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Hyperbolic Conservation Laws

Organised by

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Introduction by the Organisers

The conference was organized by Constantine Dafermos (Providence), Dietmar Kröner (Freiburg), Randy LeVeque (Seattle).

44 scientists from 9 countries participated in this workshop. During the conference we had 22 main lectures and on Tuesday and Thursday evening 10 short informal contributions. On Wednesday evening the whole group came together for a roundtable discussion on open problems. The main feature of this workshop was characterised by the collaboration of people from numerical and theoretical analysis. In the field of hyperbolic conservation laws, progress is mainly based on the input from numerical analysis to mathematical theory and vice versa. This has been shown for many new results obtained in the past and was confirmed during this workshop.

The abstracts are listed in two groups, first about "Theory" and second about "Numerics" and in each group in alphabetical order.

All participants thank the Mathematisches Forschungsinstitut Oberwolfach for making this stimulating conference possible.

AMS classification: 35Lxx, 35L65, 65Mxx, 76Nxx, 76L05

Workshop on Hyperbolic Conservation Laws

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Abstracts (Theory)

A Glimm type functional for Relaxation

Stefano Bianchini

For the simplest relaxation scheme

$$\begin{cases} f_t^- - f_x^- &= \frac{1}{2}(f^+ - f^-) \\ f_t^+ + f_x^+ &= \frac{1}{2}(f^- - f^+) \end{cases}$$

we introduce a decaying functional φ which measures interaction of waves. The functional is not local in time.
We hope that this functional may help in proving convergence of general schemes

$$F_t^\alpha + \alpha F_x^\alpha = M^\alpha(u) - F^\alpha$$

$$u = \int F^\alpha d\mu(\alpha)$$

under the assumption $DM^\alpha \; DM^\beta = DM^\beta \; DM^\alpha$ and $F^\alpha \in \Re^n$.

Order preserving solutions to the linear wave equation and application to a Chaplygin-Born-Infeld MHD-like system

Yann Brenier

A fairly well known model in gas dynamics is the so-called Chaplygin system, for which the sound speed has the unusual property of decaying to zero as the density increases to infinity. This system, which allows mass concentrations in finite time, has been advertised as a possible model for dark energy [GKMP], [Gib], and is also related to the Born-Infeld non linear theory [BI], [BDLL], [Br], [Se], for the electromagnetic field, as well as to the shallow water MHD theory [Gil]. In one space dimension, the Chaplygin system is nothing but the (formal) Eulerian formulation of the linear wave equation. It is therefore easily integrated, as long as the solutions of the wave equation are smooth and strictly monotonic (i.e. order preserving). The loss of monotonicity exactly corresponds to the blow up of the density field for the Chaplygin system The extension of solutions beyond these singularities cannot be provided any longer by solutions of the wave equations without corrections. In this talk, an order preserving modification of the wave equation is introduced which leads to a well posed reformulation (in Lagrangian coordinates) of a two dimensional system, that we call the Chaplygin-Born-Infeld (CBI) system, which includes the one-dimensional Chaplygin system as a particular case. Existence, uniqueness and stability with respect to initial conditions are established, through the analysis of a suitable order preserving numerical scheme.

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Multidimensional Transonic Shocks in Unbounded Domains

Gui-Qiang Chen

(joint work with Mikhail Feldman)

In this Note, we report some of recent developments in the study of multidimensional transonic shocks since our first paper Chen-Feldman [1] on the topic for the Euler equations for steady potential fluid flow. The Euler equations, consisting of the conservation law of mass and the Bernoulli law for velocity, can be written as a second order nonlinear equation of mixed elliptic-hyperbolic type for the velocity potential φ:

$$\operatorname{div}\left(\rho(|\nabla\varphi|^2)\nabla\varphi\right) = 0, \tag{1}$$

where the density $\rho(q^2) = \left(1-\theta q^2\right)^{\frac{1}{2\theta}}$ and $\theta = \frac{\gamma-1}{2} > 0$ with the adiabatic exponent $\gamma > 1$. The transonic shock problem can be formulated into a free boundary problem: The free boundary is the location of the multidimensional transonic shock which divides two regions of $C^{2,\alpha}$ flow, and the equation is hyperbolic in the upstream region where the $C^{2,\alpha}$ perturbed flow is supersonic.

We have developed two nonlinear approaches to deal with such free boundary problems in order to solve the transonic shock problems: one is the iteration method developed in Chen-Feldman [1, 2], which can be employed to solve physical problems with complicated boundaries as long as the corresponding fixed boundary elliptic problems can be solved; and the other is the partial hodograph method in Chen-Feldman [3], which converts the free boundary problems into the corresponding fixed boundary problems but requires special geometric forms of the boundaries in the problems.

Our results indicate that there exists a unique stable solution of the free boundary problem such that the equation is always elliptic in the downstream region and the free boundary is $C^{2,\alpha}$, provided that the hyperbolic phase is close in $C^{2,\alpha}$ to a uniform flow. The approaches have successfully been applied to solving the transonic shock problems in infinite channels [2], infinite nozzles [4], and other related problems [5]. As a concrete example, we describe the results for the infinite nozzle problem in [4] in more detail below.

Let (x', x_n) be the coordinates in $\mathbf{R}^n$, where $x_n \in \mathbf{R}$ and $x' = (x_1, \dots, x_{n-1}) \in \mathbf{R}^{n-1}$. Let $q_0^- \in \left(c, 1/\sqrt{\theta}\right)$ and $\varphi_0^-(x) := q_0^- x_n$ with the sonic speed c. Then $\varphi_0^-(x)$ is a supersonic solution in Ω, and there exists a unique $q_0^+ \in (0, c)$ such that $\rho\left((q_0^+)^2\right) q_0^+ = \rho\left((q_0^-)^2\right) q_0^-$. Thus, the function

$$\varphi_0(x) = \begin{cases} q_0^- x_n, & x \in \Omega_0^- := \Omega \cap \{x \ : \ x_n < 0\}, \\ q_0^+ x_n, & x \in \Omega_0^+ := \Omega \cap \{x \ : \ x_n > 0\} \end{cases} \tag{2}$$

is a plane transonic shock solution in Ω, Ω_0^+ and Ω_0^- are its subsonic and supersonic regions, respectively, and $S = \{x_n = 0\}$ is a transonic shock.

Consider an infinite nozzle Ω which is close in $C^{3,\alpha}$ to an infinite cylinder with arbitrary smooth cross-section: $\mathcal{C} = \Lambda \times (-a, \infty)$ with $a > 0$, where the cross-section $\Lambda \subset \mathbf{R}^{n-1}$ is an open bounded connected set with a $C^{3,\alpha}$ boundary. Such nozzles especially include the slowly varying Laval nozzle. The nozzle can generally be expressed as $\Omega = \Psi(\mathcal{C})$ with $\Psi : \mathbf{R}^n \to \mathbf{R}^n$, which is invertible and satisfies

$$\|\Psi - I\|_{3,\alpha,\mathbf{R}^n} \le \sigma \tag{3}$$

for sufficiently small $\sigma > 0$, where $\|\cdot\|_{m,\alpha,\mathcal{D}}$ is the norm in the Hölder space $C^{m,\alpha}(\overline{\mathcal{D}})$ in the domain $\mathcal{D}$. For concreteness, we also assume that there exists $L > 0$ such that $\Psi(x) = x$ for any x with $x_n > L$. Note that our assumptions imply that $\Psi(\partial\mathcal{C}) = \partial\Omega$ and $\partial\Omega = \overline{\partial_o\Omega} \cup \partial_l\Omega$, with $\partial_l\Omega := \Psi[\partial\Lambda \times (-a, \infty)]$ and $\partial_o\Omega := \Psi(\Lambda \times \{-a\})$.

Nozzle Problem. Given a supersonic upstream flow $\varphi^-(x)$ of (1) in $\Omega_1 := \{-a < x_n < 1\}$, which is a $C^{2,\alpha}$ perturbation of $\varphi_0^-(x)$ for some $\alpha \in (0,1)$:

$$\|\varphi^- - \varphi_0^-\|_{2,\alpha,\Omega_1} \le C_0\sigma, \tag{4}$$

with $\sigma > 0$ small, for some constant C_0, and satisfies $\partial_\nu\varphi^- = 0$ on $\partial_l\Omega_1$, find a transonic shock solution $\varphi(x)$ in Ω such that, denoting by $\Omega^+ := \{x \in \Omega \ : \ |D\varphi(x)| < c\}$, $\Omega^- := \Omega \setminus \Omega^+$, and $S := \partial\Omega^+ \setminus \partial\Omega$ the subsonic and supersonic regions and the shock surface of $\varphi(x)$, we have $\Omega^- \subset \Omega_1$, $\varphi = \varphi^-$ in Ω^-, and

$$\varphi = \varphi^-, \ \varphi_\nu = \varphi_\nu^- \qquad \text{on } \partial_o\Omega, \tag{5}$$

$$\partial_\nu\varphi = 0 \qquad \text{on } \partial_l\Omega, \tag{6}$$

$$\|\varphi - \omega x_n\|_{C(\Omega\cap\{x_n>R\})} \to 0 \quad \text{as } R \to +\infty, \text{ for some } \omega \in (0,c). \tag{7}$$

The supersonic upstream flow $\varphi^-(x)$ satisfying (4) can be constructed directly from the standard local existence of smooth solutions for the initial boundary value problem (5)–(6) for second order quasilinear hyperbolic equations when $(\varphi^-, \varphi_{x_n}^-)$

on $\partial_o\Omega$ is sufficiently smooth and close to $(\varphi_0^-, -q_0^-)$ with magnitude σ as σ is sufficiently small (also in (3)).

Theorem. Let $\varphi_0(x)$ be the transonic shock solution (2). Then there exist $\sigma_0 > 0$, $\hat{C}$, and C depending only on n, γ, q_0^+, Λ, Ψ, and L, such that, for every $\sigma \in (0, \sigma_0)$ and any supersonic solution $\varphi^-(x)$ of (1) satisfying the conditions stated above, there exists a global solution $\varphi \in C^{0,1}(\Omega) \cap C^{2,\alpha}(\overline{\Omega}^+)$ of the Nozzle Problem satisfying $\|D\varphi - q_0^+ e_n\|_{0;\alpha;\Omega^+} \leq \hat{C}\sigma$ and the following properties:

(i). The constant ω in (7) must be q^+: $\omega = q^+$, where q^+ is the unique solution in the interval $(0, c)$ of the equation

$$\rho((q^+)^2)q^+ = Q^+ := \frac{1}{|\Lambda|}\int_{\partial_o\Omega} \rho(|D\varphi^-|^2)D\varphi^- \cdot \nu \, dS.$$

Thus $\varphi(x)$ satisfies $\|\varphi - q^+ x_n\|_{C^1(\Omega\cap\{x_n>R\})} \to 0$ as $R \to +\infty$, and q^+ satisfies $|q^+ - q_0^+| \leq C\sigma$.

(ii). The subsonic region $\Omega^+(\varphi) := \{x \in \Omega \ : \ |D\varphi(x)| < c\}$ is of the form:

$$\Omega^+(\varphi) = \{x_n > f(x')\} \cap \Omega \quad \text{with} \quad f \in C^{2,\alpha}(\mathbf{R}^{n-1}),$$

where f satisfies $\|f\|_{2,\alpha,\mathbf{R}^{n-1}} \leq C\sigma$. Moreover, the surface $S = \{(x', f(x')) \ : \ x' \in \mathbf{R}^{n-1}\} \cap \Omega$ is orthogonal to $\partial_l\Omega$ at every point of their intersection.

