Semimartingale properties of a generalized fractional Brownian motion and its mixtures with applications in finance

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Abstract. We study the semimartingale properties for the generalized fractional Brownian motion (GFBM) introduced by Pang and Taqqu (2019) and discuss the applications of the GFBM and its mixtures to financial models, including stock price and rough volatility. The GFBM is self-similar and has non-stationary increments, whose Hurst index $H \in (0, 1)$ is determined by two parameters. We identify the region of these two parameter values where the GFBM is a semimartingale. Specifically, in one region resulting in $H \in (1/2, 1)$, it is in fact a process of finite variation and differentiable, and in another region also resulting in $H \in (1/2, 1)$ it is not a semimartingale. For regions resulting in $H \in (0, 1/2]$ except the Brownian motion case, the GFBM is also not a semimartingale. We also establish $p$-variation results of the GFBM, which are used to provide an alternative proof of the non-semimartingale property when $H < 1/2$. We then study the semimartingale properties of the mixed process made up of an independent Brownian motion and a GFBM with a Hurst parameter $H \in (1/2, 1)$, and derive the associated equivalent Brownian measure.

We use the GFBM and its mixture with a BM to study financial asset models. The first application involves stock price models with long range dependence that generalize those using shot noise processes and FBMs. When the GFBM is a process of finite variation (resulting in $H \in (1/2, 1)$), the mixed GFBM process as a stock price model is a Brownian motion with a random drift of finite variation. The second application involves rough stochastic volatility models. We focus in particular on a generalization of the rough Bergomi model introduced by Bayer, Friz and Gatheral (2016), where instead of using the standard FBM to model the volatility, we use the GFBM, and then derive an approximation for the VIX variance swaps and use numerical examples to illustrate the impact of the non-stationarity parameter.

1. INTRODUCTION

Semimartingale and non-semimartingale properties of the standard fractional Brownian motion (FBM) $B^H$ and its mixtures are well understood. These properties are important in modeling stock price [31, 41], constructing arbitrage strategies and hedging policies [40, 48, 44, 15], and modeling rough volatility [25, 7, 51]. The standard FBM $B^H$ captures short/long-range dependence, and possesses the self-similar and
stationary increment properties, as well as regular path properties. It may arise as the limit process of scaled random walks with long-range dependence or an integrated shot noise process [39].

A generalized fractional Brownian motion (GFBM) $X$, introduced by Pang and Taqqu [38], is defined via the following (time-domain) integral representation:

$$
\{X(t)\}_{t \in \mathbb{R}} \overset{d}{=} \left\{ c \int_{\mathbb{R}} \left[ (t-u)^{\alpha} - (-u)^{\alpha} \right] |u|^{-\gamma/2} B(du) \right\}_{t \in \mathbb{R}},
$$

(1.1)

where $(\alpha, \gamma)$ are constants in the region

$$
\gamma \in [0, 1), \quad \alpha \in \left( -\frac{1}{2} + \frac{\gamma}{2}, \frac{1}{2} + \frac{\gamma}{2} \right),
$$

(1.2)

$B(du)$ is a Gaussian random measure on $\mathbb{R}$ with the Lebesgue control measure $du$, and $c = c(\alpha, \gamma) \in \mathbb{R}^+$ is a normalization constant (see (2.1)). It is a $H$-self-similar Gaussian process, that is, $\{X(\kappa t) : t \in \mathbb{R}\} \overset{d}{=} \{\kappa^H X(t) : t \in \mathbb{R}\}$ for any $\kappa > 0$, with

$$
H := \alpha - \frac{\gamma}{2} + \frac{1}{2} \in (0, 1),
$$

but does not have stationary increments. The Hurst parameter $H$ is determined by two-parameters $(\alpha, \gamma)$ in the range shown in Figure 1. One may regard $\gamma \in (0, 1)$ as a scale/shift parameter. It appears that the component $|u|^{-\gamma/2}$ renders the paths rougher. On the one hand, it is somewhat surprising that the GFBM $X$ has the Hölder continuity property with the same parameter $H - \epsilon$ for $\epsilon > 0$ as the FBM $B^H$, and that the parameter $\gamma$ does affect the differentiability of the paths [27].

The GFBM $X$ defined in (1.1) can be also written as

$$
\{X(t)\}_{t \in \mathbb{R}} \overset{d}{=} \left\{ c \int_{\mathbb{R}} \left[ (t-u)^{H-\frac{1}{2}+\frac{\gamma}{2}} - (-u)^{H-\frac{1}{2}+\frac{\gamma}{2}} \right] |u|^{-\gamma/2} B(du) \right\}_{t \in \mathbb{R}},
$$

(1.3)

It is clear that when $\gamma = 0$, this becomes the standard FBM $B^H$ with Hurst parameter $H = \alpha + 1/2 \in (0, 1)$ (equivalently, $\alpha \in (-1/2, 1/2)$):

$$
\{B^H(t)\}_{t \in \mathbb{R}} \overset{d}{=} \left\{ c \int_{\mathbb{R}} \left[ (t-u)^{H-\frac{1}{2}} - (-u)^{H-\frac{1}{2}} \right] B(du) \right\}_{t \in \mathbb{R}}.
$$

(1.4)

The GFBM $X$ is derived as the limit of integrated power-law shot noise processes in [38] (see a brief review in Section 6.1). We have studied in [27] some important path properties of the GFBM $X$, including the Hölder continuity property, the differentiability and non-differentiability properties, and functional/local law of the iterated logarithm (LIL), see Section 2 for a summary of its fundamental properties. Some additional path properties such as the exact uniform modulus of continuity and Chung’s LIL are recently studied in [52, 53].

In this paper we focus on the semimartingale properties associated with the GFBM and its mixture with an independent BM. For FBM $B^H$, it is shown in [40] that $B^H$ is not a semimartingale for $H \in (0, 1/2) \cup (1/2, 1)$. When $H = 1/2$, $B^H$ is a Brownian motion, and is a martingale (thus, also a semimartingale). In [13], it is shown that the mixture of independent BM and FBM is a semimartingale for $H \in (3/4, 1)$. This
important finding is proved using a filtering approach in [12]. These results have significant implications in financial applications, in particular, arbitrage theory and pricing, see, e.g., [40, 15]. The proofs for these results rely heavily upon the stationary increments property of the FBM. The lack of stationary increments of the GFBM $X$ requires new approaches to establish similar results.

We identify the regions of the two parameter values $(\alpha, \gamma)$ in which the GFBM $X$ is a semimartingale (see the regions (I) and (II) in Figure 1(a) and Proposition 3.1). To do so, we first establish the necessary and sufficient condition for the square integrability of the derivative of the kernel of this Gaussian process. We use this to distinguish the parameter regions for the semimartingale property. We can then apply the characterization of the spectral representation of Gaussian semimartingales by Basse [5].

Another approach for establishing the semimartingale property is to study the so-called $p$-variations as was done for FBM $B^H$ in [43, 14] and for bifractional Brownian motion $B^{H,K}$ in [42]. We also establish the $p$-variation ($p < 1/H, p = 1/H, p > 1/H$) results for the GFBM $X$ (see Proposition 4.1) and use them to conclude the non-semimartingale property of the GFBM $X$ when $H < 1/2$ (see Proposition 4.2).

Recall that the standard FBM $B^H$ is a semimartingale if and only if $H = 1/2$ while the GFBM $X$ can be a semimartingale for $H \in (1/2, 1)$ in region (I) of parameters of $(\alpha, \gamma)$, and in the very special case when the GFBM $X$ becomes a BM (see Remark 3.1 and Figure 1 for more discussions, noting that there is a quadrilateral shape of $(\alpha, \gamma)$ that result in $H \in (1/2, 1)$, and there is a line segment of $(\alpha, \gamma)$ over $[0, 1]$ resulting in $H = 1/2$ but only $\gamma = 0$ gives a BM). However, although $X$ is the semimartingale in region (I), it is a process of finite variation and has an explicit expression, thanks to the result of spectral representation of Gaussian semimartingales in Basse [5].

