frac{n^{-p}}{\sqrt{m}} \sum_{t=m-1}^{n-1} \frac{v_{t+1}-v_{t+1-m}}{\sqrt{m}} \left( C(1) \left( \frac{1}{\sqrt{m}} \right) \sum_{j=0}^{m-1} \frac{v_{t+1}-v_{t+1-m}}{\sqrt{m}} + O_p\left( \frac{\sqrt{m/n}}{n} \right) \right) \theta_*$$

$$\times \left( C(1) \left( \frac{1}{\sqrt{m}} \right) \sum_{j=0}^{m-1} \frac{v_{t+1}-v_{t+1-m}}{\sqrt{m}} + O_p\left( \frac{\sqrt{m/n}}{n} \right) \right) \theta_*$$

$$= \frac{1}{\sqrt{m}} \sum_{j=0}^{m-1} \left( \frac{1}{n} \sum_{t=m-1}^{n-1} \theta^T C(1) \left( \frac{1}{\sqrt{m}} \right) \sum_{j=0}^{m-1} e_{t+1-j} F_n^j(t/n)^T \theta_* \right) + O_p\left( \frac{1}{\sqrt{m}} \right) + O_p\left( \frac{\sqrt{m/n}}{n} \right) + O_p(n^{-p}/\sqrt{m})$$

$$= O_p\left( \frac{\sqrt{m/n}}{n} \right) + O_p\left( \frac{1}{\sqrt{m}} \right),$$

which proves (5). Part (6) can easily be proved similarly to the proofs of (4) and (5).

This completes the proof of Lemma A.1. Q.E.D.

**Lemma A.2:** Let $e_i$ be a sequence of independent standard normal distributed random variables. Then

$$(1/n) \sum_{t=m}^{n} \left( \frac{1}{\sqrt{m}} \sum_{j=0}^{m-1} e_{t+j} \right)^2 = 1 + O_p\left( \frac{\sqrt{m/n}}{n} \right).$$

**Proof:** Let $N = [n/m]$. Then
\[
\frac{1}{n} \sum_{t=m}^{n} \left( \frac{1}{n} \sum_{j=0}^{m-1} e_{t-j} \right)^2 = \frac{1}{n} \sum_{i=0}^{m-1} N_{t=1}^{N} \left( \frac{1}{n} \sum_{j=0}^{m-1} e_{t_{m+i-j}} \right)^2 + \frac{1}{n} \sum_{t=mN}^{n} \left( \frac{1}{n} \sum_{j=0}^{m-1} e_{t-j} \right)^2
\]
\[
= \frac{mN}{n} \sum_{i=0}^{m-1} \left[ \frac{1}{N} \sum_{t=1}^{N} \left( \frac{1}{n} \sum_{j=0}^{m-1} e_{t_{m+i-j}} \right)^2 \right] + O_p(m/n)
\]
\[
= \frac{mN}{n} \sum_{i=0}^{m-1} \left[ 1 + O_p(1/N) + O_p(m/n) = 1 + O_p(\sqrt{m/n}), \right.
\]

due to the fact the expression between square brackets in the second line is a mean of \( N \) independent \( \chi^2(1) \) distributed random variables.

**Proof of Lemma 2**

*Proof of part (16):* It follows from (14) that

\[
\begin{pmatrix}
\frac{1}{\sqrt{n}} & 0^T \\
0 & n^p I_{k-1}
\end{pmatrix} Q^T \hat{M}_1^{-1} Q \begin{pmatrix}
\frac{1}{\sqrt{n}} & 0^T \\
0 & n^p I_{k-1}
\end{pmatrix}
\]

\[= \left( \theta^T C(1) \int \bar{W}_k(x) \bar{W}_k(x)^T dx C(1) \theta \right. \left. + \theta^T C(1) \int \bar{W}_k(x) F(x)^T dx Q \right)^{-1}
\]

\[= Q^T \int F(x) \bar{W}_k(x)^T dx C(1) \theta \Delta_\theta
\]

in distribution. It is a standard linear algebra exercise to verify that the latter matrix is

\[
\begin{pmatrix}
\tilde{\mu}^{-1} & -\theta^T C(1) \int \bar{W}_k(x) F(x)^T dx Q \Delta_\theta^{-1} \tilde{\mu}^{-1} \\
-\tilde{\mu}^{-1} \Delta_\theta^{-1} Q \int F(x) \bar{W}_k(x)^T dx C(1) \theta \Delta_\theta^{-1} + \tilde{\mu}^{-1} \Delta_\theta^{-1} Q \int F(x) \bar{W}_k(x)^T dx C(1) \theta \theta^T C(1) \int \bar{W}_k(y) F(y)^T dy Q \Delta_\theta^{-1}
\end{pmatrix}
\]
Part (16) now follows from:

