On the Front-Tracking Algorithm

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In this paper we present an improved version of the front-tracking algorithm for systems of conservation laws. The formulation and the theoretical analysis are here somewhat simpler than in previous algorithms. At the same time, our version leads to a more efficient numerical scheme. © 1998 Academic Press

1. INTRODUCTION

We are concerned with the construction of a global weak solution to the Cauchy problem for a strictly hyperbolic \( n \times n \) system of conservation laws

\[
\begin{align*}
    u_t + (f(u))_x &= 0, \\
    u(0, \cdot) &= \bar{u}.
\end{align*}
\]

(1)

The basic idea of front tracking for systems of conservation laws is to construct approximate solutions within a class of piecewise constant functions. One approximates the initial data by a piecewise constant function and solves the resulting Riemann problems. Rarefactions are replaced by many small discontinuities. One tracks the outgoing fronts until the first time two waves interact. This defines a new Riemann problem, etc. One of
the main problems in this construction is to keep the number of wavefronts finite for all times $t > 0$. For this purpose there are presently three types of front-tracking algorithms available [1, 2, 5, 9].

In [1, 2] one defines the notion of generation order which tells how many interactions were needed to produce a wavefront. In order to keep the number of waves finite, one solves in an accurate way the Riemann problems arising from interactions between waves of low order, and in a less accurate way those arising from interactions between high-order waves. This simplified solution is constructed by letting the incoming waves pass through each other, slightly changing their speeds, and by collecting all the remaining waves into a so-called "non-physical" front. All non-physical waves propagate with a constant speed greater than all characteristic speeds.

In [9] one does not use the concept of generation order. Instead, for each time where two waves interact one considers a functional depending on the future interactions. If this functional is small enough, some of the small waves are removed and the algorithm is restarted. This guarantees that the approximate solution can be prolonged for a positive time. One then shows that it is only necessary to apply this restarting procedure a finite number of times. In the special case of $2 \times 2$ systems [5], the number of fronts remains automatically finite, and no restarting procedure is needed.

By the algorithms defined in these papers piecewise constant approximate solutions $u_i(t, x)$ are defined for all $i \geq 0$. By a compactness argument one then shows that a subsequence converges to a global weak entropic solution of (1). Indeed, by the results in [3, 4] the entire sequence will converge to the unique semigroup solution.

From a theoretical point of view these methods yield the same existence result. However, from a numerical point of view they are not efficient. The method presented in [9] demands knowledge of the future history of the approximate solution, whereas in [1, 2] one has to keep track of the past history by counting the generation order of each wave.

In this paper we present a new algorithm which avoids these problems. More precisely, we introduce a threshold parameter $\varepsilon$ which dictates how to solve the Riemann problems. If the product of the strengths of the colliding waves is greater than $\varepsilon$ then we use an accurate approximation to the Riemann problem, whereas if the product is smaller than $\varepsilon$ we use a crude approximation. We prove that if $\varepsilon$ tends to zero sufficiently fast, compared with other approximation parameters (the number of initial jumps and the maximal size of rarefaction fronts), then our sequence of approximations $u_\varepsilon$ converges to an entropy weak solution of the Cauchy problem (1).
2. THE ALGORITHM

We consider a strictly hyperbolic $n \times n$ system of conservation laws (1) in which each characteristic family is either genuinely nonlinear or linearly degenerate in the sense of Lax [8], and where the flux $f$ is $C^2$ on a set $\Omega \subset \mathbb{R}^n$. The function $\tilde{u}$ is assumed to be of sufficiently small total variation. We recall that, given two states $u^-, u^+$ sufficiently close, the corresponding Riemann problem admits a self-similar solution given by at most $n+1$ constant states separated by shocks or rarefaction waves [8]. Moreover, there exist $C^2$ curves $\sigma \mapsto \psi_i(\sigma)(u^-)$, $i = 1, \ldots, n$, parametrized by arclength, such that