**Figure 1.** (a) The GFBM $X$ in (1.1) is semimartingale in the region (I): $0 < \gamma < 1$, $1/2 < \alpha < (1 + \gamma)/2$ and not semimartingale in the region (II): $0 < \gamma < 1$, $(\gamma - 1)/2 < \alpha \leq 1/2$. (b) The mixed GFBM $Y$ in (5.1) is semimartingale in the region (I) and the region (II)-1: $0 < \gamma < 1$, $(\gamma - 1)/2 < \alpha < 1/2$. The fractional Brownian motion corresponds to the line segment: $\gamma = 0$, $-1/2 < \alpha < 1/2$ with $H = \alpha + (1/2)$. The standard Brownian motion corresponds to the point $\alpha = \gamma = 0$. 


We then study the semimartingale properties of the mixed GFBM process (sum in (5.1) of an independent BM and GFBM). It is shown in [13, 12, 8] that the mixed FBM $B^H$ process is a semimartingale with respect to its own filtration if and only if $H \in \{1/2\} \cup (3/4, 1)$. We show in Proposition 5.1 that the mixed GFBM process is a semimartingale in the region of the two parameter values $(\alpha, \gamma)$ that is equivalent to $H \in (1/2, 1)$ (see regions (I) and (II)-1 in Figure 1(b)), as well as in the very special case when GFBM is also a BM. It is also worth noting that when $\gamma > 0$, the wide range of values of the Hurst index $H$ for which the mixed GFBM process is a semimartingale. In the two regions (I) and (II)-1, the mixed GFBM process has different behaviors due to the fact that in region (I) the GFBM $X$ is a process of finite variation. For both regions (I) and (II)-1, we can use the characterization of the equivalence of Gaussian measures in Shepp [47], and show that the absolute continuity of the measure of the mixed GFBM with respect to that of the standard BM, and provide an expression of the Radon–Nikodym density, using the solutions to the associated Wiener–Hopf integral equations. For that purpose, we establish that $H \in (1/2, 1)$ is the necessary and sufficient condition for the second partial derivative function of the covariance function of the GFBM $X$ to be square integrable (see Lemma 5.1). For region (I), thanks to the finite variation property of the GFBM $X$, the mixed GBFM becomes a Brownian motion plus a random drift of finite variation, and as a consequence, we provide another expression of the Radon–Nikodym density, in terms of the conditional expectation, applying the results in [30]. We also conjecture that the mixed process is not a semimartingale when $H \in (0, 1/2)$ (see further discussions in Remark 5.2).

We then use the GFBM and its mixtures in the study of financial asset models. The first application is stock price process with long range dependence in Section 6. The price models we introduce in Section 6.2 generalize those using shot noise process and FBMs in the literature [31, 2, 45, 50]. When the mixed BM and GFBM processes are semimartingales, we derive the Radon–Nikodym derivative for the equivalent martingale measure (Proposition 6.1), which can then be used for the price dynamics of various options. This occurs when the parameters $(\alpha, \gamma)$ lie in the regions (I) and (II)-1, in each of which the Hurst index of GFBM is in $(1/2, 1)$. This is in contrast with the model using the standard FBM, which only occurs when the Hurst index of the FBM is in $(3/4, 1)$, see [13, 12]. In our framework, the larger range of the Hurst index thus provides greater flexibility in modeling and for further theoretical analysis through the use of Itô’s formula and the properties of semimartingales.

We discuss the implications on arbitrage in asset pricing for stock price processes using the GFBM and its mixtures. With FBM $B^H$, arbitrage in fractional Bachelier and Black-Scholes models has been well studied in [40, 48, 44, 15]. If one uses the GFBM $X$ as a stock price process, we find that the only non-arbitrage scenario is the Brownian case (with both $\alpha = 0$ and $\gamma = 0$). Although the GFBM $X$ is a semimartingale in parameter region (I), since it is a process of finite variation, arbitrage exists as shown in [26]. On the other
hand, if one uses the mixed GFBM $Y$ as a stock price process, in the parameter region (I), the process becomes a semimartingale as a Brownian motion with a finite-variation drift, and thus, no arbitrage exists. This price process with the mixed GFBM is of particular interest, since it also exhibits the long range dependence, similar to that with the mixed FBM. Rogers [40] pointed out that it is possible to construct a process similar to the FBM to model long-range dependence of returns while avoiding arbitrage. We provide an example of a price model with these desirable properties. We identify self-financing arbitrage strategies in the Bachelier and Black-Scholes models with the GFBM $X$ for the region of parameters that results in $H \in (1/2, 1)$ in which the process $X$ is not a semimartingale, taking the approach in Shirayaev [48]. See more discussions in Section 6.3.

The second application is about rough stochastic volatility models. There have been many activities in the study of rough volatility using FBM since the seminal research initiated by Gatheral et al. [25], see the extensive literature in [51]. VIX and other volatility derivatives modeled with Brownian motion have been extensively studied, see, e.g., the books [24, 10]. Bayer, Friz and Gatheral [7] has recently proposed a rough Bergomi model using the Riemann–Liouville (R-L) FBM. In particular, they provide an approximation for the VIX variance swap which is a forward contract on realized variances. See also [28] on VIX futures using the model. We particularly focus on a generalization of the rough Bergomi model using the generalized R-L FBM, and provide an approximation for the associated VIX variance swap. We use numerical examples to illustrate the impact of the “non-stationarity” parameter $\gamma$ on the estimates of the VIX variance swaps. We observe that for the same Hurst parameter value $H$, a slight positive $\gamma$ value can dramatically affect the VIX variance. See Table 1 and Figures 2, 3 and 4. We then propose various models for the volatility process using the GFBM, the mixed GFBM process, and the generalized fractional Ornstein–Ulenbeck (fOU) processes driven by the GFBM or the mixed GFBM process. These models may be more advantageous since the GFBM and its mixture can be a semimartingale for a wide range of Hurst parameter values while possessing the long range dependence and roughness properties.

The paper is organized as follows. In Section 2, we give the precise definition of the GFBM and summarize some of its properties. In Sections 3 and 4, we study the semimartingale properties of the GFBM $X$. In Section 5, the semimartingale property of the mixed BM and GFBM process is investigated. We present the applications in financial models in Sections 6 and 7.

2. A GENERALIZED FRACTIONAL BROWNIAN MOTION

The GFBM process $X$ defined in (1.1) has the following properties:

(i) $X(0) = 0$ and $\mathbb{E}[X(t)] = 0$ for all $t \geq 0$;

(ii) $X$ is a Gaussian process and $\mathbb{E}[X(t)^2] = t^{2H}$ for $t \geq 0$;
(iii) \( X \) has continuous sample paths almost surely;
(iv) \( X \) is self-similar with Hurst parameter \( H \in (0, 1) \);
(v) the paths of \( X \) are Hölder continuous with parameter \( H - \epsilon \) for \( \epsilon > 0 \);
(vi) the paths of \( X \) is non-differentiable if \( \alpha \in (0, 1/2] \) and differentiable if \( \alpha > 1/2 \) almost surely
    (see non-differentiable region (II) \( \alpha \in (0, 1/2] \) and differentiable region (I) \( \alpha > 1/2 \) in Figure 1).

These properties are established in [38, 27]. See Proposition 5.1 [38] for properties (iii) and (iv), and Theorems 3.1 and 4.1 in [27] for (v) and (vi).