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On the Existence of Weak Solutions for the Generalized Camassa-Holm Equation

Giuseppe Maria Coclite
(joint work with Helge Holden and Kenneth H. Karlsen)

In recent years the so-called Camassa-Holm Equation [1] has caught a great deal of attention. It is a nonlinear dispersive wave equation that takes the form

$$\frac{\partial u}{\partial t} - \frac{\partial^3 u}{\partial t \partial x^2} + 3u\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + u\frac{\partial^3 u}{\partial x^3}, \qquad t > 0,\ x \in \mathbf{R}. \tag{1}$$

This equation models the propagation of unidirectional shallow water waves on a flat bottom, and $u(t,x)$ represents the fluid velocity at time t in the horizontal direction x. The Camassa-Holm Equation possesses a bi-Hamiltonian structure (and thus an infinite number of conservation laws) and is completely integrable [1]. Moreover, it has an infinite number of solitary waves called peakons (due to the discontinuity of their first derivatives at the wave peak): $u(t,x) = ce^{-|x-ct|}$, $c \in \mathbf{R}$, [1]. ¿From a mathematical point of view, the Camassa-Holm Equation is well studied. Local well-posedness results are proved in [2]. It is also known that there exist global solutions for a particular class of initial data and also solutions that blow up in finite time for a large class of initial data [2, 4] (here blow up means that the slope of the solution becomes unbounded while the solution itself stays bounded). We recall that existence and uniqueness results for global weak solutions of (1) have been proved by Constantin and Escher [3], Constantin and Molinet [5], and Xin and Zhang [8, 9], see also Danchin [6].

Herein we are interested in the Cauchy Problem for the nonlinear equation

$$\frac{\partial u}{\partial t} - \frac{\partial^3 u}{\partial t \partial x^2} + \frac{\partial}{\partial x}\Big(\frac{g(u)}{2}\Big) = 2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + u\frac{\partial^3 u}{\partial x^3}, \qquad t > 0,\ x \in \mathbf{R}, \tag{2}$$

where the function $g : \mathbf{R} \to \mathbf{R}$ is given. Observe that if $g(u) = 3u^2$, then (2) is the classical Camassa-Holm Equation. We coin (2) the *Generalized Camassa-Holm Equation.*

¿From a mathematical point of view the Generalized Camassa-Holm Equation (2) is much less studied than (1). Recently, Yin [10] has proved local well-posedness, global well-posedness for a particular class of initial data, and in particular that smooth solutions blow up in finite time (with a precise estimate of the blow-up time) for large class of initial data. Let us also mention that Lopes [7] has proved stability of solitary waves for the Generalized Camassa-Holm Equation (2).

Here we look for the existence of a global weak solution to (2) for any initial function u_0 belonging to $H^1(\mathbf{R})$. In doing so we follow closely the approach of Xin and Zhang [8] for the Camassa-Holm Equation (1). Let us be more precise about our results. We shall assume

$$g \in \mathrm{Lip}_{\mathrm{loc}}(\mathbf{R}), \qquad g(0) = 0, \qquad u(0,\cdot) = u_0 \in H^1(\mathbf{R}). \tag{3}$$

Define $h(\xi) \doteq \big(g(\xi) - \xi^2\big)/2$ for $\xi \in \mathbf{R}$. Formally, the equation (2) is equivalent to the elliptic-hyperbolic system

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial P}{\partial x} = 0, \qquad -\frac{\partial^2 P}{\partial x^2} + P = h(u) + \frac{1}{2}\Big(\frac{\partial u}{\partial x}\Big)^2. \tag{4}$$

Moreover, since $e^{-|x|}/2$ is the Green's function of the operator $-\partial_{xx}^2 + 1$, the equation (2) is equivalent to the integro-differential system

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial P}{\partial x} = 0, \qquad P(t,x) = \frac{1}{2}\int_{\mathbf{R}} e^{-|x-y|}\Big(h(u(t,y)) + \frac{1}{2}\Big(\frac{\partial u}{\partial x}(t,y)\Big)^2\Big)dy.$$

Motivated by this, we shall use the following definition of weak solution.

Definition. *We call $u : \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}$ a weak solution of the Cauchy Problem for* (2) *if*

i) $u \in C(\mathbf{R}_+ \times \mathbf{R}) \cap L^\infty\big(\mathbf{R}_+^*; H^1(\mathbf{R})\big)$;
ii) u *satisfies* (4) *in the sense of distributions*;
iii) $u(0,x) = u_0(x)$, *for every* $x \in \mathbf{R}$;
iv) $\|u(t,\cdot)\|_{H^1(\mathbf{R})} \le \|u_0\|_{H^1(\mathbf{R})}$, *for each* $t > 0$.

Our main results are collected in in the following theorem:

Theorem. *Assume* (3)*. Then there exists a weak solution $u = u(t,x)$ to the Cauchy Problem for* (2)*. Moreover, the following hold*

j) (Oleinik type Estimate) for every $t \ge 0$ and $x \in \mathbf{R}$,

$$\frac{\partial u}{\partial x}(t,x) \le \frac{2}{t} + K_1,$$

for some positive constant K_1 depending only on $\|u_0\|_{H^1(\mathbf{R})}$;

jj) (Higher Integrability) there results

$$\frac{\partial u}{\partial x} \in L^p_{\mathrm{loc}}(\mathbf{R}_+ \times \mathbf{R}), \qquad \textit{for any } 1 \le p < 3.$$

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Minimal entropy conditions for Burgers equation

Camillo De Lellis
(joint work with Felix Otto, Michael Westdickenberg)

We consider strictly convex, 1–d scalar conservation laws. We show that a single strictly convex entropy is sufficient to characterize a Kruzhkov solution. The proof uses the concept of viscosity solution for the related Hamilton-Jacobi equation.

We also show that it is sufficient to impose that the entropy production measure ν is less or equal than a nonnegative measure μ such that

$$\lim_{r\downarrow 0}\frac{\mu\big(B_r(t,x)\big)}{r} \;=\; 0 \quad \text{for every } (t,x)\in\Omega.$$

This generalization is important since it allows for the derivation of new estimates for the Kuramoto-Shivashinsky equation, see [1].

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Asymptotic Stability of Riemann Solutions for a Class of Multi-D Viscous Systems of Conservation Laws

Hermano Frid

We prove the asymptotic stability of two-states nonplanar Riemann solutions under initial and viscous perturbations for a class of multidimensional systems of conservation laws. The class considered here is constituted by those systems whose flux-functions in different directions share a common complete system of Riemann invariants, the level surfaces of which are hyperplanes. The latter are known as Temple fluxes, after Temple [7]. In particular, we obtain the uniqueness of the self-similar L^∞ entropy solution of the two-states nonplanar Riemann problem. The asymptotic stability to which the main result refers is in the sense of the convergence as $t\to\infty$ in L^1_{loc} of the space of directions $\xi = x/t$. That is, the solution $u(t,x)$ of the perturbed problem satisfies

$$u(t,t\xi)\to R(\xi), \qquad \text{as } t\to\infty, \text{ in } L^1_{loc}(\Re^n), \tag{1}$$

where $R(\xi)$ is the self-similar entropy solution of the corresponding two-states nonplanar Riemann problem.

Our analysis is motivated by the ideas developed in [1]. We first prove

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T |u(t,t\xi)-R(\xi)|\,dt = 0, \quad \text{in } L^1_{loc}(\Re^n). \tag{2}$$

An important aspect of (2) is its equivalence to the convergence in $L^1_{loc}(\Re^{n+1}_+)$ of the scaling sequence $\{u^T\}$, given by $u^T(t,x)=u(Tt,Tx)$,

to $R(x/t)$, when $T\to+\infty$. The latter is equivalent to the fact that, given any sequence $T_k\to\infty$, as $k\to\infty$, one can find a subsequence, $T_l=T_{k_l}\to\infty$, as $l\to\infty$, such that $u^{T_l}(t,x)\to R(x/t)$ in $L^1_{loc}(\Re^{n+1}_+)$ as $l\to\infty$.

This fact is frequently useful when trying to prove (2).

Once (2) is proved, a standard procedure established in [1] is then used to strengthen (2) into (1). This strengthening is similar to the ones encountered

in [2, 5].

Uniform boundedness of $u(x,t)$, due to existence of bounded invariant regions, plus existence of a strictly convex entropy give

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T |\nabla_x u(\xi t,t)|\,dt = 0, \ \text{ a.e. } \xi\in\Re^n.$$

We recall that in the one-dimensional case the idea was (cf. [4], [1]) to integrate the entropy inequality

$$\eta(u)_t + q(u)_x \le \eta(u)_{xx}, \tag{3}$$

in a region of the type

$$\Omega^{\pm}_{\xi}(T) = \{(x,t)\,:\,\pm(x-\xi t)>0, 0<t<T\},$$

where $+$ or $-$ depends on whether $\eta(u_L)=0$ or $\eta(u_R)=0$, respectively. Integration by parts gives, respectively,

$$\limsup_{T\to\infty} \pm\frac{1}{T}\int_0^T (-\xi\eta+q)(u(\xi t,t))\,dt \le 0. \tag{4}$$

Defining, for $g\in C(\Re^m)$,

$$\langle \mu^T_\xi, g(u)\rangle = \frac{1}{T}\int_0^T g(u(\xi t,t))\,dt,$$

inequality (4) combined with properties of Temple systems eventually leads to

$$\mu^T_\xi \rightharpoonup \delta_{R(\xi)}, \ \text{ as } T\to\infty, \text{ a.e. } \xi\in\Re,$$

which gives (1) in the 1D case. The latter is similar in spirit to the usual procedure in the theory of compensated compactness, since the pioneering papers of Tartar [6] and DiPerna [3], although here the probability measures have nothing to do with Young measures.

In the multi-D case we try to adapt the above procedure, but the situation now is quite more complicated, because of the geometry of the domains of integration.

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Low frequency stability of planar multi-D detonations

Helge Kristian Jenssen

(joint work with Greg Lyng & Mark Williams)

We use the classical normal mode approach of hydrodynamic stability theory to define stability determinants (Evans functions) for multidimensional strong detonations in three commonly studied models of combustion: the full reactive Navier-Stokes (RNS) model, and the simplified Zeldovitch-von Neumann-Döring (ZND), and Chapman-Jouget (CJ) models. The determinants are functions of frequencies (λ, η), where λ is a complex variable dual to the time variable, and $\eta \in \mathbb{R}^{d-1}$ is dual to the transverse spatial variables. The zeroes of these determinants in $\Re\lambda > 0$ correspond to perturbations that grow exponentially with time.

The CJ determinant, $\Delta_{CJ}(\lambda, \eta)$, turns out to be explicitly computable. The RNS and ZND determinants are impossible to compute explicitly, but we are able to compute their first-order low frequency expansions with an error term that is uniformly small as $\Re\lambda \downarrow 0$. Somewhat surprisingly, this computation yields an Equivalence Theorem: the leading coefficient in the expansions of both the RNS and ZND determinants is a constant multiple of Δ_{CJ}. In this sense the low frequency stability conditions for strong detonations in all three models are equivalent. By computing Δ_{CJ} we are able to give low frequency stability criteria valid for all three models in terms of the physical quantities: Mach number, Gruneisen coefficient, compression ratio, and heat release. The Equivalence Theorem is a step toward the rigorous theoretical justification of the CJ and ZND models as approximations to the full RNS model.

Existence to solutions of a kinetic aerosol model

Christian Klingenberg
(joint work with P.E. Jabin)

We consider the coagulation model

$$f_t + \frac{p}{m}.\nabla_x f = Q(f)$$

where we have the particle density $f(t, x, m, p)$ of particles with mass $m \in R^+$, momentum $p \in R^3$, at time $t > 0$ and position $x \in R^3$. For a general class of collision operators Q we prove existence of solutions. Under some natural restriction on the initial data we have existence without blowup of the solution.

For more details see [1] or contact one of the authors: klingenberg@mathematik.uni-wuerzburg.de, Pierre-Emmanuel.Jabin@ens.fr .

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Boundary Values for Moment Equation

Ingo Müller

The moment equations of the kinetic theory of gases furnish a quasilinear hyperbolic system which is entirely explicit. Yet until recently their solution for specific problems − even simple ones − was impossible, since boundary values for higher moments cannot be imposed and controlled.

It is obvious that the gas itself has no such problem. Indeed, the temperature field in a rarefied gas is quite reproducible even though it depends strongly on the uncontrollable boundary values.

We suggest that those boundary values fluctuate rapidly with the thermal motion and that the values, which the gas reacts to, are mean values. With that assumption the solutions of moment equations for boundary value problems become unique and they agree well with solutions obtained by other methods, e.g. molecular dynamics, kinetic schemes, or direct solutions of the Boltzmann equation.

The moment equations of the kinetic theory have been extrapolated to form the basis of Extended Thermodynamics, a classical field theory appropriate for the treatment of rarefied gases. The field equations are symmetric hyperbolic and their number is chosen so as to achieve convergence of predictions.

