

Some Monotonicity Theorems

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Introduction

A *partition* of n is a nondecreasing finite sequence of positive integers whose sum is n .

Let P be a set of positive integers.

We let $p(n)$ denote the number of partitions of n with parts in P .

A set P is called *monotone* if the sequence $p(n)$ is weakly increasing.

By a well-known theorem the generating function for integer partitions with parts in P is

$$F_P(q) = \sum_{n=0}^{\infty} p(n)q^n = \prod_{p \in P} \frac{1}{1 - q^p}$$

Thus P is monotone if all coefficients of $(1 - q)F_P(q)$ (other than $-q$) are non-negative.

Question: *Bateman, Erdős (1956)*

Which P are monotone?

It is clear that if $1 \in P$ then P is monotone.

So let us assume that n is the minimal element with $n > 1$

Since n is minimal, any monotone P must contain $n + 1$ as well. (Otherwise $p(n + 1) = 0$)

In a similar way we see that any monotone P must contain $S_n := \{n, \dots, 2n - 1\}$

We use NN to denote non-negative terms and $SP(q^a)$ for strictly positive terms past q^a .

We also let

$$f_{n,m}(q) = \frac{1 - q}{\prod_{i=n}^m (1 - q^i)}$$

A first little theorem

Theorem 1.

$$f_{n,\infty} = 1 - q + NN + SP(q^{3n+1})$$

Proof.

$$f_{n,\infty}(q) = (1 - q) \frac{1}{(q^n)_\infty}$$

where $(a)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1})$

By the q -binomial theorem we get

$$\begin{aligned} f_{n,\infty}(q) &= (1 - q) \sum_{k=0}^{\infty} \frac{q^{nk}}{(q)_k} \\ &= 1 - q + q^n + \frac{q^{2n}}{1 - q^2} + \sum_{k=3}^{\infty} \frac{q^{nk}}{(q^2)_{k-1}} \end{aligned}$$

□

The Conjecture

Conjecture 1 (Friedman-Joichi-Stanton). *For an odd positive integer $n > 1$, $f_{n,2n-1}(q) = 1 - q + NN$
If also $n \geq 7$, then $f_{n,2n-1}(q) = 1 - q + NN + SP(q^{3n+4})$*

Corollary 1. *If n is even, then $f_{n,2n+1}(q) = 1 - q + NN + SP(q^{3n+7})$*

Proof. Define $h_{n,m}(q) = f_{n,m} - 1 + q$

Then $h_{n,2n+1}(q) = \frac{1}{1-q^n}(h_{n+1,2n+1}(q) + q^n - q^{n+1})$

If n is even then $h_{n+1,2n+1}(q)$ contains q^{n+1}

Hence $h_{n,2n+1}(q) = NN + SP(q^{3n+7})$ for $n \geq 6$.

Show the cases $n = 2$ and $n = 4$ separately. □

Lemma to increase monotone sets

Lemma 1. *Suppose $H(q) = 1 - q + NN + SP(q^a)$. Let S be a set of positive integers. Now define $H(S, q) = H(q) / \prod_{s \in S} (1 - q^s)$*

If $\min\{s : s \in S\} \geq a$, then $H(S, q) = 1 - q + NN$.

Proof. Define $g(S, q) = H(S, q) - 1 + q$.

We have to show that $g(S, q) = NN$.

Consider the recurrence for $b \notin S$:

$$g(S \cup \{b\}, q) = \frac{1}{1 - q^b} (g(S, q) + q^b - q^{b+1})$$

For finite sets S use induction: Start with the smallest element in S

If $S = \emptyset$, then $g(\emptyset, q)$ is positive past q^a . Since $b \geq a$ it is positive at q^{b+1} .

Thus by the recurrence $g(\{b\}, q) = NN + SP(q^{b+1})$

Repeat this process for every element in S .

If S is infinite check the coefficient of q^j by taking the finite set $S \cap \{x : x \leq j\}$

□

Extending the set $S_n = \{n, \dots, 2n - 1\}$

By the previous lemma we immediately get:

Proposition 1. *If $n \geq 7$ and Q is a subset of $\{3n + 4, 3n + 5, \dots\}$ then $P = S_n \cup Q$ is monotone*

Thus the only other numbers we have to consider are numbers between $2n$ and $3n + 3$.