Also recall that the normalization constant \( c = c(\alpha, \gamma) \in \mathbb{R}_+ \) is given by
\[
c(\alpha, \gamma) := \left( \int_0^1 (1-v)^{2\alpha} v^{-\gamma}\, dv + \int_0^\infty [(1+v)^\alpha - v^\alpha]^2 v^{-\gamma}\, dv \right)^{-1/2}
= \left( \text{Beta}(1-\gamma, 2\alpha + 1) \right. \\
+ \left. \left( \frac{\Gamma(1-\gamma)}{\Gamma(-2\alpha)} - \frac{2\Gamma(1+\alpha-\gamma)}{\Gamma(-\alpha)} \right) \Gamma(-1-2\alpha+\gamma) \right)^{-1/2},
\]
as shown in Lemma 2.1 of [27]. Here, \( \Gamma(a) := \int_0^\infty e^{-x} x^{a-1}\, dx \), \( a > 0 \) and \( \text{Beta}(a, b) := \int_0^1 x^{a-1}(1-x)^{b-1}\, dx \), \( a, b > 0 \) are the Gamma and Beta functions, respectively, and \( \Gamma(-a) := (-a)^{-1} \Gamma(-a+1) \) for positive non-integer \( a \).

Recall that in the case \( \gamma = 0 \), the FBM \( B^H \) has stationary increments. Namely, the second moment of its increment:
\[
\mathbb{E}[(B^H(s) - B^H(t))^2] = c^2 |t - s|^{2H},
\]
and the covariance function
\[
\mathbb{E}[B^H(s)B^H(t)] = \frac{1}{2} c^2 (t^{2H} + s^{2H} - |t - s|^{2H}), \tag{2.2}
\]
where \( c = c(\alpha, 0) = c(H - 1/2, 0) \) in (2.1).

When \( \gamma \in (0, 1) \), in comparison with the FBM \( B^H \), the process \( X \) loses the stationary increment property.

In particular, the second moment of its increment is
\[
\Phi(s, t) := \mathbb{E}[(X(t) - X(s))^2] 
= c^2 \int_{\mathbb{R}} \left( (t-u)^\alpha - (s-u)^\alpha \right)^2 u^{-\gamma}\, du 
= c^2 \int_0^t (t-u)^{2\alpha} u^{-\gamma}\, du + c^2 \int_t^s ((t-u)^\alpha - (s-u)^\alpha)^2 u^{-\gamma}\, du 
+ \int_0^\infty ((t+u)^\alpha - (s+u)^\alpha)^2 u^{-\gamma}\, du, \tag{2.3}
\]
and the covariance function is
\[
\Psi(s, t) = \text{Cov}(X(t), X(s)) = \mathbb{E}[X(s)X(t)] 
= c^2 \int_{\mathbb{R}} \left( ((t-u)_+^\alpha - (-u)_+^\alpha) ((s-u)_+^\alpha - (-u)_+^\alpha) \right) u^{-\gamma}\, du
\]
\[ e^2 \int_0^s (t-u)^\alpha (s-u)^\alpha u^{-\gamma} du \]
+ \[ e^2 \int_0^\infty ((t+u)^\alpha - u^\alpha)((s+u)^\alpha - u^\alpha)u^{-\gamma} du, \]  \hspace{1cm} (2.4)

for \( 0 \leq s \leq t. \)

We also remark that generalized FBMs are stated in a more general form with the additional terms involving \((t-u)^\alpha - (-u)^\alpha\) in the integrands in [38, Sections 5.1 and 5.2]. In this paper we focus on the representations of \(X\) in (1.1) since the other forms with additional terms can be treated similarly.

3. **When is the GFBM a Semimartingale?**

Any centered Gaussian process \(X\) with right-continuous sample paths has a spectral representation in distribution [33], that is,
\[ X(t) \overset{d}{=} \int_{-\infty}^t K_t(s) dN(s), \quad t \geq 0, \]
where \(N\) is an independently scattered centered Gaussian random measure and \((t,s) \rightarrow K_t(s)\) is a square-integrable deterministic function. Basse [5, Theorem 4.6] characterizes the spectral representation of Gaussian semimartingales, identifying the family of kernels \(K_t(s)\) for which the representation \(\{ \int_{-\infty}^t K_t(s) dN(s) : t \geq 0 \}\) is a semimartingale with respect to the natural filtration \(\mathcal{F}_t^N : t \geq 0\). Specifically for one-dimensional case \(N(\cdot) \equiv B(\cdot)\) being the Brownian Gaussian random measure, it says that \(\{X(t) : t \geq 0\}\) is an \(\mathcal{F}_t^N : t \geq 0\) semimartingale if and only if for \(t \geq 0\), the kernel can be represented as
\[ K_t(s) = g(s) + \int_0^t \Psi_r(s) \mu(dr), \]  \hspace{1cm} (3.1)
where \(g : \mathbb{R}_+ \rightarrow \mathbb{R}\) is locally square integrable with respect to the Lebesgue measure, \(\mu(\cdot)\) is a Radon measure on \(\mathbb{R}_+\) and a measurable mapping \(\Psi_r(s) : (r,s) \rightarrow \mathbb{R}\) is square integrable with respect to the Lebesgue measure, \(\int_{-\infty}^\infty |\Psi_r(s)|^2 ds = 1\), and \(\Psi_r(s) = 0, r < s\).

In the case of FBM \(B^H\), we have the kernel function
\[ K_t(s) = (t-s)^{H-1/2} - (-s)^{H-1/2}, \]
and \(N(\cdot) = B(\cdot)\). Since it is a \(\mathcal{F}_t^B : t \geq 0\) semimartingale if and only if \(H = 1/2\), i.e., a Brownian motion, applying [5, Theorem 4.6], we have \(g \equiv 0\), and \(\Psi_r(s) \equiv 1\).

For the GFBM \(X\), we have the kernel function
\[ K_t(s) = [((t-s)_+)^\alpha - ((-s)_+)^\alpha] |s|^{-\gamma/2}, \]  \hspace{1cm} (3.2)
and \(N(\cdot) = B(\cdot)\).

Define the function \(\Psi_t(\cdot)\) by
\[ \Psi_t(s) := C_t^{-1} \left[ \alpha(t-s)^{\alpha-1}s^{-\gamma/2} \cdot 1_{\{0 \leq s < t\}} + \alpha(t-s)^{\alpha-1}(-s)^{-\gamma/2} \cdot 1_{\{s \leq 0\}} \right], \]
where $C_t$ is a time-dependent, normalizing constant defined by

$$C_t := \alpha t^H \left( \text{Beta}(1 - \gamma, 2\alpha - 1) + \text{Beta}(1 - \gamma, 1 - 2\alpha + \gamma) \right)^{1/2}, \quad t > 0.$$  

**Lemma 3.1.** The function $\Psi_t(\cdot)$ is square integrable with respect to the Lebesgue measure if and only if $1/2 < \alpha < (1 + \gamma)/2$ and $\gamma \in (0, 1)$ (region (I) in Figure 1(a)). In this case, $\int_{-\infty}^{\infty} |\Psi_t(s)|^2 ds = 1$.

**Proof.** If $1/2 < \alpha < (1 + \gamma)/2$ and $\gamma \in (0, 1)$, it follows from the definition that

$$\alpha^{-2} C_t^2 \int_{-\infty}^{\infty} |\Psi_t(s)|^2 ds = \int_0^r (r - s)^{2(\alpha - 1)} s^{-\gamma} ds + \int_{-\infty}^0 (r - s)^{2(\alpha - 1)} (-s)^{-\gamma} ds,$$

where the first and the second terms are rewritten by the change of variables as

$$\int_0^r (r - s)^{2(\alpha - 1)} s^{-\gamma} ds = r^{2H} \int_0^1 (1 - u)^{2(\alpha - 1)} u^{-\gamma} du = r^{2H} \text{Beta}(1 - \gamma, 2\alpha - 1),$$

$$\int_{-\infty}^0 (r - s)^{2(\alpha - 1)} (-s)^{-\gamma} ds = r^{2H} \int_0^\infty (1 + u)^{2(\alpha - 1)} u^{-\gamma} du = r^{2H} \text{Beta}(1 - \gamma, 1 + \gamma - 2\alpha).$$

Thus, in this case, $\int_{-\infty}^{\infty} |\Psi_t(s)|^2 ds = 1$. On the other hand, if the conditions on the parameters $\alpha$ and $\gamma$ are not satisfied, then these terms are not integrable. Thus we conclude the proof. \hfill \square

**Proposition 3.1.** The following properties hold:

(i) If $\gamma \in (0, 1)$ and $1/2 < \alpha < (1 + \gamma)/2$ (region (I) in Figure 1(a)), then $\{X(t), t \geq 0\}$ in (1.1) is a semimartingale with respect to $\mathcal{F}^B(\cdot)$, more specifically, a process of finite variation:

$$\frac{dX(t)}{dt} = c \int_{-\infty}^t \Psi_t(s) dB(s), \quad t \geq 0,$$

where $\Psi_t(\cdot)$ is defined by

$$\Psi_t(s) := \alpha (t - s)^{\alpha - 1} |s|^{-\gamma/2}$$

for $s < t$, and $\Psi_t(s) := 0$ for $s > t$.