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Structure of entropy solutions

Felix Otto
(joint work with C. De Lellis, M. Westdickenberg)

We show BV–structure without BV estimates for multi–d scalar conservation laws with genuinely non–linear flux function. BV structure means that the group set J is codimension 1 rectifiable and that blow–ups outside J are constant whereas they are strongest shocks $\mathcal{H}^{n-1}$ – a.e. on J. The proof involves typical arguments from geometric measure theory and starts from the kinetic formulation. Key is the classification of "'Split states"' which denote those solutions of the kinetic equation where entropy production measure factorizes : $\mu(dvdxdt) = h(dv)\ \nu(dxdt)$.

Stability of space-periodic traveling waves

Denis Serre

Consider a system of conservation laws

$$u_t + f(u)_x = u_{xx}. \tag{1}$$

A traveling wave of velocity s is a solution of the form $u = U(x - st)$. We are interested in those traveling waves for which the profile U is periodic. These are given by periodic solutions of the ODE

$$U'' = (f(U) - sU)',$$

which can be integrated once as

$$U' = f(U) - sU - q.$$

We emphasize that the parameters s and q are not given a priori. The set P of solutions is generically of dimension $n+2$ (n degrees of freedom for $U(0)$, n for q, one for each of s and the period, ℓ, and n constraints $U(\ell) = U(0)$). If we make the quotient by the translations, we obtain a manifold $\bar{P}$ of dimension $n+1$.

It turns out that slow modulation theory of such periodic traveling waves, following [4] and [3], gives rise to a first-order system of conservation laws on the manifold $\bar{P}$:

$$v_t + F(v)_x = 0. \tag{2}$$

Thus we end up with $n+1$ conservation laws in $n+1$ unknowns, instead of n. The variables v consist in the average of U (n quantities, clearly unaffected by translations), together with the frequency $\omega := 1/\ell$. The fluxes $F(v)$ are the averages of the fluxes $F(U)$, and ω times the velocity s.

Given one periodic traveling wave U_0, we are interested in its spectral stability. This is the weaker notion of stability, that tells that the spectrum of the linearized operator

$$Lw := (w_x - \mathrm{d}f(U_0)w)_x$$

lies in the left half-space $\Re\lambda \leq 0$. Using a moving frame, we may always assume that U_0 is a standing wave, that is $s(U_0) = 0$. It is not possible in general to determine explicitly this spectrum. However, Floquet's theory tells that the spectrum is essential, and the generalized eigenfunctions are periodic, up to a phase shift $\theta \in [0, 2\pi)$. At a theoretical level, it can be encoded in a so-called *Evans function* $D(\lambda, \theta)$, in the following way: The spectrum of L is the λ-projection of the zero set of D. See [1] for the construction.

The large wavelength analysis consists in computing the leading term of D in the Taylor expansion at the origin. Because of translational invariance, U' is a periodic solution of $Lw = 0$. This shows that $D(0,0) = 0$. Actually, the kernel of L is isomorphic to the tangent space to P_0 at U_0 (the submanifold of traveling waves of same period than U_0), thus is of dimension $n+1$. This translates into the fact that D vanishes at order $n+1$ at the origin.

Our main theorem is that

$$D(\lambda, \theta) = \Gamma \det(\lambda I_{n+1} - i\theta \mathrm{d}F(v_0)) + O\left(|\lambda|^{n+2} + |\theta|^{n+2}\right),$$

where Γ is a non-zero (under a transversality condition) constant, and v_0 is the point of $\bar{P}$ that corresponds to U_0. This implies immediately the necessary condition for spectral stability, that (2) be hyperbolic.

These results complete those of Oh and Zumbrun [2], which applied, roughly speaking, in the special case where all periodic traveling waves have the same velocity.

The present work will appear in full details in *Communications of Partial Differential Equations.*

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Decay of Solutions in Hyperbolic Systems of Conservation Laws with General Flux

Konstantina Trivisa
(joint work with P.G. LeFloch)

Several Glimm-type functionals for (piecewise smooth) approximate solutions of nonlinear hyperbolic systems have been introduced in recent years. In this work, following a work by Baiti and Bressan [1], (see also Bressan [4]) on genuinely nonlinear systems we provide a general framework to prove that such functionals can be extended to general functions with bounded variation and we investigate their lower semi-continuity properties with respect to the strong L^1 topology. In particular, our result applies to the functionals introduced by Iguchi-LeFloch [6] and Liu-Yang [7] for systems with general flux-functions, as well as the functional introduced by Baiti-LeFloch-Piccoli [2] for nonclassical entropy solutions. As an illustration of the use of continuous Glimm-type functionals, we also extend a result by Bressan and Colombo [5] for genuinely nonlinear systems, and establish an estimate on the spreading of rarefaction waves in solutions of hyperbolic systems with general flux-function.

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The Riemann function and oscillations in systems of two conservation laws

Athanasios E. Tzavaras

The objective of this talk is to study the structure of oscillations for approximate solutions to systems of two strictly hyperbolic conservation laws, via using singular entropy entropy-flux pairs. As a byproduct, we will obtain an existence theorem for the equations of one-dimensional elastodynamics

$$\begin{aligned} u_t - v_x &= 0 \, , \\ v_t - \sigma(u)_x &= 0 \, . \end{aligned} \tag{1}$$

under the energy norm framework. The result improves on previous studies of compactness properties by DiPerna [1], and mainly (in the energy norm setting) by Lin [2], Shearer [5] and Serre-Shearer [4]. The method exploits the existence of singular entropies for 2×2 strictly hyperbolic systems constructed in [3] and the connection with the Riemann function [6]. The full article is forthcoming [7]. Here is a short exposé of the approach and main result.

For a strictly hyperbolic 2×2 system with characteristic speeds $\lambda_1 < \lambda_2$ the equations generating entropy-entropy flux pairs are

$$\mathcal{L}_{w,z}[\eta] := \eta_{wz} - \frac{g_z}{g}\eta_w - \frac{f_w}{f}\eta_z = 0\,, \qquad \begin{aligned} q_w &= \lambda_1 \eta_w\,, \\ q_z &= \lambda_2 \eta_z\,, \end{aligned} \tag{2}$$

where $\frac{g_z}{g} = -\frac{\lambda_{1z}}{\lambda_1 - \lambda_2}$, $\frac{f_w}{f} = \frac{\lambda_{2w}}{\lambda_1 - \lambda_2}$. There exists a universal entropy-entropy flux pair $H = H(w,z;\xi,\zeta)$, $Q = Q(w,z;\xi,\zeta)$ such that any function of the type

$$\begin{aligned} &H\, \mathbb{1}^k(w,\xi)\,, \qquad (\lambda_1(\xi,\zeta) + Q)\, \mathbb{1}^k(w,\xi)\,, \\ &H\, \mathbb{1}^k(z,\zeta)\,, \qquad (\lambda_2(\xi,\zeta) + Q)\, \mathbb{1}^k(z,\zeta)\,, \\ &\text{where} \qquad \mathbb{1}^k(w,\xi) = \begin{cases} \mathbb{1}_{k<\xi<w} & \text{if } k < w \\ 0 & \text{if } k = w \\ -\mathbb{1}_{w<\xi<k} & \text{if } w < k \end{cases} \end{aligned} \tag{3}$$

with k a parameter is a singular solution of (2). The universal entropy pair $H - Q$ is precisely the generator of the fundamental solution of $\mathcal{L}$, that is

$$\mathcal{L}_{w,z}\big[H \mathbb{1}(w,\xi)\mathbb{1}(z,\zeta)\big] = \delta(w-\xi)\,\delta(z-\zeta)\,. \tag{4}$$

Oscillations of families of solutions are described through the Young measure ν. The singular entropies are an efficient tool for localizing the support of the Young measure ν and exctracting information from Tartar's commutation relation

$$\overline{\eta_1 q_2 - \eta_2 q_1} = \overline{\eta_1}\;\overline{q_2} - \overline{\eta_2}\;\overline{q_1} \tag{5}$$

This bracket can be applied to entropies of class (3) in two ways: (i) by coupling singular entropies belonging to the same characteristic family, and (ii) by coupling families belonging to different characteristic families. Coupling entropies of the same characteristic families yields the relations established by Serre:

$$\overline{\frac{\partial \lambda_1}{\partial w}(w,z) g^2(w,z) \varphi(w)} = 0\,, \qquad \overline{\frac{\partial \lambda_2}{\partial z}(w,z) f^2(w,z) \psi(z)} = 0\,. \tag{6}$$

Coupling entropies of different families and using the fundamental solution of the differential operator

$$\mathcal{N}_{\xi,\zeta} = \partial_{\xi\zeta} + \frac{\lambda_{2\zeta}}{\lambda_2 - \lambda_1}\partial_\xi - \frac{\lambda_{1\xi}}{\lambda_2 - \lambda_1}\partial_\zeta$$

yields a new formula for the coupling of oscillations between different characteristic fields, see [6].

In the second part of the talk, we consider a family of approximate solutions $(u^\varepsilon, v^\varepsilon)$ of the equations of elasticity (1) that are assumed to satisfy the uniform bounds

$$\int \frac{1}{2} v^{\varepsilon 2} + W(u^\varepsilon) dx + \varepsilon \int a^2(u^\varepsilon) u_x^{\varepsilon\, 2} + v_x^{\varepsilon\, 2} dx \leq O(1) \tag{7}$$

where $a(u) = \sqrt{\sigma'(u)}$ and $W(u) = \int^u \sigma(\tau) d\tau$. Under (7), $v^\varepsilon \rightharpoonup v$ in L^2, $u^\varepsilon \rightharpoonup u$ in L^q for some $q > 1$, and

$$\varphi(u^\varepsilon, v^\varepsilon) \rightharpoonup \int d\nu \varphi(u, v) \qquad \begin{array}{r} \forall\, |\varphi(u, v)| \leq o(1)(\frac{1}{2} v^2 + W(u)) \\ \text{as } |u|, |v| \to \infty \end{array}$$

It is assumed that the stress-strain function satisfies uniform strict hyperbolicity

$$\sigma'(u) \geq \sigma_0 > 0 \quad \forall u$$

it has at most one inflection point and (together with derivatives up to second order) satisfies certain growth conditions. It is then shown that the Young measure ν reduces to a Dirac mass, $\nu = \delta_u \otimes \delta_v$, and the convergence is strong. The result is an improvement of results in [2], [5], [4].

The proof exploits the format of singular entropies. It is based on precise estimations of the universal entropy pair $H(u, v)$, $Q(u, v)$ and its derivatives that are global in nature, and on handling Tartar's commutation bracket for singular pairs. The details are presented in [7].

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Transonic Shock Waves with Physical Boundaries

Zhouping Xin

In this talk, I discuss the existence and uniqueness of transonic shocks with slowly varying sections and given pressure at the exhaustic exit. This can be reduces to a nonlinear boundary–value problem for a mixed–type equation, which is the potential equation. Our results imply, in particular, the conjecture of Courant–Friedrichs on a phenomena of transonic shock in a de Laval nozzle. Our analysis consists of an introduction of a new partial hodograph transformation, a weighted energy estimate in a weighted H"older space. This is a joint work with Huicheng Yin (Wanjing University)

An Energy Method for Boltzmann Equation

Tong Yang

(joint work with F. Huang, T.-P. Liu, H. Zhao, H. Yu and S.-H. Yu)

The purpose of the research is on the nonlinear stability of wave patterns and non-trivial solution patterns to Boltzmann equation with or without forces.

A decomposition of the Boltzmann equation and its solution was introduced in [2] where the Boltzmann equation is rewritten into a fluid-type system coupled with an equation for the non-fluid component. In some sense, this combines the Hilbert and Chapman-Enskog expansions and gives exact Boltzmann formulation instead of approximation. Based on this, one can apply the analytic techniques in the study of conservation laws to the stability investigation on nonlinear wave patterns and profiles to Boltzmann equation, Vlasov-Poisson-Boltzmann system and Vlasov-Maxwell-Boltzmann system, etc. In fact, the energy method through the construction of entropy-entropy flux pairs becomes useful here and the behavior of the fluid components and non-fluid components are clearly analyzed.