\[
\left( \begin{array}{ccc}
\frac{1}{\sqrt{m}} & 0^T \\
0 & n^p I_{k-1}
\end{array} \right) Q \left( \begin{array}{ccc}
\frac{1}{\sqrt{n}} & 0^T \\
0 & n^p I_{k-1}
\end{array} \right) 
\]

\[
= \left( \begin{array}{ccc}
\frac{1}{\sqrt{n}} & 0^T \\
0 & n^p \sqrt{m/n} I_{k-1}
\end{array} \right) Q^T \hat{M}_1^{-1} Q \left( \begin{array}{ccc}
\frac{1}{\sqrt{n}} & 0^T \\
0 & n^p \sqrt{m/n} I_{k-1}
\end{array} \right) - \left( \begin{array}{ccc}
\tilde{\mu} & 0^T \\
0 & O
\end{array} \right) \text{ in distr.}
\]

**Proof of part (18):** This part follows trivially from (15), realizing that by Assumption 3, \(\theta^T C(1) C(1)^T \theta\) is positive, and that by Assumption 1 and the hypothesis \(\Phi(1), Q_1^T M_2 Q_1\) has rank \(k-1\), and is therefore nonsingular.

**Proof of Theorem 2**

The proof of part (24) of Theorem 2 for the case \(r = 1\) is based on Mercer's theorem. Cf. Dunford and Schwartz (1963, p. 1088), and Bierens and Ploberger (1997). Let

\[
\Gamma(x,y) = \mathbb{E}\left( \tilde{W}_1(x) \tilde{W}_1(y) \right).
\]

This function is real valued symmetric positive semi-definite, and it follows from (12) that \(\Gamma\) is continuous on \([0,1] \times [0,1]\). Now Mercer's theorem states that there exists a sequence \(\lambda_j\) of nonnegative eigenvalues and corresponding sequence \(\psi_j(x)\) of real valued continuous eigenfunctions such that

\[
\int \Gamma(x,y) \psi_j(y) dy = \lambda_j \psi_j(x), \quad j = 1, 2, \ldots, \sum_{j=1}^{\infty} \lambda_j < \infty, \quad \int \psi_j(x) \psi_j(x) dx = \mathcal{I}(i=j),
\]

where \(\mathcal{I}\) is the indicator function, and

\[
\Gamma(x,y) = \sum_{j=1}^{\infty} \lambda_j \psi_j(x) \psi_j(y).
\]

If follows straightforwardly from (12) that \(\int \tilde{W}_1(x) dx = 0\) a.s., hence \(\int \Gamma(x,y) dy = 0\). This implies that \(1\) is an eigenfunction corresponding to a zero eigenvalue. Since the eigenfunctions are
orthogonal, all the other eigenfunctions are orthogonal to 1, and therefore satisfy \( \int \psi_j(x) dx = 0 \).

Moreover, if follows trivially from (12) that \( \overline{W}_1(0) = \overline{W}_1(1) = 0 \) a.s., hence \( \Gamma(0,0) = \sum_{j=1}^{\infty} \lambda_j \psi_j(0)^2 = 0 \) and \( \Gamma(1,1) = \sum_{j=1}^{\infty} \lambda_j \psi_j(1)^2 = 0 \). Consequently, the eigenfunction \( \psi_j(x) \) corresponding to the positive eigenvalues satisfy \( \psi_j(0) = \psi_j(1) = 0 \).

We can now find sub-sequences \( j_{m(1)}, \ldots, j_{m(k-1)} \) such that for \( i, i_1, i_2 = 1, \ldots, k-1 \),

\[
\int \psi_{j_{m(i)}}(x) \psi_{j_{m(i_2)}}(x) dx = I(i_1 = i_2),
\]

\[
\int \psi_{j_{m(i)}}(x) dx = 0, \quad \int x \psi_{j_{m(i)}}(x) dx = 0, \quad \psi_{j_{m(i)}}(0) = \psi_{j_{m(i)}}(1) = 0,
\]

\[
\lim_{m \to \infty} \max_{1 \leq i \leq k-1} \lambda_{j_{m(i)}} = 0.
\]