(ii) If $\gamma = 0$ and $\alpha \in (-1/2, 0) \cup (0, 1/2)$, $\{X(t), t \geq 0\}$ in (1.1) is reduced to a fractional Brownian motion and is not semimartingale. If $\gamma \in (0, 1)$ and $\alpha \leq 1/2$ (region (II) in Figure 1(a)), then $\{X(t), t \geq 0\}$ in (1.1) is not a semimartingale with respect to $\mathcal{F}^B(\cdot)$.

(iii) Particularly, if $\gamma = 0$ and $\alpha = 0$, that is, $H = 1/2$, it is a Brownian motion and thus a semimartingale. However, for $\gamma = 2\alpha \in (0, 1)$ with $H = 1/2$, it is not a semimartingale with respect to $\mathcal{F}^B(\cdot)$.

**Remark 3.1.** In the range (region (I) of Figure 1(a)) of parameters $(\alpha, \gamma)$: $\gamma \in (0, 1)$ and $1/2 < \alpha < (1 + \gamma)/2$, the Hurst parameter $H = \alpha - \frac{\gamma}{2} + \frac{1}{2} \in \left(\frac{1}{2}, 1\right)$. Note that when $\alpha$ is close to $1/2$, and $\gamma$ is close to $1$,
the Hurst parameter $H$ is also close to $1/2$, which differs from the standard FBM case with $\gamma = 0$ and $\alpha = 0$ resulting in $H = 1/2$. The range of values of Hurst parameter $H$ possessing the semimartingale property is expanded from a single value $1/2$ for the standard FBM, to the half interval $[1/2, 1)$ for the process $X(t)$, although the process $X(t)$ is a process of finite variation in region (I) with $H \in (1/2, 1)$. Moreover, note that $H = 1/2$ here only corresponds to the singular point $\alpha = 0, \gamma = 0$. The GFBM $X$ can have $H = 1/2$ on the line segment $\alpha = \gamma/2$, which is a BM if and only if $\alpha = 0$.

**Remark 3.2** (Differentiability). It is shown in [27] that the regions (I) and (II) of Figure 1(a) correspond to the regions of almost sure differentiable and non-differentiable paths, respectively, that is, in the region (I) the sample path of GFBM is differentiable, while in region (II) the sample path of GFBM is not differentiable. This is not just a coincidence but it turns out that when $\alpha > 1/2$, it is a semimartingale and its (local) martingale part in the semimartingale decomposition is zero, and its finite variation part is the integral of a Gaussian process. When $\alpha = \gamma = 0$, it is a Brownian motion with non-differentiable sample path.

**Proof.** (i) Let us consider the case $1/2 < \alpha < (1 + \gamma)/2$ and $\gamma \in (0, 1)$. Thanks to the integrability of (3.3) in this parameter set, the square integrability of $\Psi_t(\cdot)$ is assured, by Lemma 3.1, and hence, by Theorem 4.6 of Basse (2009), we have the representation

$$X(t) = c \int_0^t \left( \int_{-\infty}^r \Psi_r(s)dB(s) \right) dr, \quad t \geq 0,$$

where $\Psi_t(\cdot)$ is defined in (3.5). Thus it is a process of finite variation with the first derivative (3.4), in particular, it is a semimartingale. See also section 4 of [27].

(ii) When $\gamma = 0$, $X(\cdot)$ in (1.1) is a FBM with Hurst index $\alpha + 1/2$, and it is not a semimartingale if $\alpha \in (-1/2, 0) \cup (0, 1/2)$. Thus, let us consider the case $\gamma \in (0, 1), \alpha \leq 1/2$ and show the claim by contradiction.

Suppose that $X(\cdot)$ in (1.1) is a semimartingale with respect to $\mathcal{F}^B(\cdot)$. We know $\mathbb{E}[X^2(t)] = t^{2\alpha - \gamma + 1}$ for $t \geq 0$. Then by Theorem 4.6 of [5] with $N = B(\cdot)$, $g(s) := 0$, $C_t = (-\infty, t]$, there is a canonical decomposition

$$X(t) = \int_{-\infty}^t K_t(s)dB(s) = \int_{-\infty}^t [(t-s)^\alpha - (s-t)^\alpha] |u|^{-\gamma/2} dB(s)$$

for $t \geq 0$, where the integrand $K_t(s)$ has the form:

$$K_t(s) := g(s) + \int_0^t \Psi_r(s)\mu(dr), \quad 0 \leq s \leq t.$$  

Here $g(\cdot)$ is square integrable with respect to the Lebesgue measure, $\mu(\cdot)$ is a Radon measure on $\mathbb{R}_+$, and $\Psi_t(s)$ is a measurable mapping satisfying

$$\int_{-\infty}^\infty |\Psi_t(s)|^2 ds = 1, \quad \text{and} \quad \Psi_t(s) \equiv 0 \ (s > t).$$
Taking derivatives with respect to $t$ in (3.8), we have
\[
\frac{dK_t(s)}{dt} = \Psi_t(s) \cdot \mu(dt),
\]
while the integrand $K_t(s)$ in (3.2) has the derivative with respect to $t$:
\[
\frac{dK_t(s)}{dt} = \alpha(t - s)^{\alpha - 1/2} |s|^{-\gamma/2}.
\]
Thus by comparing these two expressions and by setting $\mu(dr) = dr$ as the Lebesgue measure, we identify $\Psi_t(s)$ as in (3.5). However, as in Lemma 3.1, if $\alpha \leq 1/2$ and $\gamma \in (0, 1)$, $\Psi_t(\cdot)$ is not square integrable for every $t > 0$. This yields a contradiction to (3.9). Thus, we claim that $X(\cdot)$ in (1.1) is not semimartingale with respect to $\mathcal{F}_B(\cdot)$, if $\gamma \in (0, 1)$ and $\alpha \leq 1/2$.

(iii) The standard Brownian motion case $H = 1/2$ is indeed a semimartingale. The second statement on the parameter sets $\gamma = 2\alpha \in (0, 1)$ with $H = 1/2$ is proved as a special case of (ii).

\[4. \text{VARIATIONS OF THE GFBM } X\]

For the standard FBM $B^H$, it is shown in [43, Proposition 3.14] that $B^H$ has a $1/H$-variation, that is,
\[
\lim_{\varepsilon \downarrow 0} \int_0^t \frac{1}{\varepsilon} |B^H(s + \varepsilon) - B^H(s)|^{1/H} ds = \varrho_H t
\]
in the sense of convergence uniformly in compact sets, where $\varrho_H = \mathbb{E}[|Z|^{1/H}]$ for $Z \sim N(0, 1)$. Here, the limit is in the sense of convergence in probability uniformly on every compact interval (ucp). It can be also shown that the classical variation
\[
\sum_{i=0}^{n-1} |B^H_{t_{i+1}} - B^H_{t_i}|^{1/H} \xrightarrow{n \to \infty} \varrho_H t,
\]
where $0 = t_0 < \cdots < t_n = t$ is a partition of $[0, t]$, see Proposition 3.14 in [43] and Remark 1 in [42]. Then by Propositions 1.9 and 1.11 of [14] one can conclude that $B^H$ is not a semimartingale with respect to $\mathcal{F}_B(\cdot)$, if $H < 1/2$. Note that the results in [14] involve the notion of the weak semimartingales (see Definition 1.5 of [14]).

