STABILITY OF DETONATION WAVES

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We survey some recent and ongoing research using Evans function techniques to study the stability problem for detonation waves. Much of the work in this area falls into two categories. It is either (i) numerical in nature and focused on the standard (inviscid) ZND model, or (ii) restricted to the Majda or Majda-Rosales models which have the simplifying feature of scalar kinetics. In contrast the Evans function approach allows the treatment of the reacting Navier-Stokes equations.

1. Introduction

1.1. Models

We begin by considering the “abstract” combustion model

\[ U_t + \sum_{j=1}^{d} F_j(U)_{x_j} = \sum_{j,k=1}^{d} (B_{j}^{ik}(U)U_{x_k})_{x_j} + Q(R(U,z)), \]  \hspace{1cm} (1)

\[ z_t + \sum_{j=1}^{d} (G_j(U)z)_{x_j} = \sum_{j,k=1}^{d} (D_{j}^{ik}z_{x_k} + D_{j}^{ik}(U,z)U_{x_k})_{x_j} - R(U,z), \]  \hspace{1cm} (2)
where \( U, F^j \in \mathbb{R}^n; B^{jk} \in \mathbb{R}^{n \times n}; G^j \in \mathbb{R}; z, R \in \mathbb{R}^s; Q \in \mathbb{R}^{n \times s} \). We think of \( U \) as the fluid variable (mass, momentum, and energy in the case of physical combustion) while the \( z \) variable tracks the progress of the various chemical reactions. Three special cases fit in this framework. They are (i) The Majda Model \((n, d, s = 1)\) a combustion theory analogue of the viscous Burgers equation, (ii) The reacting Navier-Stokes Equations (RNSE) \((n = d + 2)\) in which \( U, F^j \) and \( B^{jk} \) are chosen as for the compressible Navier-Stokes equations, and (iii) the inviscid ZND model which is obtained from the RNSE by \( B^{jk}, D_1^{jk}, D_2^{jk} \equiv 0 \). We close the ZND equations and the RNSE by assuming (as is often done) that the equations of state are independent of the progress of the reaction. Finally we focus on the simplest possible chemical reaction, a one-step \((s = 1)\) exothermic reaction with ignition temperature kinetics. This determines the nature of the source \( R(U, z) \).

### 1.2. Evans Function and Stability

All of the three models mentioned above support a wide variety of traveling wave solutions. They are subdivided into two classes: detonations and deflagrations. Each of these classes is further subdivided into strong, weak, and Chapman-Jouget type. Given a wave, it’s natural to ask about its stability or its sensitivity to small perturbations, and in particular its spectral stability determined by the location of the spectrum of the linearized operator about the wave. Since the essential spectrum is confined to the left half plane and the origin, we may use the Evans function \( D(\lambda)^a \), an analytic function whose zeros correspond to eigenvalues, to search for possible unstable \((\Re \lambda > 0)\) point spectrum. In the case \( d = 1 \) we extract information from \( D(\lambda) \) by calculating the stability index \( \Gamma \) whose sign determines the parity of unstable eigenvalues giving useful necessary conditions for stability.

### 2. Results

#### 2.1. One Space Dimension

For the Majda model we\(^4\) have calculated stability indexes for weak and strong detonations. They are found to be consistent with stability in agree-

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\(^4\)In the case \( d > 1 \), the Evans function depends additionally on \( \xi \in \mathbb{R}^{d-1} \) arising from a Fourier transform in the transverse spatial direction(s).
ment with previous analyses of the Majda model. Moreover, in contrast to these earlier studies, our techniques apply\footnote{1} to the $1 - d$ RNSE.

**Theorem 2.1.** For a RNSE strong detonation with Lax 3-shock structure, the stability index has the form $\hat{\Gamma} := \gamma_d \det(r_1, r_2, [U] + Q)$, where $\gamma_d$ is a transversality coefficient from the traveling wave ODE, $r_j$ are the outgoing eigenvectors of the flux Jacobian, $[U]$ is a vector of jumps in the gas-dynamical quantities, and $Q := (0, 0, q)^T$. Moreover, for an ideal gas in the ZND limit of vanishing dissipative effects, $\text{sgn}\hat{\Gamma} = +1$, consistent with stability.

**Theorem 2.2.** For $q$ sufficiently small, provided that the underlying gas-dynamical shock is spectrally stable, RNSE strong detonations are spectrally stable.

### 2.2. Two Space Dimensions

For nonreactive viscous systems it has been shown\footnote{2} that the low frequency limit of the Evans function is a multiple of the Lopatinski determinant\footnote{1} for the corresponding inviscid system. That is, a necessary condition for viscous stability is that the underlying inviscid shock is stable. Interestingly, in the setting of reactive systems\footnote{2}, the situation is more involved. The low frequency limit of the the Evans function for the RNSE recovers an “inviscid” stability function which differs from the stability function of the underlying ZND detonation. That is, the low viscosity and low frequency limits do not commute. More, we can recover long wave stability criteria for both the viscous and ZND detonations by examining the low frequency limit of the Evans function and the ZND stability function. It remains to understand which of these is physically most relevant.

**References**