hence, denoting

\[
F_m(x) = Q_{+} A_{+}^{1/2} \begin{pmatrix} 
\psi_{j_{m(1)}(x)} \\
\vdots \\
\psi_{j_{m(k-1)}(x)} 
\end{pmatrix}
\]

we have

\[
\int F_m(x) F_m(x)^T dx = M_1, \quad F_m(0) = F_m(1) = 0, \quad \int F_m(x) dx = 0,
\]

and

\[
\lim_{m \to \infty} E \left[ \int \overline{W}_1(x) F_m(x)^T dx Q_{+} A_{-}^{-1} Q_{+}^T \int \overline{W}_1(y) F_m(y) dy \right]
\]

\[
= \lim_{m \to \infty} \sum_{i=1}^{k-1} \int \psi_{j_{m(i)}(x)} \Gamma(x, y) \psi_{j_{m(i)}(y)} dx dy = \lim_{m \to \infty} \sum_{i=1}^{k-1} \lambda_{j_{m(i)}} = 0.
\]

Part (24) of Theorem 2 in the case \( r = 1 \) follows now from Chebishev's inequality.

Since the components of \( \overline{W}_1(x) \) are independent, and each component is distributed as \( \overline{W}_1(x) \), the general case follows straightforwardly from the proof in the case \( r = 1 \).

Part (25) of Theorem 2 follows directly from part (24). Q.E.D.
Proof of Theorem 6

Observe that

\[
\frac{1}{\sqrt{n}}(\hat{\beta}_0 - \beta_0) = 4(1/n)\sum_{t=1}^{n} U_n(t/n) - 6(1/n)\sum_{t=1}^{n} (t/n)U_n(t/n) + o_p(1),
\]

\[
\sqrt{n}(\hat{\beta}_1 - \beta_1) = -6(1/n)\sum_{t=1}^{n} U_n(t/n) + 12(1/n)\sum_{t=1}^{n} (t/n)U_n(t/n) + o_p(1),
\]

hence

\[
\frac{z_{[nx]} - \hat{\beta}_0 - \hat{\beta}_1[nx]}{\sqrt{n}} = U_n(x) - \frac{1}{\sqrt{n}}(\hat{\beta}_0 - \beta_0) - \frac{\sqrt{n}(\hat{\beta}_1 - \beta_1)[nx]}{n}
\]

\[
= U_n(x) - 4(1/n)\sum_{t=1}^{n} U_n(t/n) + 6(1/n)\sum_{t=1}^{n} (t/n)U_n(t/n)
\]

\[
+6x(1/n)\sum_{t=1}^{n} U_n(t/n) - 12x(1/n)\sum_{t=1}^{n} (t/n)U_n(t/n) + o_p(1),
\]

\[
= C(1) \left\{ W_k(x) + (6x-4) \int W'_k(y)dy - (12x-6) \int y W'_k(y)dy \right\}
\]

\[
= C(1)W^*_k(x),
\]

say. Thus,

\[
\hat{F}(x)/\sqrt{n} = C(1) \int_{0}^{x} W^*_k(y)dy = C(1)W^{**}_k(x),
\]

say, and consequently

\[
\frac{1}{n}\hat{M}_t - C(1) \int W^{**}_k(x)W^{**}_k(x)^T dx C(1)^T
\]

in distr.

Now suppose that \( z_t \) is cointegrated with one cointegrating vector \( \Theta \). Then \( \Theta^T C(1) = 0^T \). Since
by (11), \( z_t = \beta_0 + \beta_1 t + C(1)\sum_{j=1}^t \varepsilon_j + \nu_t - \nu_0 \), with \( \nu_t = D(L)\varepsilon_t \), we have
\[
\theta^Tz_t = \theta^T(\beta_0 - \nu_0) + \theta^T\beta t + \theta^T\nu_t = \theta^T(\beta_0 - \nu_0) + \theta^T\beta t + \theta^TD(L)\varepsilon_t,
\]
and consequently, similarly to (12),
\[
\sqrt{n}\theta^T\hat{F}(x) \Rightarrow \theta^TD(1)\hat{W}_k(x).
\]