This approach of evaluating the variations is also used in [42] to show that the bifractional Brownian motion $B^{H,K}$ with parameters $(H, K), H \in (0, 1), K \in (0, 1]$ is not a semimartingale, if $HK \neq 1/2$, see Propositions 1–3 and Remark 1 there. Recall that the bifractional Brownian motion $B^{H,K}$ is a centered Gaussian process with $B^{H,K}(0) = 0$ and covariance function
\[
\mathbb{E}[B^{H,K}(s) B^{H,K}(t)] = \frac{1}{2^2H}((t^{2H} + s^{2H})^K - |t - s|^{2HK}), \quad s, t \geq 0.
\]
The bifractional Brownian motion $B^{H,K}$ is a FBM with Hurst index $H \in (0, 1)$ if $K = 1$.

We use this approach to establish the following properties of the GFBM $X$ in (1.1). The proof of the proposition below is given in Appendix 8.1.
Proposition 4.1. Let
\[ \rho_{\alpha,\gamma} := \left( e^{2 \beta(1 + 2\alpha, 1 - \gamma)} \right)^{1/(2H)} \sqrt{2^{1/H}/\pi} \Gamma((1 + (1/H))/2). \]
Then we have the convergence in \( L^1 \):
\[ \frac{1}{\varepsilon} \int_0^t |X(s + \varepsilon) - X(s)|^{1/H} ds \xrightarrow{\varepsilon \to 0} \rho_{\alpha,\gamma} t, \quad t > 0, \quad (4.1) \]
and similarly, for every partition \( \pi : 0 = t_0 < \cdots < t_n = t \) of \([0, t]\) with size \( |\pi| := \max_{1 \leq i \leq n} |t_i - t_{i-1}| \), we have
\[ \sum_{i=0}^{n-1} |X(t_{i+1}) - X(t_i)|^{1/H} \xrightarrow{n \to \infty} \rho_{\alpha,\gamma} t. \quad (4.2) \]

It is immediate from (4.2) that for \( p < 1/H \), we have
\[ \sum_{i=0}^{n-1} |X(t_{i+1}) - X(t_i)|^p \xrightarrow{n \to \infty} +\infty, \quad (4.3) \]
and for \( p > 1/H \), we have
\[ \sum_{i=0}^{n-1} |X(t_{i+1}) - X(t_i)|^p \xrightarrow{n \to \infty} 0. \quad (4.4) \]

Proposition 4.2. The process \( X \) in (1.1) is not a semimartingale with respect to its own filtration \( \mathcal{F}^X(\cdot) \) if \( H < 1/2 \), that is, \( \gamma \in (0, 1) \) and \( \alpha \in (-1/2 + \gamma/2, \gamma/2) \). Since \( \mathcal{F}^X \subset \mathcal{F}^B \), it is not a semimartingale with respect to \( \mathcal{F}^B(\cdot) \) if \( H < 1/2 \) (which is part of Proposition 3.1 (ii)).

Proof. When \( H < 1/2 \), as a consequence of (4.3) with \( p = 2 < 1/H \), by Propositions 1.9 and 1.11 of [14], \( X \) in (1.1) is not a semimartingale. Note that when \( H > 1/2 \), \( X \) in (1.1) is a process of zero quadratic variation by (4.4). \( \Box \)

5. Mixed BM and GFBM

In this section we consider the semimartingale properties of the following process
\[ Y(t) = \tilde{B}(t) + X(t), \quad t \geq 0, \quad (5.1) \]
where \( \tilde{B}(t) \) is a standard Brownian motion and \( X(t) \) is the GFBM defined in (1.1), independent of \( \tilde{B}(t) \). Let us call \( Y \) the mixed GFBM.

In the case of FBM \( B^H \), we denote
\[ Y^H(t) = \tilde{B}(t) + B^H(t), \quad H \in (0, 1), \quad t \geq 0. \]
It is shown in [13, Theorem 1.7] and [12, Theorem 2.7] (see also [8]) that \( Y^H(t) \) is a semimartingale with respect to its own filtration if and only if \( H \in \{ 1/2 \} \cup \{ \frac{1}{3}, 1 \} \). In [13], the concept of weak semimartingale and a theorem on Gaussian processes in [49] is used. On the other hand, in [12], the filtering approach is used. In particular, the mixed FBM \( Y^H \) is innovated by a martingale in its natural filtration for all \( H \in (0, 1] \).
Then the equivalence property with respect to the Wiener measure is established for $H \in (3/4, 1]$ and the equivalence property with respect to the Wiener measure is established for $H \in (1/4)$. The associated Radon-Nikodym density formulas are then derived in these ranges of the parameter $H$.

Let $\mu^{Y,H}$ be the probability measure induced by $Y^H$ on the space of its paths in $C(\mathbb{R}_+; \mathbb{R})$, and $\mu^B$ be the Wiener measure. For $H > 1/2$, the covariance function of $B^H(t)$ in (2.2) is written as

$$\Psi^H(t,s) = \mathbb{E}[B^H(t)B^H(s)] = \int_0^t \int_0^s K^H(u,v) du dv,$$

where

$$K^H(t,s) = \frac{\partial^2}{\partial t \partial s} \mathbb{E}[B^H(s)B^H(t)] = c_H |t-s|^{2H-2}.$$ (5.3)

with $c_H := c^2 H(2H - 1)$. If $H > 3/4$, $K^H(\cdot, \cdot) \in L^2([0,T]^2)$, and $\mu^{Y,H} \sim \mu^B$ (equivalence) by the general criterion in Shepp [47], and in addition, Shepp’s Radon–Nikodym derivative can be written in the form

$$\frac{d\mu^{Y,H}}{d\mu^B}(Y^H) = \exp \left( - \int_0^T \varphi_t(Y^H) dY^H(t) - \frac{1}{2} \int_0^T \varphi_t^2(Y^H) dt \right),$$

where

$$\varphi_t(Y^H) := \int_0^t L^H(s,t) dY^H(s), \quad 0 \leq t \leq T$$ (5.4)

and $L^H \in L^2([0,T]^2)$ is the unique solution of the Wiener–Hopf integral equation

$$L^H(s,t) + c_H \int_0^t L^H(r,t)|r-s|^{2H-2} dr = - c_H |s-t|^{2H}, \quad 0 \leq s \leq t \leq T.$$ (5.5)

The second partial derivative $K(u,v)$ of the covariance function $\Psi$ in (2.4) is given by

$$K(u,v) = \frac{\partial^2 \Psi}{\partial u \partial v}(u,v) = c^2 (f_1(u \wedge v, u \vee v) + f_2(u \wedge v, u \vee v)),$$ (5.5)

where

$$f_1(u,v) := \int_0^u (v-\theta)^{\alpha-1}(u-\theta)^{\alpha-1}\theta^{-\gamma} d\theta,$$

$$f_2(u,v) := \int_{v}^{\infty} (v+\theta)^{\alpha-1}(u+\theta)^{\alpha-1}\theta^{-\gamma} d\theta.$$ (5.6)

We then obtain the following square–integrability property of $K(u,v)$. Its proof is given in Section 8.2.

**Lemma 5.1.** Assume $0 < \gamma < 1$ and $(\gamma - 1)/2 < \alpha < (\gamma + 1)/2$. The function $K(\cdot, \cdot)$ in (5.5) is square integrable with respect to the Lebesgue measure in $(0,T) \times (0,T)$ for every $T > 0$, if and only if

- $\gamma/2 < \alpha < 1/2 + \gamma/2$ and $0 < \gamma < 1$ (equivalently, $H \in (1/2, 1)$, regions (I) and (II)-1 in Figure 1(b)).

