Semi-nonparametric estimation of independently and identically repeated first-price auctions via an integrated simulated moments method

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Abstract

In this paper we propose to estimate the value distribution of independently and identically repeated first-price auctions directly via a semi-nonparametric integrated simulated moments sieve approach. Given a candidate value distribution function in a sieve space, we simulate bids according to the equilibrium bid function involved. We take the difference of the empirical characteristic functions of the actual and simulated bids as the moment function. The objective function is then the integral of the squared moment function over an interval. Minimizing this integral to the distribution functions in the sieve space involved and letting the sieve order increase to infinity with the sample size then yields a uniformly consistent semi-nonparametric estimator of the actual value distribution. Also, we propose an integrated moment test for the validity of the first-price auction model, and an data-driven method for the choice of the sieve order. Finally, we conduct a few numerical experiments to check the performance of our approach.

JEL codes: C14, C15, C51, D44

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1 Introduction

As Laffont and Vuong (1993) point out, the distribution of bids determines the structural elements of auction models, provided identification is achieved. In the first-price auction model with symmetric independent private values, the structural element of interest is the value distribution. Much research has been done on the identification and the estimation of the value distribution. Donald and Paarsch (1996) apply ML estimation to first-price auctions and Dutch auctions. They use a parametric specification for the value distribution to implement ML estimation. In particular, they assume in a numerical example that the value distribution is a uniform distribution on the interval \([0, \bar{v}]\) where \(\bar{v}\) is a parametric function of auction-specific covariates. Because in this case the support of the bid distribution involved depends on parameters, the standard consistency proof of ML estimators no longer applies. Another difficulty with ML estimation of first-price auction models is that the equilibrium bid function is highly non-linear in the value and its distribution, so that the computation of the ML estimators involved is challenging. The same applies to descending price (Dutch) auctions, because they are strategically equivalent to first-price auctions. Because of the difficulty of ML estimation of these models, Laffont and Vuong (1993) suggest Simulated Non-Linear Least Squares (SNLLS) estimation for first-price auction models and Simulated Method of Moment (SMM) estimation for descending price auction models with symmetric independent private values. Their SNLLS approach requires to replace the expectation of the winning bid with a simulated one. They also suggest that the expectation of higher moments of winning bids can be used for SMM estimation if the expectation of the winning bids itself is not sufficient to identify all parameters. Both SNLLS and SMM approaches require a parametric specification for the value distribution. Laffont et al. (1995) apply the SNLLS approach suggested by Laffont and Vuong (1993) to the egg plant auction, which is a descending price auction. They specify a log-normal value distribution conditional on covariates. Li (2005) considers first-price auctions with entry and binding reservation price. This
auction consists of two stages. In the first stage the potential bidder decides whether he or she enters the auction, with payment of entry cost. In the second stage, the bidder gets to know his or her value and then decides to bid according to the equilibrium bid function, which is the same function as for the first-price auction model. Li (2005) proposes a SMM approach to estimate the entry probability and the parameters of the value distribution. One of the conditional moments is a function of the upper bound of the bid support, which can be computed via the simulation approach in Laffont et al. (1995). The other moment conditions are related to the number of active bidders, i.e., potential bidders who decide to participate in the auction.

For general nonparametric identification results of first-price auctions models with symmetric independent private values, see Athey and Haile (2002, 2006, 2007) and Guerre et al. (2000). In particular, Guerre et al. (2000) show the nonparametric identification of value distributions with bounded support $[\underline{v}, \overline{v}]$, $\overline{v} < \infty$, and propose an indirect nonparametric kernel estimation approach. In the case that the reservation price is nonbinding their approach is based on the inverse bid function $v = b + (I - 1)^{-1} \Lambda(b)/\lambda(b)$, where $I$ is the number of bidders, $v$ is a private value, $b$ is a corresponding bid, and $\Lambda(b)$ is the distribution function of bids with density $\lambda(b)$. The latter two functions are estimated via nonparametric kernel methods, as $\hat{\Lambda}(b)$ and $\hat{\lambda}(b)$, respectively. Using the pseudo-private values $\tilde{V} = B + (I - 1)^{-1} \hat{\Lambda}(B)/\hat{\lambda}(B)$, where each $B$ is an observed bid, the density of the private value distribution can now be estimated by kernel density estimation. However, the ratio $\hat{\Lambda}(b)/\hat{\lambda}(b)$ may be an unreliable estimate of $\Lambda(b)/\lambda(b)$ near the boundary of the support of $\lambda(b)$. To solve this problem, Guerre et al. (2000) use a trimming procedure which amounts to discarding pseudo-private values $\tilde{V}$ corresponding to bids $B$ that are too close to the boundary of the (known) support of the bid distribution.

Throughout this paper we confine our analysis to first-price sealed bid auctions where values are independent, private and bidders are symmetric and risk-neutral. Moreover, our asymptotic results are based on the assumption that the observed bids are generated by independently repeated identical first-price auctions. Admittedly, this type of repeated auction is rare in practice. The reason for considering this case is to lay the groundwork for the more realistic case of first-price auctions with auction-specific covariates and different numbers of potential bidders and reservation prices, like the well-known US Forest Service timber auctions.
Our main contributions to the empirical auction literature are twofold. First, we propose a direct Semi-Nonparametric Integrated Simulated Moments (SNP-ISM) estimation approach, as an alternative to the two-step nonparametric kernel estimation approach of Guerre et al. (2000), and prove that under the aforementioned weaker conditions this approach yields a uniformly consistent SNP sieve estimator of the value distribution. Second, we derive an integrated moment test of the validity of the first-price auction model.

Our SNP-ISM methodology is different from the SMM approach of Lafont and Vuong (1993) and Li (2005) in that the latter two approaches require parametric specification for the value distribution whereas ours does not. Instead, we treat the unknown value distribution function itself as a parameter, contained in a compact metric space $\mathcal{F}$ of absolutely continuous distribution functions endowed with the "sup" metric.

Based on the approach in Bierens (2008), we construct an increasing sequence $\{\mathcal{F}_n\}_{n=0}^{\infty}$ (the sieve$^1$) of subspaces of our "parameter space" $\mathcal{F}$, where the distribution functions in each subspace $\mathcal{F}_n$ can be represented by parametric functions of Legendre polynomials of order up to $n$. Given a distribution function $F \in \mathcal{F}_n$, we simulate bids according to the equilibrium bid function involved. Motivated by the well-known fact that distributions are equal if and only if their characteristic functions are identical, we take the difference of the empirical characteristic functions of the actual and simulated bids as the simulated moment function. Thus, our approach uses uncountably many moment conditions whereas Lafont and Vuong (1993) and Li (2005) use only a finite number of moment conditions. Because the condition that the value distribution has finite expectation implies that the corresponding bid distribution has bounded support, and characteristic functions of bounded random variables coincide everywhere if and only if they coincide on an arbitrary interval around zero, we take the integral of the squared simulated moment function over such an interval as our objective function, similar to the Integrated Conditional Moment (ICM) test statistic of Bierens (1982) and Bierens and Ploberger (1997). Minimizing this objective function to the distribution functions in $\mathcal{F}_n$ and letting $n$ increase with the sample size $N$ then yields a uniformly consistent SNP sieve estimator of the actual value distribution. The minimal value of this objective function

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$^1$A sequence $\{\mathcal{F}_n\}_{n=0}^{\infty}$ of subspaces of a metric space $\mathcal{F}$ is called a sieve if the closure $\bigcup_{n=0}^{\infty} \mathcal{F}_n$ of $\bigcup_{n=0}^{\infty} \mathcal{F}_n$ is equal to $\mathcal{F}$. See for example Shen(1997) and Chen (2007).
also serves as the test statistic of an Integrated Moment (IM) test of the validity of the first-price auction model. Moreover, similar to the information criteria of Hannan-Quinn (1979) and Schwarz (1978) for likelihood models, we construct an information criterion that can be used to estimate the sieve order \( n \) consistently if it is finite, and otherwise yields a data-driven sequence \( n_N \) for which the SNP sieve estimator is uniformly consistent as well. Finally, we conduct a few numerical experiments to check the performance of our SNP-ISM approach.

The paper is organized as follows. In section 2 we review the first-price auction model. In section 3 we introduce our estimation methodology and we set forth further conditions for the uniform strong consistency of the SNP-ISM sieve estimator involved. In section 4 we show how to construct the metric space \( \mathcal{F} \) and corresponding sieve spaces \( \mathcal{F}_n \) of absolutely continuous distribution functions on \((0, \infty)\). In section 5 we propose an integrated moment (IM) test of the validity of the first-price auction model with symmetric independent values and risk neutrality. In section 6 we propose an information criterion for the data-driven selection of the sieve order \( n \), similar to the order selection of autoregressions via the Hannan-Quinn (1979) and Schwarz (1978) information criteria. In section 7 we show the performance of our proposed SNP-ISM estimation and IM testing approach via a few numerical experiments. In section 8 we make some concluding remarks. Most of the proofs of our results are given in the Appendix (section 9).

2 First-Price Auctions

A first price sealed bids auction (henceforth called first-price auction) is an auction with \( I \geq 2 \) potential bidders, where the potential bidder’s values for the item to be auctioned off are independent and private, and the bidders are symmetric and risk neutral. The reservation price \( p_0 \), if any, is announced in advance and the number \( I \) of potential bidders is known to each potential bidder.

