Some Monotonicity Theorems

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Introduction

A *partition* of n is a nondecreasing finite sequence of positive integers whose sum is n.

Let P be a set of positive integers.

We let $p(n)$ denote the number of partitions of n with parts in P.

A set P is called *monotone* if the sequence $p(n)$ is weakly increasing.

By a well-known theorem the generating function for integer partions with parts in P is

$$F_P(q) = \sum_{n=0}^{\infty} p(n) q^n = \prod_{p \in P} \frac{1}{1 - q^p}$$

Thus P is monotone if all coefficients of $(1-q)F_P(q)$ (other than $-q$) are non-negative.

Question: *Bateman, Erdös (1956)*

Which P are monotone?

It is clear that if $1 \in P$ then P is monotone.

So let us assume that n is the minimal element with $n > 1$

Since n is minimal, any monotone P must contain $n+1$ as well. (Otherwise $p(n+1) = 0$)

In a similar way we see that any monotone P must contain $S_n := \{n, \ldots, 2n-1\}$

We use NN to denote non-negative terms and $SP(q^a)$ for strictly positive terms past q^a.

We also let

$$f_{n,m}(q) = \frac{1-q}{\prod_{i=n}^{m}(1-q^i)}$$

A first little theorem

Theorem 1.

$$f_{n,\infty} = 1 - q + NN + SP(q^{3n+1})$$

Proof.

$$f_{n,\infty}(q) = (1-q)\frac{1}{(q^n)_\infty}$$

where $(a)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$

By the q-binomial theorem we get

$$\begin{aligned} f_{n,\infty}(q) &= (1-q)\sum_{k=0}^{\infty}\frac{q^{nk}}{(q)_k} \\ &= 1 - q + q^n + \frac{q^{2n}}{1-q^2} + \sum_{k=3}^{\infty}\frac{q^{nk}}{(q^2)_{k-1}} \end{aligned}$$

□

The Conjecture

Conjecture 1 (Friedman-Joichi-Stanton). *For an odd positive integer $n > 1$, $f_{n,2n-1}(q) = 1 - q + NN$ If also $n \geq 7$, then $f_{n,2n-1}(q) = 1-q+NN+SP(q^{3n+4})$*

Corollary 1. *If n is even, then $f_{n,2n+1}(q) = 1 - q + NN + SP(q^{3n+7})$*

Proof. Define $h_{n,m}(q) = f_{n,m} - 1 + q$

Then $h_{n,2n+1}(q) = \frac{1}{1-q^n}(h_{n+1,2n+1}(q) + q^n - q^{n+1})$

If n is even then $h_{n+1,2n+1}(q)$ contains q^{n+1}

Hence $h_{n,2n+1}(q) = NN + SP(q^{3n+7})$ for $n \geq 6$.

Show the cases $n = 2$ and $n = 4$ separately. $\square$

Lemma to increase monotone sets

Lemma 1. *Suppose* $H(q) = 1 - q + NN + SP(q^a)$. *Let* S *be a set of positive integers. Now define* $H(S, q) = H(q)/\prod_{s \in S}(1 - q^s)$

If $\min\{s : s \in S\} \geq a$, *then* $H(S, q) = 1 - q + NN$.

Proof. Define $g(S, q) = H(S, q) - 1 + q$.
We have to show that $g(S, q) = NN$.
Consider the recurrence for $b \notin S$:

$$g(S \cup \{b\}, q) = \frac{1}{1 - q^b}(g(S, q) + q^b - q^{b+1})$$

For finite sets S use induction: Start with the smallest element in S
If $S = \emptyset$, then $g(\emptyset, q)$ is positive past q^a. Since $b \geq a$ it is positive at q^{b+1}.
Thus by the recurrence $g(\{b\}, q) = NN + SP(q^{b+1})$
Repeat this process for every element in S.
If S is infinite check the coefficient of q^j by taking the finite set $S \cap \{x : x \leq j\}$

□

Extending the set $S_n = \{n, \ldots, 2n-1\}$

By the previous lemma we immediately get:

Proposition 1. *If* $n \geq 7$ *and* Q *is a subset of* $\{3n+4, 3n+5, \ldots\}$ *then* $P = S_n \cup Q$ *is monotone*

Thus the only other numbers we have to consider are numbers between $2n$ and $3n+3$.