For the Boltzmann equation without force, we prove the stability of nonlinear rarefaction waves in the whole space in [3], and with boundary effects in [6]. Moreover, the stability of nonlinear diffusion waves corresponding to the contact discontinuity for Euler equations in [1].

As for the Boltzmann equation with self-induced electric field, or electric-magnetic fields, it also shows that the entropy-entropy flux pair similar to the one for fluid dynamics plays an important role in the lower order energy estimate. For the Cauchy problem on the Vlasov-Poisson-Boltzmann system, the dissipation coming from the electric field governed by the Poisson equation is crucial to close the a priori estimate which in turn implies that the uniform space-time integrability of the square of the difference between perturbed and unperturbed density function, cf. [4]. Notice that the later integral diverges for the Boltzmann equation or even the Navier-Stokes equations without force in the whole space. Finally for Vlasov-Maxwell-Boltzmann system, we also obtain another proof of global existence of classical solutions for the period data which was first proved by Y. Guo. In fact, our method based on the decomposition on the local Maxwellian has the advantage over the method used by Y. Guo in the sense that it gives clearer description on the time evolution of the fluid components and it could be helpful in the study on the problem about fluid limits.

In the following, we only list our papers on this subject because of the limited space. Interested readers please refer to the other references cited in the following papers.

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Hyperbolic Models for Viscous Incompressible Flows

Wen-An Yong

It is well known that viscous incompressible flows are described with incompressible Navier-Stokes equations. On the other hand, in engineering viscous incompressible flows are often simulated with certain discrete dynamical systems (the lattice Boltzmann mehod). Those dynamical systems can be regarded as a particular discretization of some hyperbolic systems with source terms—discrete-velocity or lattice Boltzmann models. Recently, it is shown that the Navier-Stokes equations can be approximated with the hyperbolic systems in the diffusive limit.

The lattice Boltzmann models are usually constructed to satisfy some physical requirements like Galilean invariance and isotropy, to possess a velocity-independent pressure and no compressible effects, and so on. These physical requirements often leave free parameters. In simulations, the free parameters were fixed through guesswork and numerical tests.

In this talk, I introduce a stability notion to characterize the hyperbolic systems approximating the incompressible Navier-Stokes equations. This notion is based on a stability condition proposed by me in 1992 for hyperbolic relaxation systems and is quite different from the well-known subcharacteristic condition and entropy dissipation conditions. In fact, because the incompressible Navier-Stokes equations are not hyperbolic, the notion of subcharacteristic condition is irrelevant here. Moreover, it has been proved recently that many lattice Boltzmann models used in practice do not admit any entropy dissipation conditions.

With our stability notion, we derive some relations of parameters for several parametrized lattice Boltzamnn models used in literature by requiring the models to be stable in our sense. Here I would like to report that the parameter values used in the literature are exactly those predicted by our stability requirement. Extensive numerical experiements show that our stability notion provides an effective criterion to fix the lattice Boltzmann models.

Furthermore, we prove that the diffusive limit of the stable hyperbolic systems is the incompressible Navier-Stokes equations at least in the regime of smooth flows. Moreover, the corresponding lattice Boltzmann schemes for Stokes flows are weighted L^2-stable under suitable restrictions on the time step.

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Abstracts (Numerics)

Residual distribution schemes for hyperbolic problems

Rémi Abgrall
(joint work with M. Mezine and M. Audiomor)

I have presented and discussed a class of compact schemes for hyperbolic problems.

In these schemes, the degrees of freedom are situated at vertices. The solution is globally represented by a continuous piecewise polynomial function. The unknown are updated in time by a technique that shares common parts with Finite element technique and Finite volume methods.

More precisely, we borrow from the FV frame work the idea of monotonicity preserving schemes and from the FE framework the idea of residual that permits to achieve high order accuracy. in the most impact possible way. To achieve the goal, we construct several monotone (first order) schemes and use them as a comparison principle (comparison with a high order residual). The current versions of the scheme are from second to fourth order in space and first to third order in time.

Examples are presented: scalar hyperbolic problems, 2D flow problems etc.

Discrete Entropy Inequalities in MHD

Tim Barth

Discrete entropy inequalities for the discontinuous Galerkin (DG) finite element discretization [12, 6, 5, 4] of first-order systems of nonlinear conservation laws are obtained subject to restrictions on the DG numerical flux function. This work generalizes to symmetrizable systems of conservation laws the cell square-entropy inequality result for scalar conservation laws given in Jiang and Shu [9]. The analysis is then expanded to include involution constraints such as those occurring in magnetohydrodynamics. Starting from the prototype conservation law system in $\mathbf{R}^d$

$$\mathbf{u}_{,t} + \operatorname{div} \mathbf{f} = 0, \quad \mathbf{u}, \mathbf{f}_i \in \mathbf{R}^m \ , \ i = 1, \ldots, d$$

with a convex entropy-entropy flux pair $\{U, F\}$ satisfying

$$U_{,t} + \operatorname{div} F \leq 0, \quad U, F_i \in \mathbf{R} \ , \ i = 1, \ldots, d \ ,$$

discrete energy analysis is performed for the DG method in terms of entropy symmetrization variables $\mathbf{v}(\mathbf{u})$ (see Barth [1]). Let $\mathcal{V}^h$ denote a finite dimensional approximation space consisting of piecewise polynomials of degree $\leq k$ in each element K of a mesh $\mathcal{T}$

$$\mathcal{V}^h = \left\{ \mathbf{v}_h \mid \mathbf{v}_h|_K \in \Big(\mathcal{P}_k(K)\Big)^m \text{ for each } K \in \mathcal{T} \right\} \ .$$

Using this approximation space, cell entropy inequalities of the form

$$\int_K U_{,t}(\mathbf{v}_h) \ dx + \int_{\partial K} \mathcal{F}(\mathbf{v}_h; \mathbf{n}) \ d s \leq 0 \ , \ \mathbf{v}_h \in \mathcal{V}^h \tag{1}$$

with conservative numerical entropy flux $\mathcal{F}(\mathbf{v}_h; \mathbf{n})$ are obtained whenever the numerical flux $\mathbf{h}(\mathbf{v}_-, \mathbf{v}_+; \mathbf{n})$ satisfies the system generalization of Osher's E-flux condition across interelement interfaces

$$[\mathbf{v}]_{x_-}^{x^+} \cdot (\mathbf{h}(\mathbf{v}_-, \mathbf{v}_+; \mathbf{n}) - \mathbf{f}(\mathbf{v}) \cdot \mathbf{n}) \leq 0 \ , \quad \forall \mathbf{v} \in [\mathbf{v}_-, \mathbf{v}_+] \ .$$

As an alternative to the system E-flux condition, we define a zero entropy dissipation (ZED) flux given by

$$\mathbf{h}_{\mathrm{ZED}}(\mathbf{v}_-, \mathbf{v}_+; \mathbf{n}) = \langle \mathbf{f} \cdot \mathbf{n} \rangle_{x_-}^{x^+} + \frac{1}{2} \int_0^1 (1 - 2\theta) \, \mathbf{f}_{,\mathbf{v}}(\mathbf{v}_{x_-} + \theta [\mathbf{v}]_{x_-}^{x^+}) \, [\mathbf{v}]_{x_-}^{x^+} \, d\theta$$

so that a cell entropy inequality is obtained for any numerical flux satisfying the comparison principle

$$[\mathbf{v}]_{x_-}^{x^+} \cdot \mathbf{h}(\mathbf{v}_-, \mathbf{v}_+; \mathbf{n}) \leq [\mathbf{v}]_{x_-}^{x^+} \cdot \mathbf{h}_{\mathrm{ZED}}(\mathbf{v}_-, \mathbf{v}_+; \mathbf{n}) \ .$$

In both analysis and practice, a suitable numerical flux is given by

$$\mathbf{h}_{\mathrm{mv}}(\mathbf{v}_-, \mathbf{v}_+; \mathbf{n}) = \langle \mathbf{f} \cdot \mathbf{n} \rangle_-^+ - \frac{1}{2} \int_0^1 |\mathbf{f}_{,\mathbf{v}}(\theta)|_{\mathbf{u},\mathbf{v}} \, [\mathbf{v}]_{x_-}^{x^+} \, d\theta$$

which is the symmetric origin of the Osher and Solomon flux [10].

We then expand the scope of the analysis to include a first-order system of conservation laws with solenoidal involution (see Dafermos [7] and Boillat [2]) with specific application to compressible magnetohydrodynamics (MHD) with involution constraint $\operatorname{div}\mathbf{B} = 0$ for the magnetic induction field. The objective is to again show how entropy analysis plays an invaluable role in designing numerical fluxes and stabilization terms. The analysis utilizes the MHD symmetrization theory due to Godunov [8] which symmetrizes the augmented MHD system

$$\mathbf{u}_{,t} + \operatorname{div} \mathbf{f} + \phi^T_{,\mathbf{v}} \operatorname{div} \mathbf{B} = 0 \ , \quad \phi(\mathbf{v}) : \mathbf{R}^m \mapsto \mathbf{R}$$

where $\phi(\mathbf{v})$ is a given function from the Godunov theory. Using this theory and tools previously developed, we prove a cell entropy inequality of the form (1) for MHD solutions computed using the discontinuous Galerkin method with numerical flux satisfying the system E-flux condition or ZED flux comparison principle whenever interelement continuity of the normal component of the magnetic induction field is enforced. This is readily accomplished using families of Raviart-Thomas [11] or Brezzi-Douglas-Marini [3] finite elements for the magnetic induction field.

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A strongly degenerate convection-diffusion equation with discontinuous coefficients modelling clarifier-thickener units

Raimund Bürger
(joint work with Kenneth H. Karlsen and John D. Towers)

Mathematical models for continuous clarification-thickening processes of ideal suspensions extend Kynch's kinematic sedimentation model [6] to continuously operated units. Within this theory, the suspension is described by a nonlinear flux density function for a scalar conservation law. In the clarifier-thickener setup, a feed source is located between the overflow and underflow levels. This gives rise to one zone located above the feed, the clarification zone with an upwards-directed bulk flow, and one zone located below the feed, the thickening zone with a donwards directed bulk flow. The overflow and underflow levels correspond to transitions from a composite nonlinear flux to a linear transport flux. The result is a nonlinear conservation law with a flux function which is discontinuous at the feed, underflow and overflow levels. This property makes standard conservation law theory inapplicable. The well-posedness of this model and the convergence of a finite-difference scheme are proved in [2, 3].

Most real-world solid-liquid suspensions do not fall within the kinematic theory. Rather, they consist of small flocs that give rise to compressible sediments as the local solids concentration exceeds a critical value. This effect is modeled by an effective solid stress function [1]. Thus, the material behaviour of the mixture is described by two functions, the flux density function and the effective solid stress, which define a strongly degenerate parabolic PDE for the solids concentration.

In a very recent analysis [4, 5], the discontinuous flux clarifier-thickener setup and the degenerate diffusion term to model sediment compressibility, have been combined into a model for clarifier-thickeners treating a flocculated suspension. The resulting governing problem can be stated as follows.

The governing equation is the convection-diffusion equation

$$u_t + f(\boldsymbol{\gamma}(x), u)_x = (\gamma_1(x)A(u)_x)_x\,,\ (x,t) \in \mathbb{R} \times (0,T), \tag{1}$$

where t is time, x is the depth variable, u is the sought volumetric solids concentration, and $A(u)$ is a monotonically increasing Lipschitz continuous function with $A(u) = 0$ for $u \leq u_\mathrm{c}$ with $0 < u_\mathrm{c} < 1$. This function models the sediment compressibility, and u_c is a critical concentration at which the solid particles are assumed to touch each other. Clearly, (1) degenerates to first-order hyperbolic type for $u < u_\mathrm{c}$.