$$u^+ = \psi_n(\sigma_n) \circ \cdots \circ \psi_1(\sigma_1)(u^-),$$

for some $\sigma_1, \ldots, \sigma_n$. We define $u_0 = u^-$ and $u_i = \psi_i(\sigma_i) \circ \cdots \circ \psi_1(\sigma_1)(u_0)$. When $\sigma_i$ is positive (negative) and the $i$th characteristic family is genuinely nonlinear, the states $u_{i-1}$ and $u_i$ are separated by an $i$-rarefaction ($i$-shock) wave. If the $i$th characteristic family is linearly degenerate the states $u_{i-1}$ and $u_i$ are separated by a contact discontinuity. The strength of the $i$-wave is defined as $|\sigma_i|$.

For given initial data $\tilde{u}$ let $\tilde{u}_\nu$ be a sequence of piecewise constant functions approximating $\tilde{u}$ in the $L^1$-norm. Let $N_\nu$ be the number of discontinuities in the function $\tilde{u}_\nu$, and choose a parameter $\delta_\nu > 0$ controlling the maximum strength of rarefaction fronts.

In order to construct a piecewise constant approximation for all positive times, we introduce various ways of solving the Riemann problems generated by wavefront interactions. More precisely, we will use an accurate solver when the product of the strengths of the incoming fronts is “large,” a simplified solver yielding non-physical waves when the product is “small,” or when one of the incoming waves is non-physical.

2.1. The Riemann Solvers

In the definition of the Riemann solvers we will introduce non-physical waves. These are waves connecting two states $u^-$ and $u^+$, say, and traveling with a fixed speed $\lambda > 0$ strictly greater than all characteristic speeds in $\Omega$. Such a wave is assigned strength $|\sigma| = |u^- - u^+|$ and is said to belong to the $(n+1)$th family. We notice that since all the non-physical fronts travel with the same speed $\lambda$, they cannot interact with each other.

Assume for a positive time $\bar{t}$ we have an interaction at $\bar{x}$ between two waves of families $i_{\alpha}, i_{\beta}$ and strengths $\sigma'_\alpha, \sigma'_\beta$, respectively, $1 \leq i_{\alpha}, i_{\beta} \leq n + 1$. Here $\sigma'_\alpha$ denotes the left incoming wave. Let $u^-, u^+$ be the Riemann problem generated by the interaction and let $\sigma_1, \ldots, \sigma_n$ and $u_0, \ldots, u_n$ be defined as in (2). We define the following approximate Riemann solvers.
(A) **Accurate solver:** if the \(i\)th wave belongs to a genuinely nonlinear family and \(\alpha_i > 0\) then we let

\[
p_i = \lceil \alpha_i/\delta_i \rceil,\tag{3}
\]

where \(\lceil s \rceil\) denotes the smallest integer number greater than \(s\). For \(l = 1, \ldots, p_i\) we define

\[
u_{i,l}^{(1)} = \psi_i(l\alpha_i/p_i)(u_{i-1}), \quad x_{i,l}(t) = \bar{x} + (t - \bar{t})\lambda_i(u_{i,l}),\tag{4}
\]

where \(\lambda_i(\cdot)\) denotes the \(i\)th characteristic speed. Otherwise, if the \(i\)th characteristic family is linearly degenerate, or it is genuinely nonlinear and \(\alpha_i \leq 0\), we define \(p_i = 1\) and

\[
u_{i,l}^{(2)} = u_i, \quad x_{i,l}(t) = \bar{x} + (t - \bar{t})\lambda_i(u_{i-1}, u_i).\tag{5}
\]

Here \(\lambda_i(u_{i-1}, u_i)\) is the speed given by the Rankine–Hugoniot conditions of the \(i\)-shock connecting the states \(u_{i-1}, u_i\). In this case the approximate solution of the Riemann problem is defined in the following way:

\[
u_a(t, x) = \begin{cases} u^- & \text{if } x < x_{1,1}(t), \\ u^+ & \text{if } x > x_{n, p_l}(t), \\ u_i & \text{if } x_{i, p_i}(t) < x < x_{i+1,1}(t), \\ u_{i,l} & \text{if } x_{i,l}(t) < x < x_{i,l+1}(t) \quad (l = 1, \ldots, p_i - 1). \end{cases} \tag{6}
\]