Let \( Q = (\Theta, Q_\Theta) \) be the orthogonal matrix of eigenvectors of \( C(1)C(1)^T \), corresponding to the increasingly ordered eigenvalues. It is now easy to verify that under the unit root hypothesis with one cointegrated vector,
\[
\frac{1}{\sqrt{n}} \begin{pmatrix} n & 0^T \\ 0 & I_{k-1} \end{pmatrix} Q^T \hat{M}_1 Q \begin{pmatrix} n & 0^T \\ 0 & I_{k-1} \end{pmatrix} \rightarrow 
\left(
\begin{pmatrix}
\theta^TD(1)\int \bar{W}_k(x)\bar{W}_k(x)^Tdx(1)^T
\theta^TD(1)\int \bar{W}_k(x)\bar{W}_k^{**}(x)^TdxC(1)^TQ
Q^TC(1)\int \bar{W}_k^{**}(x)^T\bar{W}_k(x)^TD(1)^T
Q^TC(1)\int \bar{W}_k^{**}(x)^T\bar{W}_k^{**}(x)^TdxC(1)^TQ
\end{pmatrix}
\right)
\text{in distr.}
\]
Therefore, similarly to Lemma 2, it follows that for every nonnegative sequence \( m = o(n) \) we have:
\[
\left(
\begin{pmatrix}
1/\sqrt{n} & 0^T \\ 0 & \sqrt{n}/\sqrt{m/nL_{k-1}}
\end{pmatrix} Q^T \hat{M}_1^{-1} Q \begin{pmatrix} 1/\sqrt{n} & 0^T \\ 0 & \sqrt{n}/\sqrt{m/nL_{k-1}} \end{pmatrix}
\right)
\rightarrow 
\left(
\begin{pmatrix}
\tilde{\mu}^{-1} & 0^T \\ 0 & O
\end{pmatrix}
\right)
\text{in distr., where}
\]
(A.13)
\[
\tilde{\mu}_\ast = \theta^T D(1) \int \tilde{W}_k(x) \tilde{W}_k(x)^T dx D(1)^T \theta \\
- \theta^T D(1) \int \tilde{W}_k(x) W_{k-1}^{*\ast}(x)^T dx C(1)^T Q_c \left( Q_c^T C(1) \int W_{k-1}^{*\ast}(x) W_{k-1}^{*\ast}(x) dx C(1)^T Q_c \right)^{-1} \\
\times Q_c^T C(1) \int W_{k-1}^{*\ast}(x) \tilde{W}_k(x)^T dx D(1)^T \theta \\
- \theta^T D(1) D(1)^T \theta \\
\times \left( \int \tilde{W}_1(x)^2 dx - \int \tilde{W}_1(x) W_{k-1}^{*\ast}(x)^T dx \left( \int W_{k-1}^{*\ast}(x) W_{k-1}^{*\ast}(x) dx \right)^{-1} \int W_{k-1}^{*\ast}(x) \tilde{W}_1(x) dx \right)
\]

Next we investigate the asymptotic properties of the matrix \( M_2 \) under the unit root with cointegration hypothesis. Let again \( m \) be a sequence of natural numbers converging to infinity at rate \( o(n) \). Then it follows from (11) that for \( x \in [0,1] \),

\[
\frac{1}{m} \sum_{j=0}^{m-1} \left( [nx]-j - \tilde{\beta}_0 - \tilde{\beta}_1 ([nx]-j) \right) = \\
= \frac{1}{m} \sum_{j=0}^{m-1} U_n \left( \frac{[nx]-j}{n} \right) - 4(1/n) \sum_{t=1}^{n} U_n (t/n) + 6(1/n) \sum_{t=1}^{n} (t/n) U_n (t/n) \\
+ 6 \frac{[nx]}{n} (1/n) \sum_{t=1}^{n} U_n (t/n) - 12 \frac{[nx]}{n} (1/n) \sum_{t=1}^{n} (t/n) U_n (t/n) + o_p(1).
\]

Moreover,

\[
\frac{1}{m} \sum_{j=0}^{m-1} U_n \left( \frac{[nx]-j}{n} \right) = \frac{1}{m} \sum_{j=0}^{m-1} U_n \left( \frac{[nx]-m+1+j}{n} \right) = \frac{1}{m} \sum_{j=0}^{m-1} \int U_n \left( \frac{[nx]-m+1+y}{n} \right) dy \\
= \frac{1}{m} \int U_n \left( \frac{[nx]-m+1+y}{n} \right) dy = \frac{1}{m} \int U_n \left( \frac{[nx]-m+1+y}{n} \right) dy = U_n (x) + o_p(1)
\]