The convective flux function is given by

$$f(\boldsymbol{\gamma}, u) = \gamma_1 b(u) + \gamma_2(u - u_\mathrm{F}),$$

where $b(u)$ is a material specific, non-negative Lipschitz continuous function (the so-called hindered settling function) with compact support in the interval $u \in [0,1]$ of admissible volume fractions and u_F is the concentration at which the clarifier-thickener unit (whose interior occupies the interval $(x_\mathrm{L} < 0, x_\mathrm{R} > 0)$) is fed through a feed inlet located at $x = 0$. The vector $\boldsymbol{\gamma} = (\gamma_1, \gamma_2)$ consists of the discontinuous parameters

$$\gamma_1(x) := \begin{cases} 1 & \text{for } x \in (x_{\mathrm{L}}, x_{\mathrm{R}}), \\ 0 & \text{for } x \notin (x_{\mathrm{L}}, x_{\mathrm{R}}), \end{cases} \qquad \gamma_2(x) := \begin{cases} q_{\mathrm{L}} & \text{for } x \leq 0, \\ q_{\mathrm{R}} & \text{for } x > 0, \end{cases}$$

where the parameter γ_1 describes that the nonlinear and diffusive parts of the flux are effective in the interior of the clarifier-thickener only, and the parameter γ_2 accounts for the division of the mixture feed flux into an upwards-directed transport flux with the velocity $q_{\mathrm{L}} < 0$ and a downwards-directed flux with the velocity $q_{\mathrm{R}} > 0$. The governing equation is studied together with the initial condition $u(x, 0) = u_0(x)$ for $x \in \mathbb{R}$.

We introduce a simple finite-difference scheme and prove its convergence to a weak solution that satisfies an entropy condition. A limited analysis of steady states as desired stationary modes of operation is performed. Numerical examples illustrate that the model realistically describes the dynamics of flocculated suspensions in clarifier-thickeners.

This talk is based in the papers [4, 5].

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Navier–Stokes–Korteweg equations, Augmented formulations and numerical applications

Frédéric Coquel

The present work is devoted to the numerical approximation of the solution of a mixed (hyperbolic–elliptic) non linear first order system with viscous dispersive perturbations. Such a system arises in the modelling of compressible medium undergoing phase transitions. Endpoints of travelling wave solutions are known to be sensitive to the viscous–dispersive regularisations when kept in balance. The work is precisely devoted to the development of numerical methods capable of capturing the reported sensitiveness. Following LeFloch, we propose to control the evolution in time of the discrete entropy rate of dissipation encoding precisely the sensitiveness. In that way, we propose to circumvent an unusual dependance of the entropy with respect to the gradient of the unknown when introducing an augmented formulation of the original PDEs. This augmented system possesses a consistent augmented entropy pair and restores the solutions (when smooth) of the original PDEs when the initial data are suitably prescribed. We then show how to approximate the solutions of the original PDEs by those of the augmented formulation when controlling sharply the entropy rate of dissipation.

Approximation of Hyperbolic Equations in Complex Geometries

Christiane Helzel
(joint work with Marsha J. Berger and Randall J. LeVeque)

Many applications require the approximation of hyperbolic equations in complex geometry. While often body fitted grids are used that conform to the geometry of the problem, we are particularly interested in developing numerical methods where the computational domain is embedded in a uniform Cartesian grid. Such an approach is attractive, since away from the boundary it allows the use of Cartesian grid high-resolution shock capturing methods that are by now well developed. Furthermore, even for problems with more complicated geometry a Cartesian grid embedded boundary method allows an efficient and automatic grid generation and the use of structured adaptive mesh refinement.

The numerical challenge associated with a Cartesian grid embedded boundary approach is the so-called *small cell problem*. Near the embedded boundary the irregular grid cells may be orders of magnitude smaller than regular Cartesian grid cells. Since stability theory for standard explicit finite volume methods suggests that the time step is proportional to the size of the grid cell, this would typically require small time steps near the embedded boundary. Therefore, our goal is to construct numerical methods that overcome the time step restriction at the embedded boundary and allow time steps that are appropriate for the regular part of the domain. It turns out that an even more difficult problem is to retain high accuracy near the boundary.

During the last decade several different Cartesian grid embedded boundary methods for the approximation of hyperbolic problems have been developed. Several authors use a cell merging technique where small irregular cut cells are merged together with a neighboring regular grid cell, see for instance [3], [7]. Other approaches are based on flux redistribution [6] or handle boundary cells as full cells [4]. So far these schemes have not been carried out to high-order accuracy at the boundary.

In my talk I have presented an embedded boundary method that overcomes the time step restriction while seeking an accurate approximation near the boundary as well as in the whole domain. Details of this method can be found in our papers [1] and [5]. Our approach to overcome the small cell problem is based on the so-called h-box method suggested in earlier work by Berger and LeVeque [2]. The basic idea behind this method is to approximate numerical fluxes at the interface of a small cell using initial values specified over regions of length h, where h depends on the size of a regular Cartesian grid cell. This leads to a finite volume method in which the flux difference in each grid cell is of the order of the size of the grid cell. Therefore we can hope that the size of the grid cell arising in the denominator of the finite volume method does not cause a stability problem. The accuracy of the h-box method depends strongly on the definition of the values of the conserved quantities assigned to the h-boxes. In [1] we have studied the construction of h-box methods in a relatively simple one dimensional context where most of the grid cells have the reference grid cell length h but some grid cells may be orders of magnitude smaller. For the advection equation we could show that our method leads to a second order accurate approximation of smooth solutions on non-uniform grids without any restrictions on the grid. Furthermore, we proved stability of the second order method under a CFL condition that only depends on the size of the regular grid cells. Numerical tests confirmed the same properties for the approximation of nonlinear systems, e.g. the Euler equations of gas dynamics. In [5] we have developed a second order accurate rotated grid h-box method that can handle embedded geometries. Here the calculation of numerical fluxes by using the h-box idea again leads to the required stability property. In the multi-dimensional situation a rotated grid method was necessary to also retain conservativity. Furthermore, the rotated grid method was constructed in a way that leads to a second order accurate approximation of the solution in the whole domain as well as in the cut cells at the embedded boundary. This makes our rotated grid h-box method more accurate than any other existing Cartesian grid embedded boundary method. The performance of the method was illustrated by several numerical test calculations.

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Computation of multivalued solutions to nonlinear PDEs

Shi Jin

Many physical problems arising from high frequency waves, dispersive waves or Hamiltonian systems require the computations of multivalued solutions which cannot be described by the viscosity methods. In this talk I will review several recent numerical methods for such problems, including the moment methods, kinetic equations and level set methods. Applications to the semiclassical Schroedinger equation and Euler-Piosson equations with applications to modulated electron beams in Klystrons, and general symmetric hyperbolic systems will be discussed.

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Adaptive Central-Upwind Schemes for Nonconvex Hyperbolic Conservation Laws

Alexander Kurganov
(joint work with Guergana Petrova, Bojan Popov)

We consider nonconvex hyperbolic conservation laws. The two main examples are scalar equations with nonconvex fluxes and the Euler equations of gas dynamics with general, possibly nonconvex, equation of state.

We develop Godunov-type central-upwind schemes for such class of problems. Central-upwind schemes [1], like other Godunov-type schemes, are projection-evolution methods: piecewise polynomial reconstructions are evolved in time according to the integral form of conservation laws. In the central-upwind framework, the evolution is carried out using the control volumes that include Riemann fans so that no Riemann problem solvers are required. The sizes of the corresponding Riemann fans are determined with the help of one-sided local speeds of propagation related to the largest and smallest eigen-values of the Jacobians. In the nonconvex case, a careful evaluation of these speeds helps to avoid oscillations provided a piecewise polynomial reconstruction is (essentially) non-oscillatory.

However, if the employed reconstruction is compressive, a computed non-oscillatory solution may converge to an unphysical weak solution. We have demonstrated this on several scalar examples, where the exact entropy solution is available, and on several gas dynamics examples. At the same time, using a dissipative reconstruction, which happens to lead to the convergence of the computed solution to the exact entropy solution, may significantly affect the quality of the achieved resolution. To overcome this difficulty, we propose a new class of adaptive central-upwind schemes that employ a dissipative limiter near the points where the convexity changes, and a more compressive limiter in the rest of the computational domain.

Our numerical results clearly demonstrate high resolution, accuracy, and robustness of the proposed methods.

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Geometrical Solutions Vs. Entropy Solutions

Hailiang Liu

We introduce a new level set method for computational high frequency wave propagation in dispersive media, with an application to semiclassical limit for Schrödinger equations with high frequence initial data. We discuss also level set methods for general first order equations and identify a new notion – geometrical solutions.

Finally we show both entropy solutions (for conservation laws) and viscosity solution (for Hamilton–Jacobi equations) are just part of the geometrical solution characterized by the level set formulations.

Numerical modelling of the shallow water equations and magnetohydrodynamic shallow water equations by the finite volume evolution Galerkin methods

M. Lukáčová - Medviďová
(joint work with T. Kröger and Z. Vlk)

The goal of this contribution is to present a generalization of the finite volume evolution Galerkin (FVEG) methods for the

- shallow water equations with source terms which model the bottom elevation
- shallow water magnetohydrodynamic equations.

The FVEG methods were introduced and studied extensively for hyperbolic conservation laws by Lukáčová, Morton and Warnecke, see [1]-[4] and the references therein. These methods couple a finite volume formulation with approximate evolution operators which are based on the theory of bicharacteristics for the first order systems. As a result exact integral equations for linear or linearized hyperbolic conservation laws can be derived. They take all of the infinitely many directions of wave propagation into account. For two-dimensional conservation laws this is realized by the integration along the sonic circle, i.e. for a parameter $\theta \in [0, 2\pi]$. Further integrals appearing in the exact integral equations are the integrals along time, e.g. from t_n to t_{n+1}. Since the exact integral equations are implicit in time appropriate numerical quadratures have to be applied in time in order to approximate integrals along the mantle of the so-called bicharacteristic cones. This yields the approximate evolution operators. In the finite volume framework the approximate evolution operators are used to evolve the solution along the cell interfaces in order to compute fluxes on edges. This step can be considered as a predictor step. In the corrector step the finite volume update is done. In summary, the FVEG scheme is a genuinely multi-dimensional method that is explicit in time.

• Similarly to other balance laws the shallow water equations with source term admit a steady-state solutions in which nonzero flux gradients are exactly balanced by the source terms. For correct approximation of these states a delicate treatment of the source term is necessary. In the framework of the FVEG methods we have derived the *well-balanced approximate evolution operator*. It means that the exact evolution operator, which includes the evolution of the source term, has been approximated in such a way that the resulting operator is stable with respect to small perturbations of velocity. In particular if the velocity $u \approx 0,\ v \approx 0$ then we have for the depth of the shallow water and the bottom elevation $h + b \approx const.$ Predicted values are used for the flux calculation along the cell interfaces. Further, in order to keep the well-balance character of the discretization scheme the finite volume update of the source term need to be done using the cell-interface values instead of the cell-centered ones. Numerical experiments confirm reliability of the described approach, cf. [5].

Another critical case, which need to be considered in the derivation of the well-balanced schemes is the so-called *drain on the bottom*. If depth of the shallow water $h \approx 0$ even small oscillations may result in the negative values of h and it will be impossible to compute characteristic speeds $u \pm \sqrt{gh}$. We can show that under some stablity condition our scheme is positivity preserving, i.e. $h \geq 0$.

• Further application of the FVEG scheme that we are interested in is the *shallow water magnetohydrodynamic (SMHD) equations*. These equations are used to model solar tachocline, i.e. a thin layer of the solar radius that separates the convective zone from the radiative zone in stars. In cooperation with T. Kröger [6] we have derived the exact and approximate evolution operators for the system of the SMHD equations in two space dimensions. Up to our knowledge, this is the first attempt to apply genuinely multi-dimensional EG technique to a magnetohydrodynamic model. We have studied more deeply the approximation of the spatial derivatives in the evolution operator for singular wave modes, for which the wave front concentrates to a point, as well as for non-singular wave modes. More precisely, we can show that for arbitrary hyperbolic conservation laws, the spatial derivatives of the solution can be replaced by means of the Gauß theorem with the derivatives of the eigenvectors themselves.

Due to the complex eigenstructure which arises in the SMHD system, it is still rather complicated to apply this result directly. Instead we propose to exploit this result numerically. Our numerical experiments confirm the reliability of this approach for non-singular wave modes. Treatment of the singular wave modes is more delicate. Our numerical experiments show that the approximation of the derivatives by slopes of the bilinear reconstruction yields the best results. We prove also that using a trapezoidal quadrature for cell-interface flux integrals in the magnetic part, i.e. the Maxwell equations, a discrete version of the divergence free constraint is satisfied automatically at vertices of our regular mesh.