We consider two cases:

- Numbers between $2n$ and $3n-2$
- Numbers between $3n-1$ and $3n+3$

For the first case we get the following:

Proposition 2. *If* $n \geq 7$ *and* E *is a set of even numbers,* O *a set of odd numbers, both in* $\{2n, \ldots, 3n-2\}$*, with the condition that* $E^+ = \{e+1 : e \in E\} \subset O$*, then* $P = \{n, \ldots, 2n-1\} \cup E \cup O$ *is a monotone.*

Proof of the Proposition

For any subset $S \subset \{2n, 2n+1, ...\}$ define

$$g(n,S) = f_{n,2n-1}(q) / \prod_{s \in S} (1-q^s) - 1 + q$$

We can calculate the coefficients of $f_{n,2n-1}(q)$ up to q^{3n+4}:

$$f_{n,2n-1}(q) = 1 - q + q^n + \sum_{i=0}^{(n-3)/2} q^{2n+2+2i} + q^{3n+3} + \dots$$

Using the conjecture this implies:

$$g(n,\emptyset) = q^n + \sum_{i=0}^{(n-3)/2} q^{2n+2+2i} + q^{3n+3} + SP(q^{3n+4})$$

Again we will extend S by induction, and use the recurrence:

$$g(S \cup \{b\}, q) = \frac{1}{1-q^b}(g(S,q) + q^b - q^{b+1})$$

<u>Proof of the Proposition II</u>

Since there is the term $-q^{b+1}$, we see that if b is even, we also need $b+1$.

Thus we need the following recurrence:

$$g(n, S \cup \{b, b+1\}) = \frac{q^b}{1-q^b} + \frac{q^{2b+2} - q^{b-2} + g(n, S)}{(1-q^b)(1-q^{b+1})}$$

We use both recurrences to add numbers to S:
If the number is odd we use the first recurrence.
If the number is even we use the second recurrence.

Note that the induction works, since after adding a number we still have all the even powers of q.

Combining the sets E, O and Q

The last proposition showed that

$$g(n, E \cup O) = NN + SP(q^{3n+4})$$

Using the Lemma we can combine this into the following proposition:

Proposition 3. *If $n \geq 7$ is odd, then $P = S_n \cup E \cup O \cup Q$ is monotone*

Extending by a subset of $\{3n-1,\dots,3n+3\}$

We now extend by a subset of $\{3n-1,\dots,3n+3\}$ to obtain the main result:

Theorem 2. *If $n \geq 7$ the monotone sets are $P = S_n \cup E \cup O \cup A \cup Q$ where*

- *E is a set of even integers from $\{2n,\dots,3n-2\}$*
- *O is a set of odd integers from $\{2n,\dots,3n-2\}$ and $E^+ \subset O$*
- *A is a subset of $\{3n-1,\dots,3n+3\}$ such that if $3n+i \in A$ for $i \neq 2$, then either $3n+i+1 \in P$ or $2n+i+1 \in P$*
- *Q is a subset of $\{3n+4, 3n+5,\dots\}$*

Proof of the Theorem

We begin by listing the partitions of numbers in this range with parts greater than $2n-1$:

- $\{3n-1\}$
- $\{3n\}, \{n, 2n\}$
- $\{3n+1\}, \{n, 2n+1\}, \{n+1, 2n\}$
- $\{3n+2\}, \{n, 2n+2\}, \{n+1, 2n+1\}, \{n+2, 2n\}$
- $\{3n+3\}, \{n, 2n+3\}, \{n+1, 2n+2\}, \{n+2, 2n+1\}, \{n+3, 2n\}$
- $\{3n+4\}, \{n, 2n+4\}, \{n+1, 2n+3\}, \{n+2, 2n+2\}, \{n+3, 2n+1\}, \{n+4, 2n\}$

We have to check every case individually:

If $3n-1 \in A$ then either $3n \in A$ or $2n \in E$

For the latter use the previous recurrence.