Therefore
\[
\frac{1}{m} \sum_{j=0}^{m-1} \{ z_{[nx]} - \hat{\beta}_0 - \hat{\beta}_1 ([nx] - j) \} \Rightarrow C(1)W_k^*(x).
\]

and consequently,
\[
\frac{\hat{M}_2}{n} \Rightarrow C(1) \int W_k^*(x)W_k^*(x)^T dx C(1)^T
\]

(A.14)

in distr. Furthermore, it is easy to verify that
\[
\frac{m}{n} \theta^T \hat{M}_2 \theta = \theta^T D(1)D(1)^T \theta + o_p(1),
\]

and
\[
\frac{\sqrt{m}}{n} \hat{M}_2 \theta = o_p(1).
\]

Thus,
\[
\begin{pmatrix}
\sqrt{m} & 0^T \\
0 & \frac{1}{\sqrt{n}} I_{k-1}
\end{pmatrix}
Q^T \hat{M}_2 Q
\begin{pmatrix}
\sqrt{m} & 0^T \\
0 & \frac{1}{\sqrt{n}} I_{k-1}
\end{pmatrix}^{-1}
\begin{pmatrix}
\theta^T D(1)D(1)^T \theta & 0^T \\
0 & Q_*^T C(1) \int W_k^*(x)W_k^*(x)^T dx C(1)^T Q_*
\end{pmatrix}
\]

in distr, and therefore, similarly to Lemma 2, we have
\[
\begin{pmatrix}
1/\sqrt{m} & 0^T \\
0 & \sqrt{nI}_{k-1}
\end{pmatrix}
Q^T \hat{M}_2^{-1} Q
\begin{pmatrix}
1/\sqrt{m} & 0^T \\
0 & \sqrt{nI}_{k-1}
\end{pmatrix}^{-1}
\begin{pmatrix}
(\theta^T D(1)D(1)^T \theta)^{-1} & 0^T \\
0 & Q_*^T C(1) \int W_k^*(x)W_k^*(x)^T dx C(1)^T Q_*^{-1}
\end{pmatrix}
\]

(A.15)
Comparing (13) and (15) with eigenvalue problem (20), we see that under the unit root hypothesis with single cointegration and 
\[ m = [n^\alpha], \]
with \( 0 < \alpha < 1 \), the minimum solution \( \hat{\lambda}_1 \) of eigenvalue problem (19) satisfies:
\[
n^{1-\alpha} \hat{\lambda}_1 \to \frac{\hat{\mu}}{\theta^T D(1) D(1)^T \theta}
\]
\[
= \int \left( W_1(x) \right)^2 dx - \int W_1(x) W_{k-1}^*(x) \bar{D} \left( \int W_{k-1}^*(x) W_{k-1}^*(x) dx \right)^{-1} \int W_{k-1}^*(x) W_1(x) dx
\]
in distr.

Note that the limiting random variable involved has the same upperbound as in the case of co-trending. Moreover, it follows from (12) and (14) that under the unit root hypothesis without cointegration, \( \hat{\lambda}_1 \) converges in distribution to the minimum solution of the generalized eigenvalue problem
\[
\det \left[ \int W_{k-1}^*(x) W_{k-1}^*(x) \bar{D} dx - \lambda \int W_{k-1}^*(x) W_{k-1}^*(x) \bar{D} dx \right] = 0.
\]
This completes the proof of Theorem 6 for the case of a \( k \)-variate unit root process with drift and one cointegrating vector. The general case with multiple cointegrating vectors can be shown along similar lines.

In the case of a \( k \)-variate unit root process \( z \), without drift with one cointegrating vector \( \theta \) and \( z \) demeaned rather than detrended, we have \( z_t = z_0 + \sum_{j=1}^I u_j = z_0 - v_0 + C(1) \sum_{j=1}^I c_j + v_t \), where
\[
\bar{z} = (1/n) \sum_{t=1}^n z_t \text{ and again } v_t = D(L) e_t. \text{ Hence}
\]
\[
(z_{nt} - \bar{z}) \sqrt{n} = U_n(x) - (1/n) \sum_{t=1}^n U_n(t/n) + o_p(1) \Rightarrow C(1) \left( W_k(x) - \int W_k(v) dy \right) = C(1) \tilde{W}_k^*(x),
\]
and \( \theta^T (z_t - \bar{z}) = v_t - \bar{v} \), where \( \bar{v} = (1/n) \sum_{t=1}^n v_t \). The latter implies
\[ \sqrt{n} \theta^T \hat{F}(x) = \theta^T D(1) W_k^0(x). \]

The rest of the proof is now similar to the previous case. Q.E.D.