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Weak solutions and numerical approximation of two phase flow problems with spatial discontinuity of capillary pressure

A. Michel

(joint work with G.Enchery, R. Eymard)

Reservoir or sedimentary basin simulations lead to the modelling of two-phase flows through heterogeneous porous media. The heterogeneities are related to the presence of different geological layers and can entail a phenomenon of capillary entrapment.

Thus, in mathematical models, the relative permeabilities and the capillary pressure are both functions of the saturation and of the space variable. Moreover, they can be discontinuous regardless to the space variable.

Here we consider a porous medium Ω shared in two homogeneous parts Ω_i, $i = 1, 2$, each of them being characterised by its porosity ϕ_i, its relative mobility η_i and its capillary curve π_i. Focusing on the capillary forces, the oil saturations $u_i(x,t), (x,t) \in \Omega_i \times (0,T)$, in each domain are solutions of the following equations:

$$
\begin{aligned}
(1) \qquad \phi_i \frac{\partial u_i}{\partial t} - \Delta\varphi_i(u_i) &= 0 \text{ in } \Omega_i \times (0,T), \\
(2) \qquad \eta_i \nabla \pi_i(u_i).\mathbf{n} &= 0, \text{ on } (\partial\Omega \setminus \Gamma) \times (0,T), \text{ for all } i \in \{1,2\}.
\end{aligned}
$$

Moreover, the following conditions must be satisfied on the interface Γ between Ω_1 and Ω_2:

$$
\begin{aligned}
&(3) \quad \nabla\varphi_1(u_{1,\Gamma}).\mathbf{n}_{1,\Gamma} = -\nabla\varphi_2(u_{2,\Gamma}).\mathbf{n}_{2,\Gamma} \text{ on } \Gamma \times (0,T), \text{ [flux continuity]} \\
&(4) \quad \hat{\pi}_1(u_{1,\Gamma}) = \hat{\pi}_2(u_{2,\Gamma}) \text{ on } \Gamma \times (0,T), \text{ [extended pressure continuity]}
\end{aligned}
$$

where $\varphi_i(u) = \int_0^u \eta_i(a)\pi_i'(a)da$ is a capillary diffusion term and $\hat{\pi}_1$ and $\hat{\pi}_2$ are truncated capillary pressures curves. In the usual case where π_1 and π_2 are such that $\pi_1(0) \le \pi_2(0) \le \pi_1(1) \le \pi_2(1)$ then $\hat{\pi}_1(u) = \max(\pi_1(u), \pi_2(0))$ and $\hat{\pi}_2(u) = \min(\pi_2(u), \pi_1(1))$.

Problem (1)–(4) has already been handled by Bertsch, Passo, and van Duijn [1] or van Duijn, Molenaar, and de Neef [2], especially in 1D.

In [3] we give a weak form of the problem (1)–(4) and we propose a numerical method based on a finite volume scheme with two secondary unknowns on the interface. This scheme is shown to converge to a weak solution of this problem for multidimensional bounded domains. Under strong regularity assumptions, it is easy to adapt the uniqueness proof given in [1] for a 1D problem to the multi-dimensional case.

The question of uniqueness without strong regularity assumptions is more difficult. By using Kruzkov techniques, Karlsen and Ohlberger have already obtained uniqueness and error estimates for problems with spatially dependant diffusion coefficients. Is it possible to adapt these techniques to the case where the diffusion function itself depends on the space variables ? To our knowledge, it is an open question.

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Error estimate for the approximation of non-linear conservation laws on bounded domains by the finite volume method

Mario Ohlberger
(joint work with Julien Vovelle)

Let Ω be an open convex polygonal bounded domain in $\mathbb{R}^d$, $d = 2, 3$, endowed with the Euclidean norm $|\cdot|$ and let $T \in \mathbb{R}^+$. We consider the following initial boundary value problem for non-linear scalar conservation laws

$$c_t + \nabla \cdot \mathbf{F}(x,t,c) = 0 \text{ in } \Omega \times (0,T), \tag{1}$$

$$c(\cdot,0) = c_0 \text{ in } \Omega, \tag{2}$$

$$c(x,t) = \bar{c}(t,x) \text{ in } \partial\Omega \times (0,T)\,. \tag{3}$$

Since (1) is a evolution equation, the main features of the analysis of conservation laws and of their approximations by the finite volume method already appear in the context of the Cauchy problem. The order of accuracy of the finite volume method for the Cauchy problem is one of these well-known features: the first given *a priori* error estimate is the (sharp) $h^{1/2}$ (h being the size of the mesh) estimate of Kuznetsov [6] in the 1D case. This estimate remains valid on structured meshes in $\mathbb{R}^d$ while, for finite volume schemes on unstructured meshes, the lack of an uniform BV estimate on the numerical solution leads to an error estimate of reduced order $h^{1/4}$ (see for example [2]). Still, in the context of the Cauchy problem, refined error estimates have been given (and their sharpness analyzed) according to the genuine non-linearity of the flux, to the structure of the entropy solution to (1)-(2), or to the nature of the waves in the solution. We refer to the discussion and compilation made by T. Tang on that profuse subject [11].

For practical applications a posteriori error estimates are even more important than just convergence rates. Such estimates allow to extract error indicator information that can be used in order to derive efficient self adaptive strategies for the finite volume schemes. A posteriori error estimates for finite volume approximations to the Cauchy problem were derived in [10, 3, 5, 4].

Although the study of the finite volume method applied to the Cauchy problem has led to the understanding of most of the mechanisms which govern the accuracy of this numerical method of approximation, the initial-boundary value problem has its own interest and its approximation by finite volume schemes deserves an analysis. With that purpose in mind, notice that a new and characteristic feature of the approximation of the initial-boundary value problem by a finite volume scheme is the possible creation of a numerical boundary layer. This numerical boundary layer is a sub-product of the numerical diffusion effects induced by the scheme. The study of the numerical boundary layer has been performed by C. Chainais-Hillairet and E. Grenier [1], in the 1D case and for modified Lax-Friedrichs schemes on cartesian grids in the multi-D case. Such an analysis gives a precise description of the numerical solution and, as a consequence, the speed of convergence of this solution to the entropy solution of the problem (1)–(3). In the non-characteristic case with smooth exact solutions, this speed of convergence is proved to be of order h in the $L^\infty(0,T;L^1(\Omega))$ norm, where h is the size of the mesh.

Unfortunately, the techniques of numerical boundary layer analysis seem difficult to be set when no selected direction of (discrete) derivation exists, as is the case when finite volume schemes on unstructured meshes are used. For such schemes one can therefore think to adapt the technique developed by Kuznetsov [6] for the analysis of the Cauchy problem in the framework of the initial boundary value problem to get error estimates, with the drawback that this tool is not accurate at all to take into account the special phenomena at the boundary of the domain. In the specific situation $\mathbf{F}(x,t,c) = \mathbf{u}(x,t)f(c)$ with f monotone, this drawback can be overcome, for the reason that the inflow and outflow parts of the boundary are determined *a priori* by the given velocity field $\mathbf{u}$. In [12], Vignal gives an a priori error estimate of order $h^{1/4}$ for the initial boundary value problem. However, to our knowledge, for general fluxes $\mathbf{F}$, and general schemes on possibly unstructured meshes, no results or techniques of error estimates which account for the influence of the boundary condition have been delivered. In order to fill in this gap, we adapt the technique of Kuznetsov [6] to the proof of uniqueness of the entropy solution given by F. Otto [9, 7, 13] and prove that the error can be estimated by an *a posteriori error bound* which is at least of order $h^{1/6}$ for meshes with mesh size h. In order to obtain this new result, we also prove that the exact entropy solution of problem (1)–(3) admits BV-solutions on convex polygonal bounded domains. In addition, an adaptive strategy is derived from the a posteriori result and numerical experiments are given for the resulting adaptive finite volume method. The detailed results of this work can be found in [8]

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A High-Resolution Constrained Transport Method for Astrophysical Flows

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The ideal magnetohydrodynamic (MHD) equations model the dynamics of electrically conducting fluids. These equations are important in modeling phenomena in a wide range of applications including space weather, solar physics, laboratory plasmas, and astrophysical fluid flows. In conservation form, the MHD system can be written as

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\ \rho\vec{u}\\ \mathcal{E}\\ \vec{B}\end{bmatrix} + \vec{\nabla}\cdot\begin{bmatrix}\rho\vec{u}\\ \rho\vec{u}\otimes\vec{u} + \left(p+\frac{1}{2}|\vec{B}|^2\right)\mathcal{I} - \vec{B}\otimes\vec{B}\\ \vec{u}\left(\mathcal{E}+p+\frac{1}{2}|\vec{B}|^2\right) - \vec{B}(\vec{u}\cdot\vec{B})\\ \vec{u}\otimes\vec{B} - \vec{B}\otimes\vec{u}\end{bmatrix} = 0, \tag{1}$$

$$\vec{\nabla}\cdot\vec{B} = 0\,, \tag{2}$$

where

$$p = (\gamma-1)\left(\mathcal{E} - \frac{1}{2}\rho|\vec{u}|^2 - \frac{1}{2}|\vec{B}|^2\right). \tag{3}$$

Here ρ is the mass density, $\vec{u} = (u_1, u_2, u_3)^t$ is the velocity field, $\mathcal{E}$ is the total energy, $\vec{B} = (B_1, B_2, B_3)^t$ is the magnetic field, p is the thermal pressure, $(1/2)|\vec{B}|^2$ is the magnetic pressure, and γ is the ideal gas constant. Equation (3) is the *equation of state* for an ideal gas and closes the system by relating the pressure to the other unknowns. It is also sometimes beneficial to introduce a magnetic potential, $\vec{\mathbb{A}}$, that is related to the magnetic field in the following way:

$$\vec{B} = \vec{\nabla}\times\vec{\mathbb{A}}. \tag{4}$$

The ideal MHD equations form a system of hyperbolic conservation laws, (1), with a constraint, (2). Furthermore, any exact solution to (1) automatically satisfies (2) provided that the initial magnetic field obeys the divergence-free constraint [15]. The main difficulty in numerically solving the MHD system is that most numerical schemes only satisfy constraint (2) to the truncation error of the method. Across shocks this can lead to a divergence error that is $\mathcal{O}(1)$; this in turn can lead to large spurious oscillations and negative pressures [4]. Once negative pressures are encountered the system fails to be hyperbolic and the numerical code will crash.

Several modifications to standard high-resolution shock-capturing schemes have been introduced to overcome this difficulty including the projection method [4], constrained transport [7], operator splitting [16], the 8-wave formulation [10, 11], and hyperbolic divergence-cleaning [6]. In recent years, the constrained transport method originally proposed by Evans and Hawley [7] has gained much attention (see Tóth [15] for a review). In this approach, a staggered magnetic field is introduced that can be computed from a staggered magnetic potential in such a way that it satisfies the constraint to machine precision in each grid cell for all time. Several variants of the original constrained transport approach have been introduced, each with a different strategy for updating the magnetic potential [1, 2, 5, 8, 9, 13, 14]. Unstaggered versions of a few of these methods were introduced in [15]. All of these methods automatically satisfy (2), but can sometimes fail to produce a non-oscillatory magnetic field. This is largely due to the fact that in these methods one is not able to directly apply flux limiters in the update of the magnetic field, at least in the usual sense of applying TVD limiters in high-resolution methods.

In [12] a wave propagation method is introduced that utilizes a novel constrained transport technique to keep the magnetic field divergence-free. This approach is based on directly solving a hyperbolic equation for the magnetic potential in conjunction with a new limiting strategy to obtain a non-oscillatory magnetic field. Like the methods of Tóth [15], this new approach does not use a staggered grid.

In this work we apply the constrained transport method of [12] to several test problems for ideal MHD as well as special relativistic MHD (SRMHD). Because the numerical grid is completely unstaggered, it is also relatively straightforward to incorporate it into an adaptive mesh refinement framework such as AMRCLAW [3]; results using AMRCLAW are also provided.