**Remark 5.1.** We remark that both regions (I) and (II)-1 lead to Hurst parameter $H \in (1/2, 1)$. The previous lemma shows the integrability of the function $K(\cdot, \cdot)$ in (5.5), from which we can conclude the absolute
continuity of the \( Y \) with respect to the Brownian motion \( \tilde{B}(\cdot) \) and obtain an expression of the Radon-Nikodym density as a direct consequence of Shepp’s result for general Gaussian processes in [47]. That will involve the Wiener–Hopf integral equation.

However, the two regions (I) and (II)-1 also have distinct behaviors, despite the same Hurst parameter range. In particular, in region (I), we have shown in Proposition 3.1 that the process \( X \) is of finite variation, with the representation in (3.6), which makes \( Y \) a Brownian with a random drift of finite variation. As a consequence, we are able to provide a more explicit expression of the Radon-Nikodym density using conditional expectations.

We will next state these results in two propositions. It remains open to show that the explicit Radon-Nikodym density in region (I) is equivalent to the density given by the Wiener–Hopf integral equation.

Suppose that \( \gamma / 2 < \alpha < 1/2 + \gamma / 2 \) and \( 0 < \gamma < 1 \). Let \( L(s, t) \in L^2([0, T]^2) \) be the unique solution to the Wiener–Hopf integral equation

\[
L(s, t) + \int_0^t L(r, t)K(r, s)dr = -K(s, t), \quad 0 \leq s \leq t \leq T,
\]

and define

\[
\varphi_t(Y) := \int_0^t L(s, t) dY(s), \quad 0 \leq t \leq T.
\]

Also, let \( \ell(s, t) \in L^2([0, T]^2) \) be the unique solution to the Volterra equation

\[
\ell(s, t) + \int_s^t \ell(r, t)L(s, r)dr = L(s, t), \quad 0 \leq s \leq t \leq T.
\]

Thus, by [47] and Lemma 5.1, we obtain the following proposition.

**Proposition 5.1.** Suppose that \( \gamma / 2 < \alpha < 1/2 + \gamma / 2 \) and \( 0 < \gamma < 1 \), i.e., regions (I) and (II)-1 in Figure 1(b) (both resulting \( H \in (1/2, 1) \)). The probability measure \( \mu_Y \) induced by \( Y \) in (5.1) is absolutely continuous with respect to the Wiener measure \( \mu_{\tilde{B}} \) over \([0, T]\) with the Radon–Nikodym density

\[
\frac{d\mu_Y}{d\mu_{\tilde{B}}}(Y) = \exp\left( -\int_0^T \varphi_t(Y)dY(t) - \frac{1}{2} \int_0^T [\varphi_t(Y)]^2 dt \right).
\]

By the Girsanov theorem

\[
\mathbb{W}(t) := Y(t) + \int_0^t \varphi_s(Y)ds = Y(t) + \int_0^t \int_0^s L(r, s)dY(r)ds, \quad 0 \leq t \leq T
\]

is a Brownian motion with respect to its own filtration. Moreover, \( Y(t) \) can be written as

\[
Y(t) = \mathbb{W}(t) - \int_0^t \int_0^s \ell(r, s)d\mathbb{W}(r)ds, \quad 0 \leq t \leq T.
\]

Particularly, the filtration \( \mathcal{F}_Y(\cdot) \) generated by \( Y \) and the filtration \( \mathcal{F}_W(\cdot) \) satisfy the identities \( \mathcal{F}_Y(t) = \mathcal{F}_W(t) \) for \( 0 \leq t \leq T \).
Therefore, \( Y(t) \) is a semimartingale for the pair \((\alpha,\gamma)\) values in this region.

In Region (I) of Figure 1(b), since the process \( X \) has a finite variation, as expressed in (3.6), then the mixed process \( Y \) is written as

\[
Y(t) = \tilde{B}(t) + \int_0^t \left( \int_{r=0}^{\infty} \Psi_t(s)dB(s) \right) dr = \tilde{B}(t) + \int_0^t \lambda(r)dr,
\]

(5.11)

with \( \lambda(t) := \int_0^{\infty} \Psi_t(s)dB(s), t \geq 0, \)

where \( \Psi_t(s) \) given in (3.5). Thus \( Y \) is, in fact, a Brownian motion with a random drift of finite variation, particularly, it is a semimartingale. By Theorem 2 and Lemma 4 of [30] we obtain the Radon-Nikodym density \( \frac{d\mu^Y}{d\mu^B}(Y) \) in (5.9), which is stated in the following proposition.

**Proposition 5.2.** Suppose that \( 1/2 < \alpha < 1/2 + \gamma/2 \) and \( 0 < \gamma < 1 \), i.e., region (I) in Figure 1(b) (resulting in \( H \in (1/2, 1) \)). The Radon–Nikodym density (5.9) over the time interval \([0, T]\) in Proposition 5.1 is given by

\[
\frac{d\mu^Y}{d\mu^B}(Y) = \exp \left( \int_0^T \mathbb{E}[\lambda(s)\mathcal{F}^Y(s)]dY(s) - \frac{1}{2} \int_0^T (\mathbb{E}[\lambda(s)\mathcal{F}^Y(s)])^2 ds \right).
\]

Here, \( \varphi_t(Y) \) in (5.7) is identified as \( \varphi_t(Y) \equiv -\mathbb{E}[\lambda(t)\mathcal{F}^Y(t)] \) for \( t \geq 0 \), and

\[
\mathbb{W}(\cdot) = Y(\cdot) - \int_0^T \mathbb{E}[\lambda(s)\mathcal{F}^Y(s)]ds = \tilde{B}(\cdot) + \int_0^T (\tilde{\lambda}(s) - \mathbb{E}[\lambda(s)\mathcal{F}^Y(s)])ds \]

is a Brownian motion with respect to its own filtration.

**Remark 5.2.** We conjecture that the mixture process \( Y \) is not a semimartingale with respect to its own filtration in the parameter region \( \gamma/2 - 1/2 < \alpha \leq \gamma/2 \) and \( \gamma \in (0, 1) \) (region (II)-2 including the boundary line segment \( \alpha = \gamma/2, \gamma \in (0, 1) \) in Figure 1(b)). The boundary point \( \gamma = \alpha = 0 \) represents the standard Brownian motion \( B^H \) with \( H = 1/2 \) (written as \( B^{1/2} \) without confusion) and \( Y = \tilde{B} + B^{1/2} \) becomes a semimartingale. For standard FBM \( B^H \), in Cai et al. [12], representations of the FBM with the Riemann-Liouville fractional integrals and derivatives are used to prove the innovation representations in Theorem 2.4 for \( H < 1/2 \), and equivalence of the measures for \( \tilde{B} + B^H \) and \( B^H \) for \( H < 1/4 \). However, for the GFBM \( X \), it still remains open to establish the Riemann-Liouville fractional integrals and derivatives. Therefore, we leave it as future work to prove the non-semimartingale property of the mixture process \( Y \) for the parameter pair \((\alpha,\gamma)\) in region (II)-2 of Figure 1(b).