As is well known, the equilibrium bid function of a first-price auction takes the form

\[
\beta(v|F, I) = v - \frac{1}{F(v)^{I-1}} \int_{p_0}^{v} F(x)^{I-1}dx \text{ for } v > p_0 > 0.
\]
if the reservation price $p_0$ is binding, and

$$\beta(v|F,I) = v - \frac{1}{F(v)^{I-1}} \int_0^v F(x)^{I-1} dx \text{ for } v > v_0.$$

(2)

if the reservation price $p_0$ is non-binding, where $F(v)$ is the value distribution, $I \geq 2$ is the number of potential bidders and $v \geq 0$ is the lower bound of the support of $F(v)$. See for example Riley and Samuelson (1981) or Krishna (2002). Thus, if the reservation price $p_0$ is binding then, with $V_j$ the value for bidder $j$ for the item to be auctioned off, this potential bidder issues a bid $B_j = \beta(V_j|F,I)$ according to bid function (1) if $V_j > p_0$ and does not issue a bid if $V_j \leq p_0$, whereas if the reservation price $p_0$ is not binding each potential bidder $j$ issues a bid $B_j = \beta(V_j|F,I)$ according to bid function (2). In the first-price auction model the individual values $V_j$, $j = 1, ..., I$, are assumed to be independent random drawing from the value distribution $F$. The latter is known to each potential bidder $j$, and so is the number of potential bidders, $I$.

In their derivation of the equilibrium bid function (1), Riley and Samuelson (1981) assume that the value distribution $F$ has bounded support. This condition is also adopted by Donald and Paarsch (1996) and Guerre et al. (2000), among others. However, Riley and Samuelson (1981) do not use this condition explicitly, but only its implication that then the expected revenue of the seller,

$$\overline{R} = I \int_{p_0}^\infty (vF'(v) + F(v) - 1) F(v)^{I-1} dv,$$

(3)

is finite. The argument in Riley and Samuelson (1981) also hinges on the condition that $F$ is differentiable on its support. Therefore, it suffices to assume that:

**Assumption 1.** The value distribution $F$ in (1) and (2) is absolutely continuous on $(0, \infty)$, with density $f$ satisfying

$$\int_0^\infty vf(v)dv < \infty.$$

(4)

Moreover, the support of $F(v)$ is connected.
As mentioned by Guerre et al. (2000, Footnote 8) with reference to Laftont et al. (1995), condition (4) implies that \( \sup_{v > 0} \beta(v|F,I) < \infty \), so that the bid distribution has bounded support. Moreover, using integration by parts, we can write (2) as

\[
\beta(v|F,I) = (I - 1) \int_0^v x f(x) F(x)^{I-2} dx / F(v)^{I-1},
\]

so that \( \sup_{v > 0} \beta(v|F,I) < \infty \) implies (4), and similarly if \( p_0 \) is binding. Furthermore, it is easy to verify that condition (4) is sufficient for (3) to be finite.

There are two seminal papers on the nonparametric identification of first-price auction models, namely Donald and Paarsch (1996) and Guerre et al. (2000). See also Athey and Haile (2002). Of course, parametric identification has been established earlier, in particular by Laftont et al. (1995). Donald and Paarsch (1996) and Guerre et al. (2000) show (in different ways) the nonparametric identification of first-price auction models under the assumption that the support of the value distribution is bounded.

However, the inverse bid function approach of Guerre et al. (2000) does not hinge on the assumption that the value distribution has bounded support. Therefore, we will not adopt the latter, but only that the support of the value distribution is connected. As shown in Bierens and Song (2011a), this condition is not necessary for nonparametric identification. The main reason for this condition is that then the inverse \( F^{-1}(u) \) of \( F(v) \) is continuous on \((0,1)\).

3 Integrated Simulated Moments Sieve Estimation.

3.1 Data Generating Process

As said before, in this paper we will consider the case where a first-price auction is repeated independently \( L \) times, with the same true value distribution \( F_0(v) \), the same fixed number of potential bidders \( I_0 \geq 2 \), and the same reservation price \( p_0 \). The asymptotic results will be derived for \( L \to \infty \), for the case that the reservation price \( p_0 \) is binding. Thus, in each auction \( \ell, \ell = 1, \ldots, L \), we observe \( Y_\ell \) bids \( B_{\ell,j} \), \( j = 1, \ldots, Y_\ell \), where \( Y_\ell \) may differ across auctions. However, since the \( Y_\ell \) are i.i.d. Bin\((I_0, 1 - F_0(p_0))\), we can
estimate \( I_0 \) directly by the order statistic \( \hat{I}_L = \max_{\ell=1,...,L} Y_\ell \), for which it is well known that \( \lim_{L \to \infty} P[\hat{I}_L = I_0] = 1 \). Therefore, without too much loss of generality we may assume that \( I_0 \) is given, and that the bidders with value \( V \leq p_0 \) issue a zero bid. Thus, the data consist of bids \( B_j \), including zero bids, generated according to

\[
B_j = \left( V_j - \frac{1}{F_0(V_j)^{I_0-1}} \int_{p_0}^{V_j} F_0(x)^{I_0-1} \, dx \right) \mathbf{1}(V_j > p_0),
\]

where the individual values \( V_j \) are independent random drawings from \( F_0 \).

### 3.2 The Objective Function

Given an absolutely continuous distribution function \( F \) on \((0, \infty)\), henceforth called a candidate value distribution, draw randomly values \( \tilde{V}_j(F) \), \( j = 1, ..., N \), from \( F \) and generate the corresponding bids by

\[
\tilde{B}_j(F) = \left( \tilde{V}_j(F) - \frac{1}{F(\tilde{V}_j(F))^{I_0-1}} \int_{p_0}^{\tilde{V}_j(F)} F(x)^{I_0-1} \, dx \right) \mathbf{1}(\tilde{V}_j(F) > p_0), \quad j = 1, ..., N = L \times I_0.
\]

If \( F \) does not satisfy Assumption 1 then the support of \( \tilde{B}_j(F) \) is unbounded, hence \( \tilde{B}_j(F) \sim B_j \) is not possible. Therefore, \( \tilde{B}_j(F) \sim B_j \) implies that \( \int_0^\infty v dF(v) < \infty \). Moreover, given that \( F \) satisfies Assumption 1 it follows from Theorem 1 that \( \tilde{B}_j(F) \sim B_j \) if and only if \( F(v) = F_0(v) \) for all \( v \in [p_0, \infty) \). Therefore, the main idea of our approach is to "calibrate" the candidate value distribution \( F \) such that, asymptotically, the distribution of \( \tilde{B}_j(F) \) becomes equal to the distribution of the actual bids \( B_j \). To do so, we need a measure for the distance between these two distributions. We will construct such a measure on the basis of the characteristic functions of \( B_j \) and \( \tilde{B}_j(F) \),

\[
\varphi(t) = E[\exp(i.t.B_j)] = \int_{p_0}^\infty \exp \left( i.t. \left( v - \frac{1}{F_0(v)^{I_0-1}} \int_{p_0}^{v} F_0(x)^{I_0-1} \, dx \right) \right) dF_0(v)
\]

\(^2\)In (5) and in the sequel \( \mathbf{1}(\cdot) \) denotes the indicator function.
\[ \psi(t|F) = E \left[ \exp \left( i.t.\tilde{B}_j(F) \right) \right] = \int_{p_0}^{\infty} \exp \left( i.t. \left( v - \frac{1}{F(v)^{1-1}} \int_{p_0}^{v} F(x) dx \right) \right) dF(v) \]

respectively, where \( i = \sqrt{-1} \).

Recall that under Assumption 1 the distribution of the bids \( B_j \) has bounded support. Moreover, it is well-known that distributions of bounded random variables are completely determined by the shape of their characteristic functions in an arbitrary open neighborhood \( (-\kappa, \kappa), \kappa > 0 \). In particular, if for a bounded random variable \( B \) and another random variable \( B_\ast \), \( E[\exp(i.t.B)] = E[\exp(i.t.B_\ast)] \) on \( (-\kappa, \kappa) \) then \( E[B^m] = E[B_\ast^m] \) for \( m = 1, 2, 3, \ldots, \), which implies that \( B \sim B_\ast \). Therefore, if \( \psi(t|F) = \varphi(t) \) for all \( t \in (-\kappa, \kappa) \) then \( \psi(t|F) = \varphi(t) \) for all \( t \in \mathbb{R} \), which implies that \( \tilde{B}_j(F) \sim B_j \) and thus \( F(v) = F_0(v) \) for all \( v \geq p_0 \). Consequently, denoting

\[
\overline{Q}(F|\kappa) = \frac{1}{2\kappa} \int_{-\kappa}^{\kappa} |\psi(t|F) - \varphi(t)|^2 dt
\]

\[
= \frac{1}{2\kappa} \int_{-\kappa}^{\kappa} \text{Re}[\psi(t|F) - \varphi(t)]^2 dt + \frac{1}{2\kappa} \int_{-\kappa}^{\kappa} \text{Im}[\psi(t|F) - \varphi(t)]^2 dt
\]

it follows that

**Lemma 1.** Under Assumption 1, \( \overline{Q}(F|\kappa) = 0 \) for all \( \kappa > 0 \) if \( F(v) = F_0(v) \) on \([p_0, \infty)\), whereas \( \overline{Q}(F|\kappa) > 0 \) for all \( \kappa > 0 \) if \( F(v) \neq F_0(v) \) on \([p_0, \infty)\).

Note that \( \lim_{\kappa \to 0} \overline{Q}(F|\kappa) = 0 \), as is not hard to verify, so that \( \kappa \) should not be chosen too small.