(B) **Simplified solver:** for every \(i = 1, \ldots, n\) let \(\alpha_i^{\sigma_i}\) be the sum of the strengths of all incoming \(i\)th waves. Define

\[
u'(t, x) = \psi_n(\alpha_n^{\sigma_n}) \circ \cdots \circ \psi_1(\alpha_1^{\sigma_1})(u^-).\tag{7}
\]

Let \(v_a(t, x)\) be the approximate solution of the Riemann problem \((u^-, u')\) given by (6). Observe that in general \(u' \neq u^+\) hence we introduce a non-physical front between these states. We thus define the simplified solution in the following way:

\[
u_a(t, x) = \begin{cases} v_a(t, x) & \text{if } x - \bar{x} < \hat{\lambda}(t - \bar{t}), \\ u^+ & \text{if } x - \bar{x} > \hat{\lambda}(t - \bar{t}). \end{cases} \tag{8}
\]

Notice that by construction this function contains at most two physical wavefronts and a non-physical one.
2.2. Construction of the Approximate Solutions

Given \( \nu \) we construct the approximate solution \( u_\nu(t, x) \) as follows. At time \( t = 0 \) all Riemann problems in \( \nu \) are solved accurately as in (A). By slightly perturbing the speed of a wave, we can assume that at any positive time we have at most one collision involving only two incoming fronts. Suppose that at some time \( t > 0 \) there is a collision between two waves belonging to the \( i_\alpha \)th and \( i_\beta \)th families. Let \( \sigma_\alpha \) and \( \sigma_\beta \) be the strengths of the two waves. The Riemann problem generated by this interaction is solved as follows. Let \( \epsilon \) be a fixed small parameter that will be specified later.

- if \( |\sigma_\alpha \sigma_\beta| > \epsilon \) and the two waves are physical, then we use the accurate solver (A);
- if \( |\sigma_\alpha \sigma_\beta| < \epsilon \) and the two waves are physical, or one wave is non-physical, then we use the simplified solver (B).

We claim that this algorithm yields a converging sequence of approximate solutions defined for all times \( t > 0 \), for any \( \epsilon \).

**Lemma 2.1.** The number of wavefronts in \( u_\nu(t, x) \) is finite. Hence the approximate solutions \( u_\nu \) are defined for all \( t > 0 \).

**Proof.** We introduce the standard functional \( Q \) measuring the interaction potential. For a fixed \( \nu \) and time \( t > 0 \) at which no interaction occurs in \( u_\nu(t, \cdot) \), let \( x_1(t) < \cdots < x_m(t) \) be the discontinuities in \( u_\nu(t, \cdot) \), and denote by \( \sigma_1, \ldots, \sigma_m \) and \( i_1, \ldots, i_m \) their strengths and families, respectively. Two waves \( \sigma_\alpha, \sigma_\beta \) with \( x_\alpha < x_\beta \) are said to be approaching either if \( i_\alpha = i_\beta < n + 1 \) and one of them is a shock, or if \( i_\alpha > i_\beta \).

The interaction potential \( Q \) is defined as

\[
Q(t) = \sum_{(\alpha, \beta) \in \mathcal{A}} |\sigma_\alpha \sigma_\beta|,
\]

where \( \mathcal{A} \) denotes the set of all approaching waves at time \( t \). As in [1, 2], one can prove the following. Let \( u \) be any piecewise constant approximate solution \( u \) of (1) constructed as above in the strip \( [0, T] \times \mathbb{R} \), and with sufficiently small total variation. If at time \( t \) two waves of strengths \( \sigma_\alpha \) and \( \sigma_\beta \) interact, then

\[
\Delta Q(t) \leq -\frac{|\sigma_\alpha \sigma_\beta|}{2}.
\]

For each \( \nu \) consider the set of collisions for which the interaction potential between the incoming waves is greater than \( \epsilon \). By the bound in (10), \( Q \) decreases by at least \( \epsilon \nu/2 \) in these interactions. Since new physical waves
can only be generated by this kind of interactions, it follows that their number is finite. Furthermore, since non-physical waves are introduced only when physical waves interact, the number of non-physical waves is also finite. Finally, since two waves can interact only once, the number of interactions is finite, too. This implies that the approximate solutions are defined for all positive times, i.e., \( T = \infty \) for each \( \nu \).