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Entropy Stability Theory for Nonlinear Conservation Laws

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Abstract. We introduce general families of entropy-conservative schemes, interesting in their own right. The present treatment of such schemes in [Tadmor, 2003], extends our earlier recipe for construction of entropy-conservative schemes, introduced in 1987. Entropy stability can be enforced on rarefactions while keeping the sharp resolution of shock discontinuities. A comparison with the numerical viscosities associated with these entropy-conservative schemes provides a useful framework for the construction and analysis of entropy-stable schemes.

We consider semi-discrete conservative schemes of the form

$$\frac{d}{dt}\mathbf{u}_\nu(t) = -\frac{1}{\Delta x_\nu}\Big[\mathbf{f}_{\nu+\frac{1}{2}} - \mathbf{f}_{\nu-\frac{1}{2}}\Big], \tag{2}$$

serving as consistent approximations to systems of conservation laws of the form

$$\frac{\partial}{\partial t}\mathbf{u} + \frac{\partial}{\partial x}\mathbf{f}(\mathbf{u}) = 0, \quad (x,t) \in \mathbb{R} \times [0,\infty), \tag{3}$$

where $\mathbf{f}(\mathbf{u}) = (f_1(\mathbf{u}), \ldots, f_N(\mathbf{u}))^\top$ are smooth flux functions of the N-vector of conservative variables $\mathbf{u}(x,t)$.

Let (U,F) be an entropy pair associated with the system (3). We ask whether the scheme (2) is *entropy-stable* with respect to such a pair, in the sense of satisfying a discrete entropy inequality analogous to the entropy inequality $U(\mathbf{u})_t + F(\mathbf{u})_x \leq 0$,

$$\frac{d}{dt}U(\mathbf{u}_\nu(t)) + \frac{1}{\Delta x_\nu}\Big[F_{\nu+\frac{1}{2}} - F_{\nu-\frac{1}{2}}\Big] \leq 0. \tag{4}$$

Here, $F_{\nu+\frac{1}{2}} = F(\mathbf{u}_{\nu-p+1}, \ldots, \mathbf{u}_{\nu+p})$ is a consistent numerical entropy flux, with $F(\mathbf{u}, \mathbf{u}, \ldots, \mathbf{u}) = F(\mathbf{u})$. If, in particular, equality holds in (4), we say that the scheme (2) is *entropy-conservative.*

The answer to this question of entropy stability provided in [1] consists of two main ingredients: (i) the use of the entropy variables and (ii) the comparison with appropriate *entropy-conservative* schemes.

Define the *entropy variables* $\mathbf{v} \equiv \mathbf{v}(\mathbf{u}) := \nabla_{\mathbf{u}} U(\mathbf{u})$. Thanks to the convexity of $U(\mathbf{u})$, the mapping $\mathbf{u} \to \mathbf{v}$ is one-to-one and hence we can make the change of variables $\mathbf{u} = \mathbf{u}(\mathbf{v})$, which puts the scheme (2) into the equivalent form

$$\frac{d}{dt}\mathbf{u}_\nu(t) = -\frac{1}{\Delta x_\nu}\Big[\mathbf{g}_{\nu+\frac{1}{2}} - \mathbf{g}_{\nu-\frac{1}{2}}\Big], \quad \mathbf{u}_\nu(t) = \mathbf{u}(\mathbf{v}_\nu(t)), \tag{5}$$

with a consistent numerical flux

$$\mathbf{g}_{\nu+\frac{1}{2}} = \mathbf{g}(\mathbf{v}_{\nu-p+1}, \ldots, \mathbf{v}_{\nu+p}) := \mathbf{f}(\mathbf{u}(\mathbf{v}_{\nu-p+1}), \ldots, \mathbf{u}(\mathbf{v}_{\nu+p}))$$

consistent with $\mathbf{g}(\mathbf{v}) := \mathbf{f}(\mathbf{u}(\mathbf{v}))$. The compatibility relation $U_{\mathbf{u}}^\top \mathbf{f}_{\mathbf{u}} = F_{\mathbf{u}}^\top$ implies the existence of a potential $\psi(\mathbf{v})$ given by, $\psi(\mathbf{v}) := \langle \mathbf{v}, \mathbf{g}(\mathbf{v})\rangle - F(\mathbf{u}(\mathbf{v}))$, such that $\mathbf{g}(\mathbf{v}) = \nabla_{\mathbf{v}}\psi(\mathbf{v})$. We are ready to quote the main result of [1].

Theorem 0.1. (Tadmor 1987) *The conservative scheme (5) is entropy-conservative if its numerical flux* $\mathbf{g} = \mathbf{g}^*$ *satisfies*

$$\Big\langle \Delta\mathbf{v}_{\nu+\frac{1}{2}}, \mathbf{g}^*_{\nu+\frac{1}{2}} \Big\rangle = \Delta\psi_{\nu+\frac{1}{2}}. \tag{6}$$

Next, we introduce a new general family of entropy-conservative schemes which admit an *explicit, closed-form formulation.* To this end, at each cell consisting of two neighbouring values $\mathbf{v}_\nu$ and $\mathbf{v}_{\nu+1}$, we let $\{\mathbf{r}^j_{\nu+\frac{1}{2}}\}_{j=1}^N$ be an *arbitrary* set of N linearly independent N-vectors, and let $\{\boldsymbol{\ell}^j_{\nu+\frac{1}{2}}\}_{j=1}^N$ denote the corresponding orthogonal set, $\langle \boldsymbol{\ell}^j_{\nu+\frac{1}{2}}, \mathbf{r}^k_{\nu+\frac{1}{2}}\rangle = \delta_{jk}$. Next, we introduce the intermediate states, $\{\mathbf{v}^j_{\nu+\frac{1}{2}}\}_{j=1}^N$, starting with $\mathbf{v}^1_{\nu+\frac{1}{2}} = \mathbf{v}_\nu$, and followed by $\mathbf{v}^{j+1}_{\nu+\frac{1}{2}} = \mathbf{v}^j_{\nu+\frac{1}{2}} + \Big\langle \boldsymbol{\ell}^j_{\nu+\frac{1}{2}}, \Delta\mathbf{v}_{\nu+\frac{1}{2}}\Big\rangle \mathbf{r}^j_{\nu+\frac{1}{2}}$, $j = 1, 2, \ldots, N$, thus defining a path in phase space, connecting $\mathbf{v}_\nu$ to $\mathbf{v}_{\nu+1}$,

$$\mathbf{v}^{N+1}_{\nu+\frac{1}{2}} = \mathbf{v}^1_{\nu+\frac{1}{2}} + \sum_{j=1}^N \Big\langle \boldsymbol{\ell}^j_{\nu+\frac{1}{2}}, \Delta\mathbf{v}_{\nu+\frac{1}{2}}\Big\rangle \mathbf{r}^j_{\nu+\frac{1}{2}} = \mathbf{v}_\nu + \Delta\mathbf{v}_{\nu+\frac{1}{2}} \equiv \mathbf{v}_{\nu+1}. \tag{7}$$

Since the mapping $\mathbf{u} \mapsto \mathbf{v}$ is one-to-one, the path is mirrored in the usual phase space of conservative variables, $\{\mathbf{u}^j_{\nu+\frac{1}{2}} := \mathbf{u}(\mathbf{v}^j_{\nu+\frac{1}{2}})\}_{j=1}^{N+1}$, starting with $\mathbf{u}^1_{\nu+\frac{1}{2}} = \mathbf{u}_\nu$ and ending with $\mathbf{u}^{N+1}_{\nu+\frac{1}{2}} = \mathbf{u}_{\nu+1}$.

Theorem 0.2. *The conservative scheme* $d\mathbf{u}_\nu(t)/dt = -\Big[\mathbf{g}^*_{\nu+\frac12} - \mathbf{g}^*_{\nu-\frac12}\Big]/\Delta x_\nu$, *with a numerical flux given by*

$$\mathbf{g}^*_{\nu+\frac12} = \sum_{j=1}^{N} \frac{\psi\big(\mathbf{v}^{j+1}_{\nu+\frac12}\big) - \psi\big(\mathbf{v}^{j}_{\nu+\frac12}\big)}{\big\langle \boldsymbol{\ell}^j_{\nu+\frac12}, \Delta\mathbf{v}_{\nu+\frac12}\big\rangle} \boldsymbol{\ell}^j_{\nu+\frac12}, \tag{8}$$

is an entropy-conservative approximation consistent with (3).

Proof. The entropy conservation requirement (6) follows directly from (7) for

$$\begin{aligned}\Big\langle \Delta\mathbf{v}_{\nu+\frac12}, \mathbf{g}^*_{\nu+\frac12}\Big\rangle &= \sum_{j=1}^{N} \frac{\psi\big(\mathbf{v}^{j+1}_{\nu+\frac12}\big) - \psi\big(\mathbf{v}^{j}_{\nu+\frac12}\big)}{\big\langle \boldsymbol{\ell}^j_{\nu+\frac12}, \Delta\mathbf{v}_{\nu+\frac12}\big\rangle} \Big\langle \boldsymbol{\ell}^j_{\nu+\frac12}, \Delta\mathbf{v}_{\nu+\frac12}\Big\rangle \\ &= \psi\Big(\mathbf{v}^{N+1}_{\nu+\frac12}\Big) - \psi\Big(\mathbf{v}^{1}_{\nu+\frac12}\Big) = \Delta\psi_{\nu+\frac12}.\end{aligned}$$

It remains to verify the consistency relation. Let

$$\mathbf{v}^{j+\frac12}_{\nu+\frac12}(\xi) := \frac12\Big(\mathbf{v}^{j}_{\nu+\frac12} + \mathbf{v}^{j+1}_{\nu+\frac12}\Big) + \xi\Big\langle \boldsymbol{\ell}^j_{\nu+\frac12}, \Delta\mathbf{v}_{\nu+\frac12}\Big\rangle \mathbf{r}^j_{\nu+\frac12},$$

denote the straight subpath connecting $\mathbf{v}^{j}_{\nu+\frac12}$ and $\mathbf{v}^{j+1}_{\nu+\frac12}$ for $-\frac12 \le \xi \le \frac12$; then we express the ψ-potential jump between two consecutive intermediate states as

$$\begin{aligned}\psi\Big(\mathbf{v}^{j+1}_{\nu+\frac12}\Big) - \psi\Big(\mathbf{v}^{j}_{\nu+\frac12}\Big) &= \int_{\xi=-\frac12}^{\frac12} \frac{d}{d\xi}\psi\Big(\mathbf{v}^{j+\frac12}_{\nu+\frac12}(\xi)\Big)d\xi \\ &= \Big\langle \int_{\xi=-\frac12}^{\frac12} \mathbf{g}\Big(\mathbf{v}^{j+\frac12}_{\nu+\frac12}(\xi)\Big)d\xi, \mathbf{r}^j_{\nu+\frac12}\Big\rangle \Big\langle \boldsymbol{\ell}^j_{\nu+\frac12}, \Delta\mathbf{v}_{\nu+\frac12}\Big\rangle.\end{aligned} \tag{9}$$

Inserting this into (8), we find that the entropy-conservative flux can be equivalently written as

$$\mathbf{g}^*_{\nu+\frac12} = \sum_{j=1}^{N} \Big\langle \int_{\xi=-\frac12}^{\frac12} \mathbf{g}\Big(\mathbf{v}^{j+\frac12}_{\nu+\frac12}(\xi)\Big)d\xi, \mathbf{r}^j_{\nu+\frac12}\Big\rangle \boldsymbol{\ell}^j_{\nu+\frac12}, \tag{10}$$

and consistency is now obvious, $\mathbf{g}^*(\mathbf{v},\mathbf{v}) = \sum_{j=1}^N \Big\langle \mathbf{g}(\mathbf{v}), \mathbf{r}^j_{\nu+\frac12}\Big\rangle \boldsymbol{\ell}^j_{\nu+\frac12} = \mathbf{g}(\mathbf{v})$. $\square$

A comparison with the numerical viscosities associated with (8) along the lines of [2] yields the following entropy-stability result.