In principle it is possible to compute the characteristic function \( \psi(t|F) \) in (8) by numerical integration, but that will be a computational burden. Moreover, the characteristic function \( \varphi(t) \) in (7) is unknown and has to be estimated. Therefore, we will estimate both \( \varphi(t) \) and \( \psi(t|F) \) by their corresponding empirical characteristic functions

\[
\hat{\varphi}(t) = \frac{1}{N} \sum_{j=1}^{N} \exp(i.t.B_j),
\]

\[
\hat{\psi}(t|F) = E \left[ \exp \left( i.t.\tilde{B}_j(F) \right) \right] = \int_{p_0}^{\infty} \exp \left( i.t. \left( v - \frac{1}{F(v)^{1-1}} \int_{p_0}^{v} F(x) dx \right) \right) dF(v)
\]
\[ \psi(t|F) = \frac{1}{N} \sum_{j=1}^{N} \exp\left(\mathbf{i}.t.\tilde{B}_j(F)\right), \quad (10) \]

respectively, so that \( \bar{Q}(F|\kappa) \) is then estimated by

\[ \hat{Q}_N(F|\kappa) = \frac{1}{2\kappa} \int_{-\kappa}^{\kappa} \left| \psi(t|F) - \varphi(t) \right|^2 dt. \quad (11) \]

It follows straightforwardly from Kolmogorov’s strong law of large numbers and the bounded convergence theorem that pointwise in \( F \), \( \hat{Q}_N(F|\kappa) \xrightarrow{a.s.} \bar{Q}(F|\kappa) \). This result suggests to estimate \( F_0 \) by minimizing the objective function \( \hat{Q}_N(F|\kappa) \) for an a priori chosen \( \kappa > 0 \) in some way, to be discussed in the next subsection.

Note that the objective function \( \hat{Q}_N(F|\kappa) \) has a closed form expression in terms of the actual bids \( B_j \) and the simulated bids \( \tilde{B}_j(F) \), namely

\[ \hat{Q}_N(F|\kappa) = \frac{2}{N} + \frac{2}{N^2} \sum_{k=2}^{N} \sum_{m=1}^{k-1} \frac{\sin(\kappa(B_k - B_m))}{\kappa(B_k - B_m)} \]
\[ + \frac{2}{N^2} \sum_{k=2}^{N} \sum_{m=1}^{k-1} \frac{\sin(\kappa(\tilde{B}_k(F) - \tilde{B}_m(F)))}{\kappa(\tilde{B}_k(F) - \tilde{B}_m(F))} \]
\[ - \frac{2}{N^2} \sum_{k=1}^{N} \sum_{m=1}^{N} \frac{\sin(\kappa(B_k - \tilde{B}_m(F)))}{\kappa(B_k - \tilde{B}_m(F))} \quad (12) \]

which is not hard to verify, using the well-known formulas \( \exp(\mathbf{i}.a) = \cos(a) + \mathbf{i}.\sin(a) \) and \( \cos(a - b) = \cos(a)\cos(b) + \sin(a)\sin(b) \).

### 3.3 Semi-Nonparametric Sieve Estimation

#### 3.3.1 Strong Consistency of Sieve Estimators

The standard consistency proof for parameter estimators of nonlinear parametric models requires that the parameters are confined to a compact subset of a Euclidean space. Because the true value distribution \( F_0 \) plays now the role of parameter, we need to construct a compact metric space \( \mathcal{F} \) of absolutely continuous distribution functions \( F \) on \((0, \infty)\) containing the true
value distribution \( F_0 \): \( F_0 \in \mathcal{F} \). Because \( F_0 \) is unknown, the latter has to be assumed, though. We will endow the space \( \mathcal{F} \) with the "sup" metric, i.e., the metric

\[
\|F_1 - F_2\| = \sup_{v \geq p_0} |F_1(v) - F_2(v)|
\]

(13)

if the reservation price is binding and

\[
\|F_1 - F_2\| = \sup_{v > 0} |F_1(v) - F_2(v)|
\]

(14)

if not. Note that (13) is not a genuine metric on the space of distribution functions, but may be interpreted as a metric on the space of conditional distribution functions \( F(v | p_0) = P[V \leq v | V \geq p_0] = 1(v \geq p_0)(F(v) - F(p_0))/(1 - F(p_0)) \) because \( \sup_{v \geq p_0} |F_1(v) - F_2(v)| = 0 \) implies \( F_1(v | p_0) = F_2(v | p_0) \) and vice versa. Anyhow, since in the binding reservation price case the shape of the value distribution for \( v < p_0 \) does not matter, we may without loss of generality treat (13) as a metric, and interpret pairs \( F_1, F_2 \) for which \( \sup_{v \geq p_0} |F_1(v) - F_2(v)| = 0 \) as belonging to an equivalence class.

If it were possible to compute

\[
\hat{F}_N = \arg \min_{F \in \mathcal{F}} \hat{Q}_N(F|\kappa)
\]

(15)

then similar to the standard strong consistency proof for parametric M-estimators, the strong consistency of \( \hat{F}_N \), i.e.,

\[
\|\hat{F}_N - F_0\| \overset{a.s.}{\longrightarrow} 0,
\]

can be established by verifying the following three conditions:

\[
\sup_{F \in \mathcal{F}} \left| \hat{Q}_N(F|\kappa) - \overline{Q}(F|\kappa) \right| \overset{a.s.}{\longrightarrow} 0, 
\]

(16)

\[
F_0 = \arg \min_{F \in \mathcal{F}} \overline{Q}(F|\kappa) \text{ is unique.} 
\]

(17)

\[
\overline{Q}(F|\kappa) \text{ is continuous on } \mathcal{F}. 
\]

(18)

The latter means that for each \( F \in \mathcal{F} \) and an arbitrary \( \varepsilon > 0 \) there exists a \( \delta \) depending on \( F \) and \( \varepsilon \) such that \( |\overline{Q}(F|\kappa) - \overline{Q}(F_*|\kappa)| < \varepsilon \) if \( \|F - F_*\| < \delta \) for \( F_* \in \mathcal{F} \), where the latter metric is either (13) or (14).

\[3\)See for example Bierens (2004, Theorem 6.11, p.146).\]
Of course, in practice it is not possible to compute (15). The standard approach to get around this problem is sieve estimation,\(^4\) as follows. Let \(\{\mathcal{F}_n\}_{n=0}^\infty\) be an increasing sequence of compact subsets of \(\mathcal{F}\) which is dense in \(\mathcal{F}\), i.e.,

\[
\mathcal{F} = \overline{\bigcup_{n=1}^\infty \mathcal{F}_n},
\]

where the bar denotes the closure. The sequence \(\{\mathcal{F}_n\}_{n=0}^\infty\) is known as the sieve. Moreover, suppose that for each \(n\) the computation of

\[
\tilde{F}_n = \arg \min_{F \in \mathcal{F}_n} \hat{Q}_N(F|\kappa)
\]

is feasible. Then the following result holds:

**Lemma 2.** Let \(\mathcal{F}\) be a compact metric space of absolutely continuous distribution functions, endowed with the sup metric (13) or (14). Let \(n_N\) be an arbitrary subsequence of \(N\). Under the conditions (16)-(19), \(\|\tilde{F}_{n_N} - F_0\| \xrightarrow{a.s.} 0\) as \(n_N \to \infty\).

**Proof.** Similar to Bierens (2008, Theorem 11).

Note that by (7) and (9) and Kolmogorov’s strong law of large numbers, \(\hat{\varphi}(t) \xrightarrow{a.s.} \varphi(t)\) pointwise in \(t\). Therefore, the uniform convergence condition (16) follows by bounded convergence if pointwise in \(t\),

\[
\sup_{F \in \mathcal{F}} \left| \hat{\psi}(t|F) - \psi(t|F) \right| \xrightarrow{a.s.} 0,
\]

where \(\hat{\psi}(t|F)\) is defined by (10) and \(\psi(t|F)\) by (8). The condition (20) follows from Jennrich’s (1969) uniform strong law of large numbers, adapted to random functions on compact metric spaces, provided that it can be shown that \(\hat{\psi}(t|F)\) and \(\psi(t|F)\) are (a.s) continuous in \(F\). The latter is true if \(\hat{B}_j(F)\) is a.s. continuous in \(F\), which will be shown in the next subsection. Given this continuity condition, (18) then holds as well.

### 3.3.2 Continuity of Simulated Values and Bids

To show that the simulated bids \(\hat{B}_j(F)\) are a.s. continuous in \(F\), it matters how the simulated values are generated:

\(^4\)See for example Shen (1997) and Chen (2007).
**Assumption 2.** Given a sequence $U_1, ..., U_N, ...$ of independent random drawings from the uniform $[0, 1]$ distribution, for each candidate value distribution $F$ the corresponding simulated values $\tilde{V}_j(F) = F^{-1}(U_j)$ are generated by solving $U_j = F(\tilde{V}_j(F))$ for $j = 1, ..., N$.

Note that $F^{-1}(u)$ is defined as $F^{-1}(u) = \inf_{v=F(u)} v$ for $u \in (0, 1)$. See for example Devroye (1986, Theorem 2.1, p.28). The latter theorem also implies that $\tilde{V}_j(F)$ is distributed as $F$.

It follows now from (6) that the corresponding simulated bids can be generated according to

$$
\tilde{B}_j(F) = \left( \tilde{V}_j(F) - \frac{1}{U_j^{t_0-1}} \int_{p_0}^{\tilde{V}_j(F)} F(x)^{t_0-1} dx \right) \mathbf{1}\left( \tilde{V}_j(F) > p_0 \right)
$$

$$
= \left( \tilde{V}_j(F) - \frac{\tilde{V}_j(F) - p_0}{U_j^{t_0-1}} \int_0^{\tilde{V}_j(F)} F(p_0 + x(\tilde{V}_j(F) - p_0))^{t_0-1} dx \right)
$$

$$
\times \mathbf{1}\left( \tilde{V}_j(F) > p_0 \right)
$$

(21)

if the reservation price $p_0$ is binding, and

$$
\tilde{B}_j(F) = \tilde{V}_j(F) - \frac{\tilde{V}_j(F)}{U_j^{t_0-1}} \int_0^{1} F(x.\tilde{V}_j(F))^{t_0-1} dx
$$

if the reservation price $p_0$ is non-binding. The integrals involved can be computed numerically or via Monte Carlo integration. However, the asymptotic theory in this paper will be based on the assumption that these integrals are computed exactly.