For the case $3n-1, 3n \in A$ use the other recurrence.

This will lead to more cases. To eventually check all cases we will also need two more recurrences (to add 3/4 consecutive numbers)

Result for even n

By carrying out a similar evaluation of all cases if $n \geq 4$ is even we get the following theorem:

Theorem 3. *If $n \geq 4$ is even, then the monotone subsets are $P = S_n^* \cup E \cup O \cup A \cup Q$ where*

- $S_n^* = \{n, \ldots, 2n+1\}$
- *E is a set of even integers from $\{2n+2, \ldots, 3n+1\}$*
- *O is a set of odd integers from $\{2n+2, \ldots, 3n+1\}$ such that $E^+ \subset O$*
- *A is a subset of $\{3n+2, \ldots, 3n+6\}$ such that if $3n+2i \in A$ then either $3n+2i+1 \in P$ or $2n+2i+1 \in P$*
- *Q is a subset of $\{2n, 3n+7, 3n+8, \ldots\}$*

Proof of the Conjecture

We first show the following lemma:

Lemma 2.

$$\frac{1}{(q^n;q)_n} = 1+\sum_{m=0}^{n-1} q^{(n+m)(2m+1)}\frac{(q^{n-1};q^{-1})_m}{(q;q)_{2m+1}} + \sum_{m=0}^{n-2} q^{2(n+m+1)(m+1)}\frac{(q^{n-1};q^{-1})_{m+1}}{(q;q)_{2m+2}}$$

Proof. First apply the q-binomial theorem:

$$\frac{1}{(q^n;q)_n} = 1+\sum_{j=1}^{\infty}\frac{(q^n;q)_j}{(q;q)_j}q^{nj}$$

Now repeat the following process:

1. Split the sum into two sums, using the first factor of the numerator
2. Pull the first term out of the first sum and shift the second sum by one
3. Combine those two sums
4. Pull out the first term of the sum

After repeating this $t+1$ times we get:

$$\begin{aligned}\frac{1}{(q^n;q)_n} &= 1+\sum_{m=0}^{t} q^{(n+m)(2m+1)}\frac{(q^{n-1};q^{-1})_m}{(q;q)_{2m+1}} \\ &+\sum_{m=0}^{t} q^{2(n+m+1)(m+1)}\frac{(q^{n-1};q^{-1})_{m+1}}{(q;q)_{2m+2}} \\ &+(q^{n-1};q^{-1})_{t+1}\sum_{j=2t+3}^{\infty}\frac{(q^{n+t+1};q)_{j-2t-2}}{(q;q)_j}q^{(n+t+1)j}\end{aligned}$$

Choosing $t=n-1$ we get the claim after noting that $(q^{n-1};q^{-1})_n=0$

$$\begin{aligned}\frac{1}{(q^n;q)_n} &= 1+\sum_{m=0}^{n-1} q^{(n+m)(2m+1)}\frac{(q^{n-1};q^{-1})_m}{(q;q)_{2m+1}} \\ &+\sum_{m=0}^{n-2} q^{2(n+m+1)(m+1)}\frac{(q^{n-1};q^{-1})_{m+1}}{(q;q)_{2m+2}}\end{aligned}$$

□

Now note that the last two terms in each sum add to

$$\frac{q^{4n^2-6n+2}}{(q^n;q)_n}$$

We subtract this term from both sides to get

$$\frac{1-q^{4n^2-6n+2}}{(q^n;q)_n} = 1+\sum_{m=0}^{n-2} q^{(n+m)(2m+1)}\frac{(q^{n-1};q^{-1})_m}{(q;q)_{2m+1}} + \sum_{m=0}^{n-3} q^{2(n+m+1)(m+1)}\frac{(q^{n-1};q^{-1})_{m+1}}{(q;q)_{2m+2}}$$