Theorem 0.3. . *Given a complete path in phase space,* $\Big\{\mathbf{u}^j_{\nu+\frac12} := \mathbf{u}\Big(\mathbf{v}^j_{\nu+\frac12}\Big)\Big\}_{j=1}^{N+1}$, *associated with left and right orthogonal sets* $\langle \boldsymbol{\ell}^j_{\nu+\frac12}, \mathbf{r}^k_{\nu+\frac12}\rangle = \delta_{jk}$, *where* $\mathbf{r}^j_{\nu+\frac12}$ *is* *in the direction of* $\mathbf{v}^{j+1}_{\nu+\frac12} - \mathbf{v}^{j}_{\nu+\frac12}$. *Then, the semi-discrete scheme*

$$
(11)\quad \frac{d}{dt}\mathbf{u}_\nu(t) = -\frac{1}{2\Delta x_\nu}\left[\sum_{j=1}^{N}\Big\langle \mathbf{f}\Big(\mathbf{u}^{j}_{\nu+\frac12}\Big) + \mathbf{f}\Big(\mathbf{u}^{j+1}_{\nu+\frac12}\Big), \mathbf{r}^{j}_{\nu+\frac12}\Big\rangle \boldsymbol{\ell}^{j}_{\nu+\frac12} - \Big\langle \mathbf{f}\Big(\mathbf{u}^{j}_{\nu-\frac12}\Big) + \mathbf{f}\Big(\mathbf{u}^{j+1}_{\nu-\frac12}\Big), \mathbf{r}^{j}_{\nu-\frac12}\Big\rangle \boldsymbol{\ell}^{j}_{\nu-\frac12}\right]
$$
$$
+ \frac{1}{2\Delta x_\nu}\left[\sum_{j=1}^{N} q^{j+\frac12}_{\nu+\frac12}\Big\langle \boldsymbol{\ell}^{j}_{\nu+\frac12}, \Delta\mathbf{v}_{\nu+\frac12}\Big\rangle \boldsymbol{\ell}^{j}_{\nu+\frac12} - \sum_{j=1}^{N} q^{j+\frac12}_{\nu-\frac12}\Big\langle \boldsymbol{\ell}^{j}_{\nu-\frac12}, \Delta\mathbf{v}_{\nu-\frac12}\Big\rangle \boldsymbol{\ell}^{j}_{\nu-\frac12}\right],
$$

is entropy-stable if it contains more numerical viscosity than the entropy-conservative one in the sense that the following holds
(12)

$$
q^{j+\frac12}_{\nu+\frac12} \ge \Big\langle \mathbf{r}^{j}_{\nu+\frac12}, Q^{j+\frac12,*}_{\nu+\frac12}\mathbf{r}^{j}_{\nu+\frac12}\Big\rangle,\ Q^{j+\frac12,*}_{\nu+\frac12} := \int_{\xi=-\frac12}^{\frac12} 2\xi B\Big(\mathbf{v}^{j+\frac12}_{\nu+\frac12}(\xi)\Big)d\xi,\ B(\mathbf{v}) := \mathbf{g}_{\mathbf{v}}(\mathbf{v}).
$$

Remark. *Choice of path.* The new ingredient here is the choice of a proper subpath in phase space. We demonstrate the advantage of using such a subpath in the context of second-order accurate schemes. Let $\Big\{\mathbf{w}^k(\mathbf{v}(\xi)) = \mathbf{w}^k\Big(\mathbf{v}^{j+\frac12}_{\nu+\frac12}(\xi)\Big)\Big\}$ be the orthonormal eigensystem of the symmetric $B =$ so that $B\Big(\mathbf{v}^{j+\frac12}_{\nu+\frac12}(\xi)\Big)\mathbf{w}^k(\mathbf{v}(\xi)) = b_k(\mathbf{v}(\xi))\mathbf{w}^k(\mathbf{v}(\xi))$. Expanding $\mathbf{r}^{j}_{\nu+\frac12} = \sum_k \langle \mathbf{w}^k(\mathbf{v}(\xi)), \mathbf{r}^{j}_{\nu+\frac12}\rangle \mathbf{w}^k(\mathbf{v}(\xi))$, we rewrite the amount of entropy-conservative viscosity corresponding to a typical subpath on the left of (12)

$$
(13)\quad \Big\langle \mathbf{r}^{j}_{\nu+\frac12}, Q^{j+\frac12,*}_{\nu+\frac12}\mathbf{r}^{j}_{\nu+\frac12}\Big\rangle = \int_{\xi=-\frac12}^{\frac12} 2\xi\Big\langle \mathbf{r}^{j}_{\nu+\frac12}, B\Big(\mathbf{v}^{j+\frac12}_{\nu+\frac12}(\xi)\Big)\mathbf{r}^{j}_{\nu+\frac12}\Big\rangle d\xi
$$
$$
= \sum_{k=1}^{N}\int_{\xi=-\frac12}^{\frac12} 2\xi b_k(\mathbf{v}(\xi))\Big\langle \mathbf{w}^k(\mathbf{v}(\xi)), \mathbf{r}^{j}_{\nu+\frac12}\Big\rangle^2 d\xi.
$$

Simple upper bounds, for instance, $2\xi b_k(\mathbf{v}(\xi)) \le \sup_\xi |b_k(\mathbf{v}(\xi))|$, characterize the first-order Roe-type schemes. For second-order accuracy, we perform one more integration by parts,

$$
(14)\quad \Big\langle \mathbf{r}^{j}_{\nu+\frac12}, Q^{j+\frac12,*}_{\nu+\frac12}\mathbf{r}^{j}_{\nu+\frac12}\Big\rangle =
$$
$$
\sum_{k=1}^{N}\int_{\xi=-\frac12}^{\frac12}\Big(\frac14 - \xi^2\Big)\Big[\Big\langle \nabla_{\mathbf{v}} b_k(\mathbf{v}(\xi)), \mathbf{r}^{j}_{\nu+\frac12}\Big\rangle\Big\langle \mathbf{w}^k(\mathbf{v}(\xi)), \mathbf{r}^{j}_{\nu+\frac12}\Big\rangle^2 d\xi +
$$
$$
+ 2b_k(\mathbf{v}(\xi))\Big\langle \mathbf{r}^{j}_{\nu+\frac12}, \nabla_{\mathbf{v}}\mathbf{w}^k(\mathbf{v}(\xi))\mathbf{r}^{j}_{\nu+\frac12}\Big\rangle\Big]d\xi.
$$

Here, the second-order accuracy is reflected by the viscosity amplitudes of order $\mathcal{O}\big(|\Delta \mathbf{v}_{\nu+\frac{1}{2}}|\big)$ along each subpath (being entropy-conservative, the amount of entropy dissipation is zero). How should we choose an appropriate subpath? To simplify matters we consider the symmetric case where the entropy and conservative variables coincide, $B(\mathbf{v}) = A(\mathbf{u}) := \mathbf{f_u}(\mathbf{u})$. We let $\big\{\mathbf{u}^j_{\nu+\frac{1}{2}}\big\}_{j=1}^N$ be the breakpoints along the path of (approximate) solutions to the Riemann problem. It is well known that each subpath is directed along the eigensystem of $A\big(\mathbf{u}^j_{\nu+\frac{1}{2}}\big)$, that is, $\mathbf{u}^{j+1}_{\nu+\frac{1}{2}} - \mathbf{u}^j_{\nu+\frac{1}{2}} \sim \mathbf{r}^j_{\nu+\frac{1}{2}}$, so that $\{\mathbf{w}^j \sim \mathbf{r}^j_{\nu+\frac{1}{2}}, a_j\}$ is the normalized eigensystem of A. With this choice, all but one of the terms on the right of (14) vanish to higher order (in $|\Delta \mathbf{u}_{\nu+\frac{1}{2}}|$) and the leading term governing entropy dissipation is given by

$$\Big\langle \mathbf{r}^j_{\nu+\frac{1}{2}}, Q^{j+\frac{1}{2},*}_{\nu+\frac{1}{2}} \mathbf{r}^j_{\nu+\frac{1}{2}} \Big\rangle \approx \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{4} - \xi^2\right) \Big\langle \nabla_{\mathbf{u}} a_j \Big(\mathbf{u}^{j+\frac{1}{2}}_{\nu+\frac{1}{2}}(\xi)\Big), \mathbf{r}^j_{\nu+\frac{1}{2}} \Big\rangle d\xi.$$

The last expression captures the essence of the entropy-conservative schemes that balance between entropy dissipation along j-shocks, where $\Big\langle \nabla_{\mathbf{u}} a_j(\mathbf{u}(\xi)), \mathbf{r}^j_{\nu+\frac{1}{2}} \Big\rangle > 0$, and the entropy production along j-rarefactions, where $\Big\langle \nabla_{\mathbf{u}} a_j(\mathbf{u}(\xi)), \mathbf{r}^j_{\nu+\frac{1}{2}} \Big\rangle < 0$.

To enforce entropy stability, we need to increase the amount of numerical viscosity. The use of different subpaths allows us to stabilize rarefactions while avoiding spurious entropy dissipation with shocks. Turning off the entropy production along the rarefactions, leading to viscosity amplitude, $q^{j+\frac{1}{2}}_{\nu+\frac{1}{2}}$, acting along the j-wave,

$$q^{j+\frac{1}{2}}_{\nu+\frac{1}{2}} = \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{4} - \xi^2\right) \Big\langle \nabla_{\mathbf{u}} a_j \Big(\mathbf{u}^{j+\frac{1}{2}}_{\nu+\frac{1}{2}}(\xi)\Big), \mathbf{r}^j_{\nu+\frac{1}{2}} \Big\rangle^+ d\xi. \tag{15}$$

Next, we note that if the path connecting $\mathbf{u}^j_{\nu+\frac{1}{2}}$ and $\mathbf{u}^{j+1}_{\nu+\frac{1}{2}}$ is chosen along the (approximate) Riemann solution, then the integrand on the right of (15) does not change sign. A simple upper bound of the entropy-conservative amplitude on the right of (15) yields an entropy-stable Lax–Wendroff-type viscosity

$$\int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{4} - \xi^2\right) \Big\langle \nabla_{\mathbf{u}} a_j(\mathbf{u}(\xi)), \mathbf{r}^j_{\nu+\frac{1}{2}} \Big\rangle^+ d\xi \le \frac{1}{4} \frac{\big[a_j\big(\mathbf{u}^{j+1}_{\nu+\frac{1}{2}}\big) - a_j\big(\mathbf{u}^j_{\nu+\frac{1}{2}}\big)\big]^+}{\big\langle \boldsymbol{\ell}^j_{\nu+\frac{1}{2}}, \Delta v_{\nu+\frac{1}{2}} \big\rangle}. \tag{16}$$

Corollary 0.4. (Tadmor 2003) *The following Lax–Wendroff-type difference scheme is a second-order accurate entropy-stable approximation of (3):*

$$\frac{d}{dt}\mathbf{u}_\nu(t) = -\frac{1}{2\Delta x_\nu}\left[\sum_{j=1}^{N}\left\langle \mathbf{f}\left(\mathbf{u}^j_{\nu+\frac{1}{2}}\right) + \mathbf{f}\left(\mathbf{u}^{j+1}_{\nu+\frac{1}{2}}\right), \mathbf{r}^j_{\nu+\frac{1}{2}}\right\rangle \boldsymbol{\ell}^j_{\nu+\frac{1}{2}} - \left\langle \mathbf{f}\left(\mathbf{u}^j_{\nu-\frac{1}{2}}\right) + \mathbf{f}\left(\mathbf{u}^{j+1}_{\nu-\frac{1}{2}}\right), \mathbf{r}^j_{\nu-\frac{1}{2}}\right\rangle \boldsymbol{\ell}^j_{\nu-\frac{1}{2}}\right] + \frac{1}{8\Delta x_\nu}\left[\sum_{j=1}^{N}\left[a_j\left(\mathbf{u}^{j+1}_{\nu+\frac{1}{2}}\right) - a_j\left(\mathbf{u}^j_{\nu+\frac{1}{2}}\right)\right]^+ \boldsymbol{\ell}^j_{\nu+\frac{1}{2}} - \sum_{j=1}^{N}\left[a_j\left(\mathbf{u}^{j+1}_{\nu-\frac{1}{2}}\right) - a_j\left(\mathbf{u}^j_{\nu-\frac{1}{2}}\right)\right]^+ \boldsymbol{\ell}^j_{\nu-\frac{1}{2}}\right]. \tag{17}$$

No artificial dissipation is added in shocks and in particular, it has the desirable property of keeping the sharpness of shock profiles.

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