Note that to guarantee smoothness of the empirical characteristic function $\hat{\psi}(t|F)$ of the simulated bids $\tilde{B}_j(F)$ in $t$ and $F$, the same sequence $\{U_k\}_{k=1}^N$ of independent uniformly $[0, 1]$ distributed random variables should be used to generate the simulated bids for different candidate value distributions $F$.

An alternative approach to generate simulated values $\tilde{V}_j(F)$ from $F$ is the well-known accept-reject method. See for example Devroye (1986) or Rubinstein (1981). However, the simulation procedure in Assumption 2 has the advantage that it is easier to prove that the simulated values and bids involved are continuous in $F$, in the following sense:
Lemma 3. Let $F_n$ and $F$ be absolutely continuous candidate value distributions in $\mathcal{F}$ such that
\[ \lim_{n \to \infty} ||F_n - F|| = 0 \] (22)

For a given random drawing $U$ from the uniform $[0, 1]$ distribution, let $\tilde{V}(F_n) = F_n^{-1}(U)$ and $\tilde{V}(F) = F^{-1}(U)$, with corresponding simulated bids $\tilde{B}(F_n)$ and $\tilde{B}(F)$, respectively. Then
\[ P \left[ \lim_{n \to \infty} \tilde{V}(F_n) = \tilde{V}(F) \right] = 1 \] (23)
and
\[ P \left[ \lim_{n \to \infty} \tilde{B}(F_n) = \tilde{B}(F) \right] = 1. \] (24)

Proof. Appendix.

4 Compactness and Uniform Consistency

4.1 Series Representation of Density Functions on the Unit Interval

Any absolutely continuous distribution function $F(v)$ on $[0, \infty)$ can be expressed as
\[ F(v) = H(G(v)), \]
where $G(v)$ is an a priori chosen absolutely continuous distribution function with support $(0, \infty)$, for example the exponential distribution, and $H$ is an absolutely continuous distribution function on the unit interval, namely $H(u) = F(G^{-1}(u))$. The density $f(v)$ of $F(v)$ then takes the form
\[ f(v) = h(G(v))g(v), \] (25)
where $g(v)$ is the density of $G(v)$ and $h(u)$ is the density of $H(u)$, i.e.,
\[ H(u) = \int_0^u h(x)dx, \text{ with } h(u) = f(G^{-1}(u))/g(G^{-1}(u)). \]

Therefore, we can estimate $f$ and $F$ by estimating $h$ given $G.$
As shown in Bierens (2008), any density function \( h(u) \) on \([0, 1]\) can be represented by an infinite series expansion of the form

\[
h(u) = \frac{(1 + \sum_{k=1}^{\infty} \delta_k \rho_k(u))^2}{1 + \sum_{k=1}^{\infty} \delta_k^2}
\]  

(26)
a.e. on \([0, 1]\), where \( \sum_{k=1}^{\infty} \delta_k^2 < \infty \) and the \( \rho_k(u) \) are orthonormal Legendre polynomials on \([0, 1]\).

### 4.2 Compact Spaces of Density and Distribution Functions

Because, indirectly, the density \( h \) in (25) plays the role of unknown parameter, we will first construct a compact metric space of densities on the unit interval. This can be done by imposing restrictions on the parameters \( \delta_k \) in (26), as follows.

**Lemma 4.** Given an a priori chosen sequence \( \delta_k \) of positive numbers such that \( \sum_{k=1}^{\infty} \delta_k^2 < \infty \), let \( D(0, 1) \) be the space of density function \( h(u) \) on \([0, 1]\) of the form (26), where the parameters \( \delta_k \) are restricted by the inequality

\[ |\delta_k| \leq \overline{\delta}_k, \quad k = 1, 2, 3, ..... \]

If we endow \( D(0, 1) \) with the \( L^1 \) metric

\[ ||h_1 - h_2||_1 = \int_0^1 |h_1(u) - h_2(u)| \, du, \]

then \( D(0, 1) \) is a compact metric space. Consequently, the corresponding space of absolutely continuous distribution functions on \([0, 1]\),

\[ \mathcal{H}(0, 1) = \left\{ H(u) = \int_0^u h(x) \, dx, \quad h \in D(0, 1) \right\}, \]

endowed with the "sup" metric \( \sup_{0 \leq u \leq 1} |H_1(u) - H_2(u)| \), is a compact metric space as well.

**Proof.** Bierens (2008, Theorem 8).

As to the choice of the \( \overline{\delta}_k \)'s, we may for example choose

\[ \overline{\delta}_k = c \left( 1 + \sqrt{k \ln(k)} \right)^{-1}, \quad k = 1, 2, 3, ...., \]
for some large constant \( c > 0 \), as then \( \sum_{k=1}^{\infty} \delta_k^2 < \infty \).

To construct compact spaces of densities and distribution functions on \((0, \infty)\),

**Assumption 3.** Choose an absolutely continuous distribution function \( G(v) \) with density \( g(v) \), finite expectation \( \int_0^\infty vg(v)dv < \infty \), and support \((0, \infty)\), as initial guess of the true value distribution \( F_0 \).

The reason for requiring that \( \int_0^\infty vg(v)dv < \infty \) is that if the initial guess \( G(v) \) of \( F_0(v) \) is right then \( \int_0^\infty vdF_0(v) < \infty \). C.f. Assumption 1.

It follows now straightforwardly from Lemma 4 that:

**Lemma 5.** With \( G(v) \) and \( g(v) \) as in Assumption 3, the space

\[ D(G) = \{ f(v) = h(G(v))g(v), \ h \in D(0,1) \} \]

of densities on \((0, \infty)\), endowed with the \( L^1 \) metric \( \int_0^\infty |f_1(v) - f_2(v)|dv \), is a compact metric space. Moreover, the corresponding space

\[ F = \left\{ F(v) = \int_0^v f(x)dx, \ f \in D(G) \right\} \]

of absolutely continuous distribution functions on \((0, \infty)\), endowed with one of the sup metrics (14) or (13), is a compact metric space as well.

The space \( F \) is now the ”parameter” space of candidate value distributions \( F \), provided that

**Assumption 4.** The sequence \( \delta_k \) in Lemma 4 is chosen such that the density \( f_0 \) of the true value distribution \( F_0 \) is contained in \( D(G) \).

Of course, this condition is not verifiable in practice, but neither is the standard condition in nonlinear M-estimation\(^{5}\) that the true parameter vector \( \theta_0 \) is contained in a pre-specified compact parameter space \( \Theta \).

\(^{5}\)See for example Bierens (2004, Ch. 6).
4.3 The Sieve Spaces and the Uniform Strong Consistency of the Sieve Estimator

For a density function \( h(u) \) in (26) and its associated parameter sequence \( \{\delta_k\}_{k=1}^{\infty} \), let

\[
h_n(u) = h(u|\delta_n) = \frac{(1 + \sum_{k=1}^{n} \delta_k \rho_k(u))^2}{1 + \sum_{k=1}^{n} \delta_k^2}, \quad \text{where } \delta_n = (\delta_1, \ldots, \delta_n),
\]

be the \( n \)-th order truncation of \( h(u) \). The case \( n = 0 \) corresponds to the uniform density: \( h_0(u) = 1 \). Following Gallant and Nychka (1987) we will call this truncated density a SNP density function. It has been shown by Bierens (2008) that

\[
\lim_{n \to \infty} \int_0^1 |h_n(u) - h(u)| \, du = 0.
\]  

(29)

Thus, defining the space of \( n \)-th order truncations of \( h(u) \) by

\[
D_n(0, 1) = \left\{ h_n(u) = \frac{(1 + \sum_{k=1}^{n} \delta_k \rho_k(u))^2}{1 + \sum_{k=1}^{n} \delta_k^2}, \sup_{k \geq 1} |\delta_k|/\delta_k \leq 1 \right\},
\]

it follows that for each \( h \in D(0, 1) \) there exists a sequence \( h_n \in D_n(0, 1) \) of SNP densities such that (29) holds. This implies that

\[
D(0, 1) = \bigcup_{n=0}^{\infty} D_n(0, 1).
\]

Similarly, defining

\[
H_n(0, 1) = \left\{ H_n(u) = \int_0^u h_n(v) \, dv, \ h_n \in D_n(0, 1) \right\},
\]

it follows that for each distribution function \( H \in H(0, 1) \) there exists a sequence of SNP distribution functions \( H_n \in H_n(0, 1) \) such that

\[
\lim_{n \to \infty} \sup_{0 \leq u \leq 1} |H_n(u) - H(u)| = 0,
\]

hence

\[
H(0, 1) = \bigcup_{n=0}^{\infty} H_n(0, 1).
\]

It is now easy to verify that the following result holds.
Lemma 6. Choose $G$ as in Assumption 3 and denote
\[ \mathcal{F}_n = \{ F_n(v) = H_n(G(v)), \ H_n \in \mathcal{H}_n(0, 1) \} . \] (30)

Under the conditions of Lemma 4, for each distribution function $F \in \mathcal{F}$ and each truncation order $n$ there exists a distribution function $F_n \in \mathcal{F}_n$ such that
\[ \lim_{n \to \infty} \sup_{v > 0} |F(v) - F_n(v)| = 0. \]

Consequently, $\{ \mathcal{F}_n \}_{n=0}^{\infty}$ is dense in $\mathcal{F}$.