We can now state the main result of this paper.

**Theorem 2.2.** Let \( \bar{u} \) be of small total variation, and let \( \bar{u}_\nu \) converge to \( \bar{u} \) in the \( L^1 \)-norm. Let \( N_\nu \) be the number of jumps in \( \bar{u}_\nu \), \( \delta_\nu \) the parameter controlling the maximum strengths of rarefaction fronts, and \( \varepsilon_\nu \) the threshold parameter. If

\[
\lim_{\nu \to \infty} \delta_\nu = 0, \quad \lim_{\nu \to \infty} \varepsilon_\nu \left( N_\nu + \frac{1}{\delta_\nu} \right)^k = 0,
\]

for every positive integer \( k \), then the sequence of piecewise constant approximations \( u_\nu \) constructed by the above algorithm converges to an entropy weak solution of the Cauchy problem (1).

### 3. Proof of the Theorem

We recall here that a weak solution \( u \) of (1) is a distributional solution, i.e., for any fixed smooth function \( \phi \) with compact support in \( \mathbb{R} \times \mathbb{R} \) it satisfies

\[
\int_{-\infty}^{+\infty} \bar{u}(x) \phi(0, x) \, dx + \int_{0}^{+\infty} \int_{-\infty}^{+\infty} (u(t, x) \phi_t(t, x) + f(u(t, x)) \phi_x(t, x)) \, dx \, dt = 0.
\]

As in [1, 2], if the approximate initial data \( \bar{u}_\nu \) have sufficiently small total variation, by Helly’s theorem it follows that there exists a subsequence of \( u_\nu(t, x) \) which converges in \( L^1_{\text{loc}} \) to some function \( u(t, x) \). To prove that \( u \) is a weak solution of (1) it suffices to show that

\[
\int_{-\infty}^{+\infty} (\bar{u}_\nu(x) - \bar{u}(x)) \phi(0, x) \, dx + \int_{0}^{T} \sum_{\alpha} (\dot{x}_\alpha[u_\nu(t, x_\alpha)] - [f(u_\nu(t, x_\alpha))] \phi(t, x_\alpha) \, dt
\]

(13)
tends to zero as \( n \to \infty \), for any fixed smooth function \( \phi \) with compact support. Here the \( x_\alpha = x_\alpha(t) \) denote the lines of discontinuity of \( u_\nu \) in the strip \([0, T] \times \mathbb{R} \), and \( [\cdot] \) denotes the jump across these discontinuities. By assumption the first term tends to zero as \( n \to \infty \).

To estimate the second term, let \( \mathcal{R}(t) \) and \( \mathcal{M}(t) \) be the sets of indices \( \alpha \) corresponding to rarefactions and non-physical fronts at time \( t \), respectively, and let \( \sigma_\alpha \) be the strength of the wave at \( x_\alpha \). Proceeding as in \([2]\), since the total variation of \( u_\nu(t, \cdot) \) is uniformly bounded in \( t \) and \( n \), we obtain

\[
\left| \int_0^T \sum_\alpha \left( \dot{x}_\alpha[u_\nu(t, x_\alpha)] - [f(u_\nu(t, x_\alpha))] \phi(t, x_\alpha) \right) dt \right|
\leq C(\max|\phi|) \int_0^T \left( \sum_{\alpha \in \mathcal{R}(t)} |\sigma_\alpha(t)|^2 + \sum_{\alpha \in \mathcal{M}(t)} |\sigma_\alpha(t)| \right) dt
\leq CT(\max|\phi|) \left( \sup_{t \in [0, T]} |\sigma_\alpha(t)| + \sup_{t \in [0, T]} \sum_{\alpha \in \mathcal{M}(t)} |\sigma_\alpha(t)| \right),
\]

where \( C \) denotes constants independent of \( \nu \).