We consider two cases:

- Numbers between $2n$ and $3n - 2$
- Numbers between $3n - 1$ and $3n + 3$

For the first case we get the following:

Proposition 2. *If $n \geq 7$ and E is a set of even numbers, O a set of odd numbers, both in $\{2n, \dots, 3n - 2\}$, with the condition that $E^+ = \{e + 1 : e \in E\} \subset O$, then $P = \{n, \dots, 2n - 1\} \cup E \cup O$ is a monotone.*

Proof of the Proposition

For any subset $S \subset \{2n, 2n + 1, \dots\}$ define

$$g(n, S) = f_{n, 2n-1}(q) / \prod_{s \in S} (1 - q^s) - 1 + q$$

We can calculate the coefficients of $f_{n, 2n-1}(q)$ up to q^{3n+4} :

$$f_{n, 2n-1}(q) = 1 - q + q^n + \sum_{i=0}^{(n-3)/2} q^{2n+2+2i} + q^{3n+3} + \dots$$

Using the conjecture this implies:

$$g(n, \emptyset) = q^n + \sum_{i=0}^{(n-3)/2} q^{2n+2+2i} + q^{3n+3} + SP(q^{3n+4})$$

Again we will extend S by induction, and use the recurrence:

$$g(S \cup \{b\}, q) = \frac{1}{1 - q^b} (g(S, q) + q^b - q^{b+1})$$

Proof of the Proposition II

Since there is the term $-q^{b+1}$, we see that if b is even, we also need $b + 1$.

Thus we need the following recurrence:

$$g(n, S \cup \{b, b + 1\}) = \frac{q^b}{1 - q^b} + \frac{q^{2b+2} - q^{b-2} + g(n, S)}{(1 - q^b)(1 - q^{b+1})}$$

We use both recurrences to add numbers to S :

If the number is odd we use the first recurrence.

If the number is even we use the second recurrence.

Note that the induction works, since after adding a number we still have all the even powers of q .

Combining the sets E , O and Q

The last proposition showed that

$$g(n, E \cup O) = NN + SP(q^{3n+4})$$

Using the Lemma we can combine this into the following proposition:

Proposition 3. *If $n \geq 7$ is odd, then $P = S_n \cup E \cup O \cup Q$ is monotone*

Extending by a subset of $\{3n - 1, \dots, 3n + 3\}$

We now extend by a subset of $\{3n - 1, \dots, 3n + 3\}$ to obtain the main result:

Theorem 2. *If $n \geq 7$ the monotone sets are $P = S_n \cup E \cup O \cup A \cup Q$ where*

- *E is a set of even integers from $\{2n, \dots, 3n - 2\}$*
- *O is a set of odd integers from $\{2n, \dots, 3n - 2\}$ and $E^+ \subset O$*
- *A is a subset of $\{3n - 1, \dots, 3n + 3\}$ such that if $3n + i \in A$ for $i \neq 2$, then either $3n + i + 1 \in P$ or $2n + i + 1 \in P$*
- *Q is a subset of $\{3n + 4, 3n + 5, \dots\}$*

Proof of the Theorem

We begin by listing the partitions of numbers in this range with parts greater than $2n - 1$:

- $\{3n - 1\}$
- $\{3n\}, \{n, 2n\}$
- $\{3n + 1\}, \{n, 2n + 1\}, \{n + 1, 2n\}$
- $\{3n + 2\}, \{n, 2n + 2\}, \{n + 1, 2n + 1\}, \{n + 2, 2n\}$
- $\{3n + 3\}, \{n, 2n + 3\}, \{n + 1, 2n + 2\}, \{n + 2, 2n + 1\}, \{n + 3, 2n\}$
- $\{3n + 4\}, \{n, 2n + 4\}, \{n + 1, 2n + 3\}, \{n + 2, 2n + 2\}, \{n + 3, 2n + 1\}, \{n + 4, 2n\}$

We have to check every case individually:

If $3n - 1 \in A$ then either $3n \in A$ or $2n \in E$

For the latter use the previous recurrence.

For the case $3n - 1, 3n \in A$ use the other recurrence.

This will lead to more cases. To eventually check all cases we will also need two more recurrences (to add 3/4 consecutive numbers)

Result for even n

By carrying out a similar evaluation of all cases if $n \geq 4$ is even we get the following theorem:

Theorem 3. *If $n \geq 4$ is even, then the monotone subsets are $P = S_n^* \cup E \cup O \cup A \cup Q$ where*

- $S_n^* = \{n, \dots, 2n + 1\}$
- E is a set of even integers from $\{2n+2, \dots, 3n+1\}$
- O is a set of odd integers from $\{2n+2, \dots, 3n+1\}$ such that $E^+ \subset O$
- A is a subset of $\{3n + 2, \dots, 3n + 6\}$ such that if $3n + 2i \in A$ then either $3n + 2i + 1 \in P$ or $2n + 2i + 1 \in P$
- Q is a subset of $\{2n, 3n + 7, 3n + 8, \dots\}$

Comments+Bibliography

Is there a combinatorial proof of the conjecture?

→ Find an injection from partitions of $k - 1$ into partitions of k .

What about the cases $n = 2, 3, 5$?

What about strict monotonicity?

Bibliography

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