The sequence of spaces $\mathcal{F}_n$ now forms the sieve. Because the distribution functions in $\mathcal{F}_n$ are parametric, with parameters $\delta_n = (\delta_1, \ldots, \delta_n)'$, the computation of $\hat{F}_n = \arg \min_{F \in \mathcal{F}_n} \hat{Q}_N(F|\kappa)$ is feasible. In particular, $\hat{F}_n$ can be computed via the simplex method of Nelder and Mead (1965).

Summarizing, it has been shown that,

**Theorem 1.** With $\mathcal{F}$ defined by (27) with sieve spaces $\mathcal{F}_n$ defined by (30), it follows from Assumptions 1-4 that for an arbitrary subsequence $n_N$ of $N$ satisfying $\lim_{N \to \infty} n_N = \infty$ the SNP-ISM sieve estimator
\[ \tilde{F}_{n_N} = \arg \min_{F \in \mathcal{F}_{n_N}} \hat{Q}_N(F|\kappa) \]

is uniformly strongly consistent, i.e., $\sup_{v \geq p_0} |\tilde{F}_{n_N}(v) - F_0(v)| \stackrel{a.s.}{\to} 0$ if the reservation price $p_0$ is binding, and $\sup_{v \geq 0} |\tilde{F}_{n_N}(v) - F_0(v)| \stackrel{a.s.}{\to} 0$ if not.

5 An Integrated Moment Test of the Validity of the First-Price Auction Model

5.1 The Test

If the independent private values paradigm and/or the risk neutrality assumption do not hold, the bid functions (1) and (2) no longer apply to the actual bids. Because the simulated bids are derived from these bid functions, we then have, by Lemma 3, that
\[ \hat{Q}_N(\tilde{F}_{n_N}|\kappa) \stackrel{a.s.}{\to} \inf_{F \in \mathcal{F}} Q(F|\kappa) > 0, \]
where $\hat{F}_{n_N}$ is the sieve estimator. This suggests to use $\hat{Q}_N(\hat{F}_{n_N}|\kappa)$ as a basis for a consistent integrated moment (IM) test of the null hypothesis that

$H_0$: the actual bids come from a first-price sealed bid auction where the values are independent, private and bidders are symmetric and risk-neutral,

against the general alternative that

$H_1$: the null hypothesis $H_0$ is false.

The IM test we propose is based on the fact that similar to the results in Bierens (1990) and Bierens and Ploberger (1997) for the Integrated Conditional Moment (ICM) test, the following result holds.

**Theorem 2.** Under $H_0$ and Assumptions 1-3,

$$\hat{W}_N(t) = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \left( \exp \left( i.t.\hat{B}_j(F_0) \right) - \exp (i.t.B_j) \right) \Rightarrow W(t)$$

on $[-\kappa, \kappa]$, where $W(t)$ is a complex-valued zero-mean Gaussian process on $[-\kappa, \kappa]$ with covariance function

$$\Gamma(t_1, t_2) = E[W(t_1)\overline{W(t_2)}] = E[\hat{W}_N(t_1)\overline{\hat{W}_N(t_2)}].$$

Hence by the continuous mapping theorem,

$$N.\hat{Q}_N(F_0|\kappa) = \frac{1}{2\kappa} \int_{-\kappa}^{\kappa} \left| \hat{W}_N(t) \right|^2 dt \Rightarrow \frac{1}{2\kappa} \int_{-\kappa}^{\kappa} |W(t)|^2 dt.$$

**Proof.** Appendix.

If it were true that

$$\lim_{N \to \infty} P \left[ \hat{Q}_N(\hat{F}_{n_N}|\kappa) \leq \hat{Q}_N(F_0|\kappa) \right] = 1 \quad (31)$$

then for all $c > 0$,

$$\limsup_{N \to \infty} P \left[ N.\hat{Q}_N(\hat{F}_{n_N}|\kappa) > c \right] \leq \lim_{N \to \infty} P \left[ N.\hat{Q}_N(F_0|\kappa) > c \right] \leq \lim_{N \to \infty} P \left[ N.\hat{Q}_N(F_0|\kappa) > c \right]$$

$$= P \left[ \frac{1}{2\kappa} \int_{-\kappa}^{\kappa} |W(t)|^2 dt > c \right] \quad (32)$$
so that upper bounds of the critical values of the test based on $N\hat{Q}_N(\bar{F}_{n_N}|\kappa)$ can be derived from the last probability in (32). However, in general there is no guarantee that (31) holds, except if

**Assumption 5.** The true value distribution $F_0$ is of the SNP type itself: $F_0 \in \cup_{n=1}^{\infty} \mathcal{F}_n$.

Then there exists a smallest natural number $n_0$ such that $F_0 \in \mathcal{F}_{n_0}$, hence $N\hat{Q}_N(\bar{F}_{n_N}|\kappa) \leq N\hat{Q}_N(F_0|\kappa)$ for $n_N \geq n_0$ and thus for $c > 0$,

$$\limsup_{N \to \infty} P \left[ N\hat{Q}_N(\bar{F}_{n_N}|\kappa) > c \right] \leq P \left[ \frac{1}{2\kappa} \int_{-\kappa}^{\kappa} |W(t)|^2 dt > c \right].$$

### 5.2 Bootstrap Critical Values

The problem in approximating the limiting process $W(t)$ by bootstrapping is two-fold, namely that we cannot increase $N \to \infty$ because the $B_j$’s are only observable for $j = 1, \ldots, N$, and $F_0$ is unknown. To overcome these problems, generate for large $M$ simulated bids $\tilde{B}_j$, $j = 1, 2, \ldots, 2M$, from the bid distribution corresponding to the sieve estimator $\hat{F}_{n_N}$ of $F_0$, according to the approach in Assumption 2. Thus, draw $U_j$, $j = 1, 2, \ldots, 2M$, independently from the uniform $[0,1]$ distribution, and generate the corresponding simulated bids $\tilde{B}_j$ similar to (21).

Next, denote

$$\tilde{W}_M \left( t|\tilde{F}_{n_N} \right) = \frac{1}{\sqrt{M}} \sum_{j=1}^{M} \exp \left( i.t.\tilde{B}_j \right) - \frac{1}{\sqrt{M}} \sum_{j=M+1}^{2M} \exp \left( i.t.\tilde{B}_j \right).$$

Then

**Theorem 3.** Under the conditions of Theorem 1,

$$\tilde{W}_M \left( t|\tilde{F}_{n_N} \right) \Rightarrow W \left( t|\tilde{F}_{n_N} \right) \text{ on } [-\kappa, \kappa], \text{ conditional on } \tilde{F}_{n_N}$$

(33)

for fixed $N$ and $M \to \infty$, where $W \left( t|\tilde{F}_{n_N} \right)$ is a complex-valued zero-mean Gaussian process with conditional covariance function

$$\tilde{\Gamma}(t_1, t_2|\tilde{F}_{n_N}) = E \left[ \tilde{W}_M \left( t_1|\tilde{F}_{n_N} \right) \tilde{W}_M \left( t_2|\tilde{F}_{n_N} \right) \tilde{F}_{n_N} \right]$$
Moreover, under $H_0$,
\[
\sup_{(t_1, t_2) \in [-\kappa, \kappa] \times [-\kappa, \kappa]} \left| \widetilde{\Gamma}(t_1, t_2 | \widetilde{F}_n) - \Gamma(t_1, t_2) \right| \xrightarrow{a.s.} 0
\]  
(34)
as $N \to \infty$, and consequently
\[
\frac{1}{2\kappa} \int_{-\kappa}^{\kappa} \left| \widetilde{W}(t | \widetilde{F}_n) \right|^2 \, dt \xrightarrow{d} \frac{1}{2\kappa} \int_{-\kappa}^{\kappa} |W(t)|^2 \, dt.
\]  
(35)
Hence, for $M \to \infty$ first, and then $N \to \infty$,
\[
\frac{1}{2\kappa} \int_{-\kappa}^{\kappa} \left| \widetilde{W}_M(t | \widetilde{F}_n) \right|^2 \, dt \xrightarrow{d} \frac{1}{2\kappa} \int_{-\kappa}^{\kappa} |W(t)|^2 \, dt.
\]  
(36)

Proof. Appendix.

Therefore, bootstrap critical values of $\frac{1}{2\kappa} \int_{-\kappa}^{\kappa} |W(t)|^2 \, dt$ can be computed as follows. First, choose a large $M$, say $M = 1000$. Next, generate $\widetilde{T}_k = \frac{1}{2\kappa} \int_{-\kappa}^{\kappa} \left| \widetilde{W}_M(t | \widetilde{F}_n) \right|^2 \, dt$ independently for $k = 1, ..., K$, say $K = 500$, and sort the statistics $\widetilde{T}_k$ in increasing order. The $\alpha \times 100\%$ bootstrap critical value is then $\widetilde{T}_{(1-\alpha)K}$.

6 Determination of the Sieve Order Via an Information Criterion

Recall that under Assumption 5 there exists a smallest natural number $n_0$ such that $F_0 \in \mathcal{F}_{n_0}$. The question now arises how to estimate $n_0$ consistently.