So, to have convergence to a weak solution of \((1)\), we need to prove that both the maximal size of refraction waves and the total amount of non-physical waves present in \( u_\nu \) tend to zero as \( n \to \infty \).

By \((3)\) it is clear that the strength of any rarefaction wave in \( u_\nu \) is bounded by \( \delta_\nu \). To estimate the second term in the right-hand side of \((14)\) we prove the following lemma.

**Lemma 3.1.** The total strength of non-physical waves in \( u_\nu \) at time \( t \) tends to zero uniformly in \( t \) as \( n \to \infty \).

**Proof.** We introduce the notion of generation order of wavefronts. Fix a \( \nu \). First we assign order 1 to all the waves at time \( t = 0 \). For waves generated at times \( t > 0 \) we assign orders inductively as in \([1, 2]\); i.e., the order of an outgoing wave is the minimum order of the incoming waves of the same family, if any, and is one more than the maximum order of all incoming waves otherwise. Observe that, since the numbers of waves is finite, there is a maximal generation order for the waves in \( u_\nu \), call it \( s_\nu \).

Let \( V_k(t) \) be the sum at time \( t \) of the strengths of all waves in \( u_\nu(t, \cdot) \) of order \( \geq k \). As in \([1, 2]\), it follows that for all \( k \geq 2 \),

\[
V_k(t) \leq 4C_1 2^{-k}, \quad \forall t \geq 0,
\]

uniformly in \( \nu \), for some constant \( C_1 \). This bound will be used to estimate the total strength of higher-order non-physical fronts.
For lower-order non-physical waves we need another estimate. Let \( M_i, S_i \) denote the total number of fronts and the number of non-physical fronts of order \( i \), respectively, in the approximate solution \( u \). Since a \( k \)th-order wave can be generated only from an interaction between one of order \( k - 1 \) and one of order \( j \leq k - 1 \) and since only one non-physical wave can be generated from each interaction, we have the two relations (see also [1, 2])

\[
M_k \leq n \delta^{-1}_\nu (M_1 + \cdots + M_{k-1}) M_{k-1},
\]

(16)

\[
S_k \leq (M_1 + \cdots + M_{k-1}) M_{k-1}.
\]

(17)

Now define \( P_1 = M_1 \) and \( P_k = n \delta^{-1}_\nu (P_1 + \cdots + P_{k-1}) P_{k-1} \). It is easily established that \( M_k \leq P_k \leq P_{k+1} \leq n \delta^{-1}_\nu (k + 1) P_k^2 \) for every \( k \) and that \( P_k \) satisfies the bounds

\[
P_k \leq \left( k \delta^{-1}_\nu P_1 \right)^{2^{k-1}} \leq \left( k n^2 \delta^{-2}_\nu N_\nu \right)^{2^{k-1}}.
\]

(18)

In turn, using (17), this implies that

\[
S_k \leq k P_{k-1}^{2^{k-1}} \leq k \left( k n^2 \delta^{-2}_\nu N_\nu \right)^{2^{k-1}} \leq C(\nu) \left( N_\nu + \frac{1}{\delta_\nu} \right)^{p(k)},
\]

(19)

where \( C(\nu) = (kn)^{2^k} \) and \( p(k) = 2^{k+1} \). We note that the estimate (17) is useful only for each fixed \( k \); since the constants \( C(s_\nu) \) and \( p(s_\nu) \) grow too fast as \( \nu \to \infty \), we cannot use this to estimate the total amount of non-physical waves of all orders.