5.1. **Generalized Riemann-Liouville FBM and its Mixture.** In this section, we discuss the semimartingale properties of the generalized Riemann–Liouville (R-L) FBM and its mixtures. The generalized R-L FBM is introduced in Remark 5.1 in [38], and further studied in Section 2.2 in [27]. It is defined by

\[
X(t) = c \int_0^t (t-u)^\alpha u^{-\gamma/2}B(du), \quad t \geq 0,
\]

(5.12)
where \( B(du) \) is a Gaussian random measure on \( \mathbb{R} \) with the Lebesgue control measure \( du \) and \( c \in \mathbb{R}, \gamma \in [0, 1) \) and \( \alpha \in (\gamma/2 - 1/2, \gamma/2 + 1/2) \). It is a continuous self-similar Gaussian process with Hurst parameter \( H = \alpha - \gamma/2 + 1/2 \in (0, 1) \). When \( \gamma = 0 \), it reduces to the standard R-L FBM

\[
B^H(t) = c \int_0^t (t - u)^\alpha B(du), \quad t \geq 0.
\]

It is clear that the semimartingale properties in Proposition 3.1 hold for the process \( X \) in (5.12). In particular, by letting the natural kernel \( K_t(s) := (t - s)^\alpha s^{-\gamma/2} \), we have the spectral representation \( X(t) = \int_0^t K_t(s) N(ds), \) for a Gaussian measure \( N(\cdot) \). Define

\[
\Psi_t(s) = C_t^{-1} \alpha(t - s)^{\alpha-1} s^{-\gamma/2},
\]

for a time-dependent normalization constant \( C_t \). As shown in the proof of Lemma 3.1, the function \( \Psi_1(\cdot) \) is square integrable with respect to the Lebesgue measure if and only if \( 1/2 < \alpha < 1/2 + \gamma/2 \) and \( \gamma \in (0, 1) \), and thus by Basse’s characterization of the spectral representation of Gaussian semimartingales ([5, Theorem 4.6]), we can conclude the semimartingale property in part (ii) of Proposition 3.1. The non-semimartingale property in part (i) of Proposition 3.1 also follows from a similar argument as in the proof of the proposition.

In addition, the variation properties in Proposition 4.1 also hold for the process \( X \) in (5.12), from which we can also conclude the non-semimartingale property as in the proof of Proposition 4.2.

We next discuss the mixed process \( Y = \tilde{B} + X \) with \( X \) in (5.12) and an independent BM \( \tilde{B} \). Let \( L(s, t) \in L^2([0, T]^2) \) be the unique solution to the Wiener–Hopf integral equation (5.6) with

\[
K(u, v) = c^2 \int_0^u (v - \theta)^{\alpha-1}(u - \theta)^{\alpha-1}\theta^{-\gamma} d\theta,
\]

and let \( \ell(s, t) \in L^2([0, T]^2) \) be the unique solution to the Volterra equation (5.8). Then the properties in Proposition 5.1 hold, in particular, \( Y = \tilde{B} + X \) with \( X \) in (5.12) is a semimartingale with respect to the filtration generated by itself in the parameter range \( \gamma/2 < \alpha < 1/2 + \gamma/2 \) and \( 0 < \gamma < 1 \). In addition, Proposition 5.2 holds for \( \gamma/2 < \alpha < 1/2 + \gamma/2 \) and \( 0 < \gamma < 1 \), where

\[
\tilde{\lambda}(t) = \int_0^t \Psi_t(s) dB(s)
\]

in (5.11).

6. Stock Price Processes Associated with GFBM

6.1. Shot noise process and integrated shot noise. In modeling of financial markets, Brownian motion and compound Poisson processes or more generally, Lévy processes are widely utilized to capture effects of various noises. Recently, Pang and Taqqu [38] studied a non-stationary, power-law shot noise process \( Z^* = \{Z^*(y) : y \in \mathbb{R}\} \) on the whole real line defined by

\[
Z^*(y) := \sum_{j=-\infty}^{\infty} g^*(y - \tau_j) R_j, \quad y \in \mathbb{R},
\]

(6.1)

where \( \{\tau_j : j \in \mathbb{Z}\} \) is a sequence of Poisson arrival times of shots with rate \( \lambda \) on the whole real line, and each \( R_j \) is the noise associated with shot \( j \) at time \( \tau_j \) for \( j \in \mathbb{Z} \). It is a generalization of the compound Poisson
process on the positive half line. The variables \( \{R_j, j \in \mathbb{Z}\} \) are conditionally independent, given \( \{\tau_j\} \), and the marginal distribution of \( R_j \) depends on the shot arrival time \( \tau_j \), that is, \( \mathbb{P}(R_j \leq r \mid \tau_j = u) =: F_u(r), \) \( r \in \mathbb{R}, u \in \mathbb{R} \) for every \( j \in \mathbb{Z} \). Assume that

(i) (the power-law property) the function \( g^* \) satisfies \( g^*(y) = y^{-(1-\alpha)} L^*(y) \) for \( y \geq 0 \) and \( g^*(y) = 0 \) for \( y < 0 \) and \( \alpha \in (0, 1/2) \), where \( L^* \) is a positive slowly varying function at \( +\infty \), and

(ii) (the moment conditions) the common conditional distribution \( F_t \) of the noises \( R_j \), given \( \tau = t \), satisfies the zero mean \( K_1(t) := \int_{\mathbb{R}} r dF_t(r) = 0 \) for every \( u \in \mathbb{R} \) and finite variance \( K_2(t) = \int_{\mathbb{R}} r^2 dF_t(r) = t^{-\gamma} \tilde{L}_+(t) \) for \( t > 0 \) and \( K_2(t) = |t|^{-\gamma} \tilde{L}_-(t) \) for \( t < 0 \), where \( \gamma \in (0, 1) \), and \( \tilde{L}_\pm \) are some positive slowly varying functions.

The shot noise process \( Z^* \) in (6.1) is a generalization of the compound Poisson process, because if \( g^*(y) := 1_{\{y > 0\}} \), \( y \in \mathbb{R} \), then \( Z^* \) is a compound process. The integrated shot noise process \( Z = \{Z(t) : t \in \mathbb{R}_+\} \) is defined by

\[
Z(t) := \int_0^t Z^*(y) dy = \sum_{j=-\infty}^{\infty} (g(t-\tau_j) - g(-\tau_j)) R_j, \quad t \geq 0,
\]

where the shot shape function \( g(\cdot) \) is differentiable with its derivative \( g^* \), i.e., \( g(t) := \int_0^t g^*(y) dy \) for \( t \geq 0 \).

6.2. Stock price models driven by the noise with the long-range dependence. Stock pricing models with a shot-noise component have been developed to study credit and insurance risks [2, 45, 50]. In particular, the stock price \( P(t) \) is modeled as

\[
P(t) := \mathbb{P}(0) \exp \left( \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma \tilde{B}(t) + \sigma \int_0^t \sum_{\tau_i \leq s} \hat{f}(s-\tau_i, R_i) ds \right), \tag{6.3}
\]

for \( t \geq 0 \), where \( \{(\tau_i, R_i) : i \in \mathbb{N}\} \) is a marked point process, independent of the Brownian motion \( \tilde{B} \), with arrival times \( \tau_i \) and marks (noises) \( U_i \in \mathbb{R}^d \), and the function \( \hat{f} : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R} \) is the deterministic shot shape function. \( \mu \in \mathbb{R} \) and \( \sigma > 0 \) are some real constants. Equivalent martingale measures for this price process are studied in [45, 46]. In [45], it is also discussed when the shot-noise component is Markovian or a semimartingale. This is usually when the function \( \hat{f} \) takes a particular form (exponential function for the Markovian property). In these studies, the noises \( \{R_i\} \) are assumed to be i.i.d. with finite variance. Since it is usually more difficult to work with the shot noise process directly, one may use the diffusion approximations.

For example, Klëppelberg and Kühn [31] showed that under regular variation conditions, a Poisson shot noise process can be approximated by an FBM (under proper scaling and validated by a functional central limit theorem), and then used the limiting FBM as a stock pricing model.