Now multiply by $\frac{1-q}{1-q^{4n^2-6n+2}} = \frac{1-q^{4n^2-6n+3}}{1-q^{4n^2-6n+2}} - q$ to get

$$\frac{1-q}{(q^n;q)_n}+q = \frac{1}{1-q^{4n^2-6n+3}}\Bigg(1-q^{4n^2-6n+3} + \sum_{m=0}^{n-2} q^{(n+m)(2m+1)}\frac{(q^{n-1};q^{-1})_m}{(q;q)_{2m}} + \sum_{m=0}^{n-3} q^{2(n+m+1)(m+1)}\frac{(q^{n-1};q^{-1})_{m+1}}{(q;q)_{2m+1}}\Bigg)$$

In order to show the conjecture we now have to do three things:

1. Show that the first sum has non-negative coefficients
2. Show that the second sum has non-negative coefficients
3. Show that in one of the sums the coefficient of q^{4n^2-6n+3} is positive

Let us look at the first sum:

The term $m = 0$ is q^n

Now look at the terms for which $2m + 1 \geq n - 1$

We obtain:

$$\frac{(q^{n-1};q^{-1})_m}{(q;q)_{2m}} = \frac{1}{(q^n;q)_{2m+2-n}(q^2;q)_{n-m-2}}$$

For the terms with $2m+1 < n-1$ we choose s such that $2^s < m + 1 \leq 2^{s+1}$, define $k_s := 2^{s+1}$ and obtain:

$$\frac{(q^{n-1};q^{-1})_m}{(q;q)_{2m}} = \frac{1-q}{1-q^n} \begin{bmatrix} n \\ k_s \end{bmatrix}_q \frac{1}{(q^{k_s+1};q)_{2m+1-k_s}(q^{n-k_s+1};q)_{k_s-m-1}}$$

The term $\frac{1-q}{1-q^n} \begin{bmatrix} n \\ k_s \end{bmatrix}_q$ is a polynomial with non-negative coefficients. [Andrews]

For the second sum we carry out a similar evaluation:

The first term is $q^{2(n+1)}\frac{1-q^{n-1}}{1-q^2}$

Since n is odd, this has non-negative coefficients.

For the terms with $2m+2 \geq n-1$ we have:

$$\frac{(q^{n-1};q^{-1})_{m+1}}{(q;q)_{2m+1}} = \frac{1}{(q^n;q)_{2m+3-n}(q^2;q)_{n-m-3}}$$

For the terms with $2m+2 < n-1$ we choose s such that $2^s \leq m+1 \leq 2^{s+1}-1$, define $k_s := 2^{s+1}$ and obtain:

$$\frac{(q^{n-1};q^{-1})_{m+1}}{(q;q)_{2m+1}} = \frac{1-q}{1-q^n}\begin{bmatrix} n \\ k_s \end{bmatrix}_q \frac{1}{(q^{k_s+1};q)_{2m+2-k_s}(q^{n-k_s+1};q)_{k_s-m-2}}$$

Finally, we have to show that q^{4n^2-6n+3} appears in one of the sums.

The last term of the first sum is

$$q^{(n+n-2)(2(n-2)+1)}\frac{(q^{n-1};q^{-1})_{n-2}}{(q^2;q)_{2(n-2)}} = \frac{q^{4n^2-10n+6}}{(q^n;q)_{n-2}}$$

The denominator has the factors $(1-q^n)$ and $(1-q^{2n-3})$, so we get the term $q^{4n^2-10n+6+2n+(2n-3)}$

We also have to show strict positivity past q^{3n+4}

We get this from the $m=1$ term of the first sum:

$$\begin{aligned} q^{(n+1)(2\cdot 1+1)}\frac{(q^{n-1};q^{-1})_1}{(q^2;q)_{2\cdot 1}} &= q^{3(n+1)}\frac{1-q^{n-1}}{(1-q^2)(1-q^3)} \\ &= q^{3(n+1)}\frac{1+q^2+q^4+\cdots+q^{n-3}}{1-q^3} \end{aligned}$$

Comments+Bibliography

Is there a combinatorial proof of the conjecture?
$\rightarrow$ Find an injection from partitions of $k-1$ into partitions of k.

What about the cases $n = 2, 3, 5$?

What about strict monotonicity?

Bibliography

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