For nested likelihood models this can be done via information criteria, for example the Hannan-Quinn (1979) or Schwarz (1978) information criteria. These information criteria are of the form
\[
C_N(n) = -\frac{2}{N} \ln (L_N(n)) + n \cdot \frac{\phi(N)}{N}
\]
where $L_N(n)$ is the maximum likelihood of a model with $n$ parameters, with $\phi(N) = \ln (N)$ for the Schwarz criterion and $\phi(N) = 2 \ln (\ln (N))$ for the Hannan-Quinn criterion. Then for $2 \leq n \leq n_0$,
\[
p \lim_{N \to \infty} (C_N(n) - C_N(n-1)) = p \lim_{N \to \infty} \frac{2}{N} \ln (L_N(n-1)) - p \lim_{N \to \infty} \frac{2}{N} \ln (L_N(n)) < 0
\]
whereas for \( n > n_0 \), 
\[-2 \ln (L_N(n_0)) - \ln (L_N(n)) \xrightarrow{d} \chi^2_{n-n_0}, \]
hence
\[
p \lim_{N \to \infty} \frac{N}{\phi(N)} (C_N(n) - C_N(n_0)) = n - n_0.
\]
Note that the latter result only hinges on
\[-2 \ln (L_N(n_0)) - \ln (L_N(n_0)) = O_p(1).
\]
Because by Theorem 2,
\[
N \left( \inf_{F \in \mathcal{F}_n} \hat{Q}_N(F | \kappa) - \inf_{F \in \mathcal{F}_{n_0}} \hat{Q}_N(F | \kappa) \right) = O_p(1) \text{ if } n > n_0,
\]
whereas for \( 2 \leq n \leq n_0 \),
\[
p \lim_{N \to \infty} \left( \inf_{F \in \mathcal{F}_n} \hat{Q}_N(F | \kappa) - \inf_{F \in \mathcal{F}_{n-1}} \hat{Q}_N(F | \kappa) \right) < 0
\]
it seems that in our case we may replace \(-2 \ln (L_N(n))\) by \( \inf_{F \in \mathcal{F}_n} \hat{Q}_N(F | \kappa)\):
\[
C_N(n) = \inf_{F \in \mathcal{F}_n(G)} \hat{Q}_N(F | \kappa) + n \frac{\phi(N)}{N},
\]
and estimate \( n_0 \) by \( \hat{n}_N = \arg \min C_N(n) \). Asymptotically that will work: \( \lim_{N \to \infty} P[\hat{n}_N = n_0] = 1 \). However, in practice it will not, due to the fact that \( \hat{Q}_N(F | \kappa) \) is bounded: \( \sup_F \hat{Q}_N(F | \kappa) \leq 4 \), and that \( \inf_{F \in \mathcal{F}_n} \hat{Q}_N(F | \kappa) \) will be close to zero if \( n < n_0 \) but not too far away from \( n_0 \), so that in small samples the penalty term \( n \phi(N)/N \) may dominate \( \inf_{F \in \mathcal{F}_n} \hat{Q}_N(F | \kappa) \) too much. Therefore, we propose the following modification of \( C_N(n) \):
\[
\tilde{C}_N(n) = \inf_{F \in \mathcal{F}_n} \hat{Q}_N(F | \kappa) + \Phi(n) \frac{\phi(N)}{N}, \tag{37}
\]
where \( \Phi(n) \) is an increasing but bounded function of \( n \). For example, let for some \( \alpha \in (0, 1) \),
\[
\Phi(n) = 1 - (n + 1)^{-\alpha}.
\]
Then similar to the Hannan-Quinn and Schwarz information criteria we have:
Theorem 4. Let $\tilde{n}_N = \max_{s.t.} \tilde{c}_N(n) \leq c_N(n-1) \ n$ and $\tilde{F}_n = \arg\min_{F \in \mathcal{F}_n} \hat{Q}_N(F|\kappa)$. Under Assumption 5, $\lim_{N \to \infty} P[\tilde{n}_N = n_0] = 1$, hence $||\tilde{F}_n - F_0|| \overset{a.s.}{\to} 0$. If Assumption 5 is not true then $\lim_{N \to \infty} \tilde{n}_N = \infty$, hence $p\lim_{N \to \infty} ||\tilde{F}_n - F_0|| = 0$.

Proof. Appendix.

7 Some Numerical Experiments

In this section we check the performance of the IM test of the validity of the first-price auction model, and the fit of SNP-ISM density estimator with estimated the truncation order $\tilde{n}_N$, via a few numerical experiments. In all experiments we use the exponential distribution

$$G(v) = 1 - \exp(-v/3), \ g(v) = \frac{1}{3} \exp(-v/3)$$

as the initial guess for the value distribution, and the truncation order $\tilde{n}_N$ is determined via the approach in Theorem 4, with information criterion

$$\tilde{C}_N(n) = \inf_{F \in \mathcal{F}_n} \hat{Q}_N(F|\kappa) + (1 - (n + 1)^{-1/3}) \cdot \frac{\ln(\ln(N))}{N}.$$  

Moreover, we have chosen $\kappa = 1$ in (11) and (12). The 5% and 10% bootstrap critical values of the IM test will be based on $K = 500$ bootstrap samples.

7.1 The IM Test

In this subsection we check the performance of the IM test by a few numerical examples. In all cases we have generated bids from 500 identical and independent auctions, each with two sealed bids and no reservation price. In the first case the observed bids come from a second price auction with independent private values and value distribution $F_0(v)$ to be specified below. As is well known, in a second price auction it is a weakly dominant strategy to bid the true value. See Krishna (2002). Therefore, in this case the null hypothesis that the observed bids can be rationalized by a first-price auction model is false. The second case is a first-price auction with two risk-averse bidders, utility function $U(x) = \sqrt{x}$ and the same value distribution $F_0(v)$ as
in the first case. As pointed out by Krishna (2002), this auction is observationally equivalent to a first-price auction with two risk-neutral bidders and value distribution $F_0(v)^2$, so that in this case the null hypothesis is true.

In the first instance we have chosen for $F_0(v)$ the standard exponential distribution

$$f_0(v) = 1 - \exp(-v), \quad h_0(v) = \exp(-v),$$

with initial guess (38). Recall that $f_0(v) = h_0(G(v))g(v)$, where $G$ and $g$ are specified in (38), and therefore

$$h_0(u) = \frac{f_0(3 \ln(1/(1 - u)))}{g(3 \ln(1/(1 - u)))} = 3(1 - u)^2 = \frac{(1 + \delta_1 \rho_1(u))^2}{1 + \delta_1^2} = h_1(u|\delta_1), \quad (40)$$

where $\rho_1(u) = \sqrt{3}(2u - 1)$ is the first-order Legendre polynomial and $\delta_1 = -1/\sqrt{3}$. Consequently, $F_0 \in F_1$, so that Assumption 5 is satisfied. In case 2 the actual value distribution is $F_\ast(v) = F_0(v)^2 = (1 - \exp(-v))^2$ with density $f_\ast(v) = 2f_0(v)F_0(v) = h_\ast(G(v))g(v)$. It is easy to verify that $h_\ast(u) = 6(1 - u)^2 - 6(1 - u)^5$, so that similar to (40), $F_\ast \in F_3$. Thus, $F_\ast$ satisfies Assumption 5 as well.

In case 1 the IM test statistic is $\tilde{T} = 3.0531$, with estimated truncation order $\tilde{n}_N = 4$ and 5% and 10% bootstrap critical values $\tilde{T}_{0.95K} = 1.1882$ and $\tilde{T}_{0.90K} = 0.9447$, respectively. Thus, as expected, the false null hypothesis is firmly rejected at the 5% significance level. In case 2 the IM test statistic takes the value $\tilde{T} = 0.0004$, with estimated truncation order $\tilde{n}_N = 2$ (note that the true order is 3) and 5% and 10% bootstrap critical values $\tilde{T}_{0.95K} = 0.3630$ and $\tilde{T}_{0.90K} = 0.2661$, respectively. Thus, the true null hypothesis is not rejected at the 10% significance level.

To check how essential Assumption 5 is for the IM test, we have also conducted the IM tests for the two cases involved with $\chi_3^2$ distributed true values, so that Assumption 5 no longer holds. In the second price auction case the IM test statistic takes the value $\tilde{T} = 12.8454$, with estimated truncation order $\tilde{n}_N = 1$ and 5% and 10% bootstrap critical values $\tilde{T}_{0.95K} = 3.3847$ and $\tilde{T}_{0.90K} = 2.4939$, respectively. Thus, again, the false null hypothesis is firmly rejected at the 5% significance level. In the other case the IM test statistic is $\tilde{T} = 1.2417$, with estimated truncation order $\tilde{n}_N = 1$ and 5% and 10% bootstrap critical values $\tilde{T}_{0.95K} = 1.0671$ and $\tilde{T}_{0.90K} = 2.4939$, respectively. Now the true null hypothesis is not rejected at the 5% significance level but it is rejected the 10% level. The latter indicates that Assumption 5 matters for the size of the IM test.
7.2 The Fit

In the previous experiments the estimated truncation orders are small, so the question arises whether for such a small truncation order the value density can be adequately approximated. In this section we check this for three cases. In each case we generate independently 200 auctions without a reservation price, where each auction consists of 5 bids whose private values come from a chi-square distribution, so in each case we have a sample of 1000 i.i.d. bids. The three cases only differ with respect to the degrees of freedom $r$ of the chi-square distribution, namely $r = 3, 4, 5$, respectively.

In these cases the true value densities $f_0(v)$ are quite different from the density $g(v)$ of the initial guess (38), in particular the left tails, as shown in Figure 1. Thus, the SNP density $h_n(u)$ needs to convert the exponential density $g(v)$ into an approximation $f_n(v) = h_n(G(v))g(v)$ of a $\chi^2_r$ density, so that $h_n(u)$ needs to bend down the left tail of $g(v)$ towards zero. This seems challenging. However, it appears that the SNP density $h_n(u)$ has no problem doing that, even for small values of $n$.

In Figures 2-4 we compare the SNP sieve density estimators $f_n(v) = h_n(G(v))g(v)$ with the true $\chi^2_r$ value densities $f_0(v)$ for $r = 3, 4, 5$, and estimated truncation orders $\tilde{n}_N = 4, 2, 4$ respectively. These figures show that our SNP-ISM estimation approach works remarkably well, certainly in view of the bad choice of the initial guess $g(v)$ for $f_0(v)$ (see Figure 1) and the small truncation orders. On the other hand, it seems from Figure 3 that the truncation order $\tilde{n}_N = 2$ is too small, as the fit of $f_{\tilde{n}_N}(v)$ for $\tilde{n}_N = 4$ in Figures 2 and 4 looks better than in Figure 3.