By standard interaction estimates [1, 2], the strength of a non-physical wave generated by an interaction between two physical waves is bounded by \( C_1 \delta_\nu \). As a non-physical front interacts with physical ones, its strength can increase. However as in [2, Lemma 2, pp. 115–116], there exists a constant \( C_2 \) such that for all times the strength of the wave remains bounded by \( C_1 C_2 \delta_\nu \).

Now we can estimate the total strength of the non-physical waves at time \( t \) in the following way. By (15) and (17) it follows that

\[
\sum_{a \in M(t)} |\sigma_a| \leq \sum_{k \leq k_0} C(k) \left( N_\nu + \frac{1}{\delta_\nu} \right)^{p(k)} \cdot C_1 C_2 \delta_\nu + \sum_{k \geq k_0} 4C_1 2^{-k},
\]

(20)

for some integer number \( k_0 \).

Given \( \rho > 0 \), choose \( k_0 \) such that \( \sum_{k \geq k_0} 4C_1 2^{-k} \leq \rho/2 \). Next, by the second condition in (11), take \( \nu \) so large that

\[
\sum_{k \leq k_0} C(k) \left( N_\nu + \frac{1}{\delta_\nu} \right)^{p(k)} \cdot C_1 C_2 \delta_\nu < \frac{\rho}{2}.
\]

(21)
By (20) and (21) it follows
\[ \sum_{\sigma \in \mathcal{A}(t)} |\sigma| \leq \rho, \]  
for large \( \nu \), uniformly in \( t \). This completes the proof of the lemma.

Since both the maximal size of rarefaction waves and the total amount of non-physical waves present in \( u_\nu \) tend to zero, then (13) and (14) show that \( u \) is a weak solution of (1).

In a similar way one can prove that if \( (\eta, q) \) is a flux, entropy-flux pair, then we have
\[ \eta(u), + q(u), \leq 0, \]
in the distributional sense (see also [1, 2]). This completes the proof of the theorem.

Remark 1. The approximation error in the above scheme consists of two different parts, due to the approximation of rarefaction waves and the introduction of non-physical waves. The error from approximation of rarefaction waves is \( \mathcal{E}(\delta_\nu) \), whereas the error from the non-physical waves is split into two parts. By (19), the non-physical waves of generation \( \geq k \) contribute by an amount \( \mathcal{E}(2^{-k}) \). From (17) the error due to non-physical waves of generation \( \leq k \) is bounded by
\[ \mathcal{E}\left((kn)^{2^k} \delta_\nu^{-2^{k+1}} \epsilon_\nu\right), \]
if we assume that \( N_\nu \) is \( \mathcal{E}(\delta_\nu^{-1}) \). By asking the three errors to be of the same order of magnitude, it follows that \( k = \mathcal{E}(\log \delta_\nu) \) and that an appropriate choice for \( \epsilon_\nu \) is given by
\[ \epsilon_\nu = \frac{\delta_\nu^{1+2/\delta_\nu}}{(\log \delta_\nu^{-1})^{1/\delta_\nu}}. \]

Remark 2. The second condition in (11) is also necessary. Assume that both \( \delta_\nu \) and the strength of any wave at time \( t = 0 \) are \( \mathcal{E}(N_\nu^{-1}) \), which is the case for smooth initial data approximated by their values at equally spaced intervals. By the interaction estimates one gets that the strength of a wave of generation \( k \) is \( \mathcal{E}(C_1N_\nu^{-1}) \). If (11) fails, i.e., \( \epsilon_\nu \) has only polynomial growth w.r.t. \( N_\nu^{-1} \), then for \( \nu \) large enough the approximate solution \( u_\nu \) can contain waves of only a finite number of orders, independent of \( \nu \). Hence, in general this cannot yield a weak solution as \( \nu \to \infty \).
Remark 3. The algorithm presented in this paper is numerically more efficient than the previous theoretical algorithms since one does not consider the generation order and keeps track only of the “big waves.” This last feature—of keeping only the waves which have a large potential for influencing the solution—is the main advantage with respect to computational effort. This reflects what is actually done in practice when one implements front tracking for systems of conservation laws (see [7] and references therein).

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