7.3 Comparison with the Nonparametric Approach

Next we compare our SNP-ISM estimation approach with the nonparametric approach of Guerre et al. (2000), hereafter called the GPV case. The data-generating process is the same as in Guerre et al. (2000). In particular, the value distribution is now truncated log-normal, with support $[0.055, 2.5]$. Again, we consider 200 replications of first-price auctions with five bidders and no binding reservation price, resulting in a random sample.
of $N = 1000$ i.i.d. bids. The two-step nonparametric kernel estimation with trimming of the observed bids in the first step has been conducted in exactly the same way as in Guerre et al. (2000). Also the SNP-ISM estimation has been conducted in the same way as before, with the exponential distribution (38) as initial guess of the value distribution, and information criterion (39) for the estimation of the truncation order. The result of the latter is $\tilde{n}_N = 3$.

Of course, in both the GPV and SNP-ISM cases the support of the value distribution is treated as unknown. The common assumption in both cases is that the bid distribution has bounded support, which in the SNP-ISM case is due to Assumption 1 and in the GPV case is due to the assumption that the value distribution has bounded support.

As we see from Figure 5, neither approach does a good job in estimating the truncated log-normal density. Despite (or perhaps due to) the trimming as in Guerre et al. (2000), their kernel estimator does not capture the tails of the true value density well, in particular the right tail, although the mode is close. However, the SNP-ISM estimation result is even worse.

A possible reason for the latter is the following. Recall that the density of the truncated log-normal distribution involved takes the form

$$f_0(v) = \frac{\exp(-\ln(v)^2/2)}{v (\Phi(\ln(v)) - \Phi(\ln(\underline{v}))) \sqrt{2\pi}} 1(v \in [\underline{v}, \overline{v}]),$$

where $\Phi(x)$ is the c.d.f. of the standard normal distribution. Moreover, recall that we can write $f_0(v)$ as $f_0(v) = h_0(G(v))g(v)$, where $G$ and $g$ are given in (38) and $h_0$ is a density function on $[0, 1]$. It is easy to verify that in this case $h_0(u)$ takes the form

$$h_0(u) = \frac{f_0(3 \ln(1/(1 - u)))}{g(3 \ln(1/(1 - u)))} = \frac{\exp(-\ln(3 \ln(1/(1 - u))))^2/2)}{(1 - u) \ln(1/(1 - u)) (\Phi(\ln(\overline{v})) - \Phi(\ln(\underline{v}))) \sqrt{2\pi}} 1(u \leq u \leq \overline{u}),$$

where

$$\underline{u} = 1 - \exp(-\underline{v}/3) \approx 0.0182$$
$$\overline{u} = 1 - \exp(-\overline{v}/3) \approx 0.5654$$
Now for SNP densities of the form (28) the SNP-ISM approach minimizes indirectly the metric
\[ Z_{10} |h(u|\delta_n) - h_0(u)| \, du = Z_{10} (h(u|\delta_n) - h_0(u)) \, du + Z_{10} h(u|\delta_n) \, du + Z_{10} h_0(u) \, du \]
(41)
to \( \delta_n = (\delta_1, ..., \delta_n)' \). In view of the values of \( \overline{\nu} \) (close to 0.5654) and \( n = 3 \) the term \( \int_{\overline{\nu}}^{1} h(u|\delta_n) \, du \) in (41) is quite dominant, which pushes \( h(u|\delta_n) \) downwards for \( u \) close to \( \overline{\nu} \), whereby in the process \( h(u|\delta_n) \) is tilted to the right. The main culprit seems to be the information criterion (39), which is too conservative: we need a much larger truncation order \( n \) to get a satisfactory approximation of \( h_0(u) \) if its support is smaller than \((0, 1)\).

For the same reason we may expect that, with initial guess (38) and information criterion (39), the SNP-ISM approach will perform poorly if the reservation price is binding, although we have not verified this.

A possible solution to this problem is to choose for \( G \) a distribution function with bounded but unknown support \([\underline{\nu}, \overline{\nu}]\) and treat the lower and upper bounds \( \underline{\nu} \) and \( \overline{\nu} \) as unknown parameters. For example, in the non-binding reservation price case we may specify the value distribution as
\[ F_0(v) = H_0(G(v|\underline{\nu}, \overline{\nu})) \]
where \( G(v|\underline{\nu}, \overline{\nu}) \) is the uniform distribution function on \([\underline{\nu}, \overline{\nu}]\) and \( H_0 \) is a distribution function on \([0, 1]\). The latter can be estimated semi-nonparametrically as before, together with the parameters \( \underline{\nu} \) and \( \overline{\nu} \). In the case of a binding and known reservation price \( p_0 \) we may specify \( F_0(v) \) as
\[ F_0(v) = q_0 + (1 - q_0) H_0(G(v|p_0, \overline{\nu})) \]
where now \( G(v|p_0, \overline{\nu}) \) is the uniform distribution function on \([p_0, \overline{\nu}]\), and \( q_0 = F_0(p_0) \). Again, \( H_0 \) can be estimated semi-nonparametrically via the SNP-ISM approach, together with the parameters \( \overline{\nu} \) and \( q_0 \).

8 Concluding Remarks

In this paper we have made two major contributions to the literature on empirical auctions. First, we have proposed a new semi-nonparametric sieve
estimation method for the value distribution of a first-price auction, based on a comparison of the empirical characteristic functions of the actual bids and simulated bids. Our SNP-ISM approach differs fundamentally from the nonparametric estimation approach of Guerre et al. (2000) in that we estimate the value distribution directly, whereas in the nonparametric auction literature, the value distribution is estimated indirectly via kernel estimation of the inverse bid function. Second, our approach yields as by-product an integrated moment test for the validity of the first-price auction model.

The few numerical experiments we have conducted indicate that the SNP-ISM approach works surprisingly well if the support of the value distribution is the whole positive real line \((0, \infty)\) and the reservation price is non-binding, but our approach performs poorly if the support of the value distribution is bounded. The main reason for the latter seems to be the data driven selection of the sieve truncation order via an information criterion, which is too conservative for value distributions with bounded support. How to fix that will be left for future research.

The approach in this paper has been confined to the case of independently and identically repeated first-price auctions, but can be extended to auctions with auction-specific heterogeneity. As to the latter, we have already made a start with this. See Bierens and Song (2011b).

9 Appendix

9.1 Proof of Lemma 3

We only consider the binding reservation price case. First, we show that

\[
\lim_{n \to \infty} \tilde{B}(F_n) = 0 \text{ if and only if } \tilde{B}(F) = 0
\]

as follows. Let \(\tilde{B}(F) = 0\), which is equivalent to \(F(p_0) < U\), and suppose that \(\limsup_{n \to \infty} \tilde{B}(F_n) \geq p_0\). The latter implies that there exists a subsequence \(n_m\) such that \(\tilde{B}(F_{n_m}) \geq p_0\) for \(m = 1, 2, 3, \ldots\), which implies that \(U \leq F_{n_m}(p_0)\) for \(m = 1, 2, 3, \ldots\). However, this is not possible because then by (13), \(\|F_{n_m} - F\| > U - F(p_0)\) for \(m = 1, 2, 3, \ldots\), whereas (22) implies that \(\lim_{m \to \infty} \|F_{n_m} - F\| = 0\). Thus, \(\tilde{B}(F) = 0\) implies \(\lim_{n \to \infty} \tilde{B}(F_n) = 0\). Similarly, \(\lim_{n \to \infty} \tilde{B}(F_n) = 0\) implies \(\tilde{B}(F) = 0\).

Next we show that (23) is true, by contradiction. Suppose that \(\limsup_{n \to \infty} \tilde{V}(F_n) > \tilde{V}(F)\), and note that \(\tilde{V}(F)\) is a.s. a point in the support of \(F\).
Then there exists a subsequence \( n_m \) and an \( \varepsilon > 0 \) such that for all \( m, \)
\[
\tilde{V}(F_{n_m}) > \tilde{V}(F) + \varepsilon.
\]
But then \( U = F(\tilde{V}(F)) < F(\tilde{V}(F) + \varepsilon) \) a.s. and
\[
U = F_{n_m}(\tilde{V}(F_{n_m})) \geq F_{n_m}(\tilde{V}(F) + \varepsilon) \overset{a.s.}{\to} F(\tilde{V}(F) + \varepsilon)
\]
so that
\[
U = F(\tilde{V}(F)) < F(\tilde{V}(F) + \varepsilon) \leq U \text{ a.s.,}
\]
which is impossible. Thus, \( \limsup_{n \to \infty} \tilde{V}(F_n) \leq \tilde{V}(F) \). Similarly, it follows that \( \liminf_{n \to \infty} \tilde{V}(F_n) \geq \tilde{V}(F) \). Thus (23) is true.

Finally, it follows straightforwardly from (22) and (23) that
\[
\lim_{n \to \infty} \int_{0}^{1} F_n \left( p_0 + u(\tilde{V}(F_n) - p_0) \right)^{I_0-1} du
\]
\[
= \int_{0}^{1} F \left( p_0 + u(\tilde{V}(F) - p_0) \right)^{I_0-1} du.
\]
(43)

The result (24) now follows from (21), (23), (42) and (43).

9.2 Proof of Theorem 2

It suffices to show that \( \hat{W}_N(.) \) is tight on \([-\kappa, \kappa]\), which can be done by verifying that the following two conditions hold (see Billingsley 1968, Theorem 8.2):

(i) For each \( \delta > 0 \) and an arbitrary \( t_0 \in [-\kappa, \kappa] \), there exists an \( \varepsilon \) such that
\[
\sup_N P \left( \hat{W}_N(t_0) > \varepsilon \right) \leq \delta.
\]

(ii) For each \( \delta > 0 \) and \( \varepsilon > 0 \), there exists an \( \xi > 0 \) such that for \( t_1, t_2 \in [-\kappa, \kappa] \),
\[
\sup_N P \left( \sup_{|t_1 - t_2| < \xi} |\hat{W}_N(t_1) - \hat{W}_N(t_2)| \geq \varepsilon \right) \leq \delta.
\]

Condition (i) follows from the fact that for arbitrary \( t \in [-\kappa, \kappa] \),
\[
(\text{Re} \hat{W}_N(t), \text{Im} \hat{W}_N(t))'
\]
converges in distribution to a bivariate normal distribution. Condition (ii) follows from Chebyshev’s inequality for first moments:

\[
P \left( \sup_{|t_1 - t_2| < \xi} |\hat{W}_N(t_1) - \hat{W}_N(t_2)| \geq \varepsilon \right) \leq \varepsilon^{-1} E \left[ \sup_{|t_1 - t_2| < \xi} |\hat{W}_N(t_1) - \hat{W}_N(t_2)| \right]
\]

and the fact that, with \( \hat{B}_j = \tilde{B}_j (F_0) \),

\[
E \left[ \sup_{|t_1 - t_2| < \xi} |\hat{W}_N(t_1) - \hat{W}_N(t_2)| \right]
\]

\[
\leq E \left[ \sup_{|t_1 - t_2| < \xi} |\exp(i.t_1 B_1) - \exp(i.t_2 B_1)| \right]
+ E \left[ \sup_{|t_1 - t_2| < \xi} |\exp(i.t_1 \hat{B}_1) - \exp(i.t_2 \hat{B}_1)| \right]
= 2.E \left[ \sup_{|t_1 - t_2| < \xi} |\exp(i.t_1 B_1) - \exp(i.t_2 B_1)| \right] \leq 2.\xi \overline{b}
\]

where \( \overline{b} \) is the upper bound of the support of \( B_1 \). Thus, condition (ii) holds for \( \xi = \delta \varepsilon / (2.\overline{b}) \).

### 9.3 Proof of Theorem 3

Part (33) of Theorem 3 follows similarly to Theorem 2. To prove part (34), denote for \( u \in (0, 1) \)

\[
\eta(u|F) = \left( F^{-1}(u) - \frac{1}{u^{I_0-1}} \int_{p_0}^{F^{-1}(u)} F(v)^{I_0-1} dv \right) 1(u > F(p_0))
\]

\[
= \frac{p_0 F(p_0)^{I_0-1} + (I_0 - 1) \int_{p_0}^{u} z^{I_0-2} F^{-1}(z) dz}{u^{I_0-1}} 1(u > F(p_0)),
\]

where the latter equality follows from integration by parts. Observe that

\[
\tilde{\Gamma}(t_1, t_2|\tilde{F}_{nN})
\]

\[
= \frac{1}{M} \sum_{j=1}^{M} E \left[ \left( \exp(i.t_1 \eta(\tilde{U}_j|\tilde{F}_{nN})) - \exp(i.t_1 \eta(\tilde{U}_{j+M}|\tilde{F}_{nN})) \right) \right]
\]

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\[
\times \left( \exp (-i.t_2.\eta \left( \tilde{U}_j | \tilde{F}_{nN} \right)) - \exp (-i.t_2.\eta \left( \tilde{U}_{j+M} | \tilde{F}_{nN} \right)) \right) \left| \tilde{F}_{nN} \right]
\]
\[
= 2 \int_0^1 \exp \left( i \left( t_1 - t_2 \right) \eta \left( u | \tilde{F}_{nN} \right) \right) du
\]
\[
-2 \int_0^1 \exp \left( it_1.\eta \left( u | \tilde{F}_{nN} \right) \right) du \int_0^1 \exp \left( -it_2.\eta \left( u | \tilde{F}_{nN} \right) \right) du
\]
\[
= 2 \int_0^1 \cos \left( \left( t_1 - t_2 \right) \eta \left( u | \tilde{F}_{nN} \right) \right) du
\]
\[
+ 2i \int_0^1 \sin \left( \left( t_1 - t_2 \right) \eta \left( u | \tilde{F}_{nN} \right) \right) du
\]
\[
-2 \left( \int_0^1 \cos \left( t_1.\eta \left( u | \tilde{F}_{nN} \right) \right) du + i \int_0^1 \sin \left( t_1.\eta \left( u | \tilde{F}_{nN} \right) \right) du \right)
\]
\[
\times \left( \int_0^1 \cos \left( t_2.\eta \left( u | \tilde{F}_{nN} \right) \right) du - i \int_0^1 \sin \left( t_2.\eta \left( u | \tilde{F}_{nN} \right) \right) du \right).
\]

The result (34) now follows from the bounded convergence theorem and fact that similar to Lemma 3, \( \tilde{F}_{nN}^{-1}(u) \to F_0^{-1}(u) \) a.s. pointwise in \( u \in [0,1] \), hence \( \eta \left( u | \tilde{F}_{nN} \right) \to \eta \left( u | F_0 \right) \) a.s. pointwise in \( u \in [0,1] \). Moreover, the results (35) and (36) follow from the continuous mapping theorem and the fact that zero-mean Gaussian processes are completely determined by their covariance functions.

### 9.4 Proof of Theorem 4

The event \( \tilde{n}_N = n_0 \) is equivalent to

\[
\max_{1 \leq n \leq n_0} \left( \tilde{C}_N(n) - \tilde{C}_N(n - 1) \right) \leq 0 \text{ and } \tilde{C}_N(n_0 + 1) - \tilde{C}_N(n_0) > 0.
\]

so that

\[
P \left[ \tilde{n}_N \neq n_0 \right] \leq P \left[ \max_{1 \leq n \leq n_0} \left( \tilde{C}_N(n) - \tilde{C}_N(n - 1) \right) > 0 \right]
\]
\[
+ P \left[ \tilde{C}_N(n_0 + 1) - \tilde{C}_N(n_0) \leq 0 \right]
\]

(44)

For fixed \( n \leq n_0 \),

\[
\tilde{C}_N(n) - \tilde{C}_N(n - 1) \to \inf_{F \in \mathcal{F}_n(G)} Q(F|\kappa) - \inf_{F \in \mathcal{F}_{n-1}(G)} Q(F|\kappa) \leq 0 \text{ a.s.}
\]
hence
\[
\lim_{N \to \infty} P \left[ \max_{1 \leq n \leq n_0} \left( \tilde{C}_N(n) - \tilde{C}_N(n - 1) \right) > 0 \right] = 0. \tag{45}
\]

For \( n = n_0 + 1 \),
\[
\left| N \left( \tilde{C}_N(n_0 + 1) - \tilde{C}_N(n_0) \right) - \phi(N) (\Phi(n_0 + 1) - \Phi(n_0)) \right|
\leq N \left( \inf_{F \in \mathcal{F}_{n_0}} \hat{Q}_N(F|\kappa) + \inf_{F \in \mathcal{F}_{n_0+1}} \hat{Q}_N(F|\kappa) \right)
\leq 2. N. \hat{Q}_N(F_0|\kappa)
\]

so that with probability 1,
\[
\frac{N}{\phi(N)} \left( \tilde{C}_N(n_0 + 1) - \tilde{C}_N(n_0) \right) \geq \Phi(n_0 + 1) - \Phi(n_0) - 2. N. \hat{Q}_N(F_0|\kappa)
\]

Therefore,
\[
\lim_{N \to \infty} P \left[ \tilde{C}_N(n_0 + 1) - \tilde{C}_N(n_0) \leq 0 \right]
\leq \lim_{N \to \infty} P \left[ \frac{N. \hat{Q}_N(F_0|\kappa)}{\phi(N)} \geq \frac{1}{2} (\Phi(n_0 + 1) - \Phi(n_0)) \right] = 0 \tag{46}
\]

because \( N. \hat{Q}_N(F_0|\kappa)/\phi(N) = O_p \left( 1/\phi(N) \right) = o_p(1) \). It follows now from (44), (45) and (46) that \( \lim_{N \to \infty} P[\bar{n}_N = n_0] = 1 \).

In the case \( n_0 = \infty \) it follows from (45) that for any \( \bar{n} \geq 1 \),
\[
\lim_{N \to \infty} P[\bar{n}_N \geq \bar{n}] = 1,
\]

which implies that \( p \lim_{N \to \infty} \bar{n}_N = \infty \). Because for each \( n \) we can choose an \( F_n \in \mathcal{F}_n \) such that \( \lim_{n \to \infty} ||F_n - F_0|| = 0 \), it follows that for this sequence \( F_n \), \( p \lim_{N \to \infty} ||F_{\bar{n}_N} - F_0|| = 0 \). Hence
\[
p \lim_{N \to \infty} \overline{Q} (F_{\bar{n}_N}|\kappa) = \overline{Q} (F_0|\kappa)
\]

The result in Theorem 4 for the case \( n_0 = \infty \) now follows from the proof of Theorem 2, adapted to the "plim" case.
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References


Figures

Figure 1: $g(v) = \exp(-v/3)/3$ compared with the $\chi^2_r$ densities for $r = 3, 4, 5$

Figure 2: $f_{\bar{n}N}(v)$ (dashed curve) compared with the true $\chi^2_3$ density $f_0(v)$
Figure 3: $f_{\bar{N}}(v)$ (dashed curve) compared with the true $\chi^2_4$ density $f_0(v)$

Figure 4: $f_{\bar{N}}(v)$ (dashed curve) compared with the true $\chi^2_5$ density $f_0(v)$
Figure 5: Comparison of the GPV and SNP-ISM approaches