Here, as a pre-limit, we consider the usual random walk noise and the shock noises on the stock price. We evaluate the effects of these noises, when the frequency of arrival of shot noises is very high with appropriate
scaling. Given a scaling parameter $\varepsilon > 0$, we model a pre-limit of price process by

$$P_\varepsilon(t) := P(0) \exp \left( \left( \frac{\mu - \sigma^2}{2} \right) t + \sigma \varepsilon^{1/2} \sum_{j=1}^{[t/\varepsilon]} \xi_j + \sigma \int_0^{t/\varepsilon} \frac{1}{\varepsilon^{1-H}} Z^* \left( \frac{u}{\varepsilon} \right) du \right)$$

$$= P(0) \exp \left( \left( \frac{\mu - \sigma^2}{2} \right) t + \sigma \varepsilon^{1/2} \sum_{j=1}^{[t/\varepsilon]} \xi_j + \sigma \varepsilon^H \int_0^{t/\varepsilon} Z^* \left( u \right) du \right)$$

$$= P(0) \exp \left( \left( \frac{\mu - \sigma^2}{2} \right) t + \sigma \varepsilon^{1/2} \sum_{j=1}^{[t/\varepsilon]} \xi_j + \sigma \varepsilon^H \mathcal{Z} \left( \frac{t}{\varepsilon} \right) \right), \quad (6.4)$$

for $t \geq 0$, where $\{\xi_j, j \in \mathbb{N}\}$ are i.i.d. random variables with zero mean and unit variance, independent of the shot noise $Z^*$ in (6.1), and $\mathcal{Z}$ is the integrated shot noise process in (6.2). Here, $\mu$ and $\sigma > 0$ are some real constants. (One may also choose a model without the random walk component, in which case the model in (6.5) will have only the process $X$ instead of the mixed GFBM.)

Under certain regularity conditions and with a proper scaling, Pang and Taqqu [38] have shown that the scaled process $\tilde{Z}^{\varepsilon}(t) := \varepsilon^H \mathcal{Z}(t/\varepsilon)$ converge weakly to the GFBM $X$, as $\varepsilon \to 0$. The random walk term $\varepsilon^{1/2} \sum_{j=1}^{[t/\varepsilon]} \xi_j$, $t \geq 0$ converges weakly to the standard BM, independent of $X$.

Suppose that the parameters $(\alpha, \gamma)$ are in the semimartingale region: $\gamma/2 < \alpha < 1/2 + \gamma/2$ and $0 < \gamma < 1$ (i.e., $H \in (1/2, 1)$ for the GFBM $X$), as in the assumption of Proposition 5.1. As a scaling limit of (6.4), we propose a stock price model using the mixed GFBM as follows:

$$P(t) = P(0) \exp \left( \left( \frac{\mu - \sigma^2}{2} \right) t + \sigma \tilde{B}(t) + X(t) \right)$$

$$= P(0) \exp \left( \left( \frac{\mu - \sigma^2}{2} \right) t + \sigma Y(t) \right), \quad t \geq 0,$$  

where $Y(\cdot) = \tilde{B} + X$ is the mixed GFBM in (5.1) with $X$ in (5.1) (For a recent account of weak convergence in financial models, we refer to Kreps [32].) Under the above parameter range, $Y$ is a semimartingale. The price dynamics is determined as the unique strong solution of the linear stochastic differential equation

$$dP(t) = P(t)(\mu dt + \sigma dY(t)); \quad t \geq 0,$$  

(6.6)

driven by the semimartingale $Y$, where $\mu$ is a drift and $\sigma$ is volatility of stock price under a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. Thus, we take the filtration $\mathbb{F} := (\mathcal{F}_t, t \geq 0) = (\mathcal{F}^Y(t), t \geq 0)$.

As an example with this stock price model (6.5), we consider an investor who trades this stock with price (6.5) and money market account with an instantaneous interest rate $r(>0)$. We first recall what is known in the case of standard FBM $B^H$, as shown in [40, 13, 12], the mixed process $Y^H = \tilde{B} + B^H$ is a semimartingale if and only if $H = 1/2$ (the Brownian case) and $H \in (3/4, 1)$. Of course, with a BM, i.e., $H = 1/2$, the standard results of stock pricing and equivalence of martingale measure can be applied.
On the other hand, with $H \in (3/4, 1)$, one also obtain the Radon–Nikodym derivative in (6.7) where $Y$ is replaced by $Y^H$, and the function $\varphi_t(Y)$ is replaced by $\varphi_t(Y^H)$ in (5.4).

Also, recall that for the GFBM $X$ with $H = 1/2$, the parameters $(\alpha, \gamma)$ lies on the line segment $\alpha = \gamma/2$ in Figure 1. It is a semimartingale, only when $\gamma = 0$, which becomes the special case of Black-Sholes pricing model; otherwise, there does not exist an equivalent martingale measure.

In the following proposition, we give the expressions of the equivalent martingale measures for the discounted stock price process modeled with the mixed process $Y$.

**Proposition 6.1.** Assume $\gamma/2 < \alpha < 1/2 + \gamma/2$ and $0 < \gamma < 1$ (regions (I) and (II)-1 in Figure 1(b)). Under the stock price process dynamics (6.6), the discounted stock price process $e^{-rt}P(t)$, $0 \leq t \leq T$ is a martingale under the new measure $\mathbb{Q}$ defined by

$$
\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} := \exp \left( -\int_0^T (\theta - \varphi_t(Y))dY(t) - \frac{1}{2} \int_0^T (\theta^2 - |\varphi_t(Y)|^2)dt \right),
$$

(6.7)

where $\theta := (\mu - r)/\sigma$ is the market price of risk and $\varphi(Y)$ is defined in (5.7). Particularly, if $1/2 < \alpha < 1/2 + \gamma/2$ and $0 < \gamma < 1$, i.e., region (I) in Figure 1(b), then $\varphi(Y) = -E[\tilde{\lambda}(\cdot)\mid \mathcal{F}'(\cdot)] = -E[\tilde{\lambda}(\cdot)\mid \mathcal{F}]$, where $\tilde{\lambda}(t) = \int_0^t \psi_t(s)dB(s)$ for $t \geq 0$.

**Proof.** It follows from Proposition 5.1 and the Girsanov theorem that the process $\overline{W}(\cdot) = Y(\cdot) + \int_0^\cdot \varphi_s(Y)ds$ in (5.10) is a Brownian motion for $0 \leq t \leq T$ under $\mathbb{P}$.

By the simple application of the product rule to (6.6), we have the discounted stock price process

$$
e^{-rt}P(t) = P(0) + \int_0^t \sigma e^{-rs}P(s)d\left( Y(s) + \frac{\mu - r}{\sigma}s \right)
$$

$$= P(0) + \int_0^t \sigma e^{-rs}P(s)d\left( \overline{W}(s) - \int_0^s \varphi_u(Y)du + \theta s \right),
$$

for $t \geq 0$, with $\theta := (\mu - r)/\sigma$. By another application of the Girsanov theorem, $Y(t) + \theta t$, $0 \leq t \leq T$ is a Brownian motion under the measure $\mathbb{Q}$. In particular, the discounted stock price process $e^{-rt}P(t)$, $0 \leq t \leq T$ is a martingale under $\mathbb{Q}$.

If $1/2 < \alpha < 1 + \gamma/2$ and $0 < \gamma < 1$, i.e., region (I) in Figure 1(b), then $\varphi(Y) = -E[\tilde{\lambda}(\cdot)\mid \mathcal{F}'(\cdot)] = -E[\tilde{\lambda}(\cdot)\mid \mathcal{F}]$, because $\mathcal{F}' \equiv \mathcal{F}P = \mathcal{F}$.

Consequently, the time- $t$ price of European option on this stock with payoff function $g$ and with maturity $T$ is given by

$$E^Q[e^{-r(T-t)}g(P(T))\mid \mathcal{F}_t] = E^P\left[ e^{-r(T-t)}g(P(T)) \cdot \frac{d\mathbb{Q}}{d\mathbb{P}} |_{\mathcal{F}_t} \right],
$$

where the conditional expectations $E^Q$ and $E^P$ are calculated under $\mathbb{Q}$ and $\mathbb{P}$, respectively, given the filtration $\{\mathcal{F}_t\}_{t \geq 0}$. Under the stock price model (6.6) with the mixed GFBM $Y$ in the regions (I) and (II)-1 of the parameter sets $\{ \gamma/2 < \alpha < 1/2 + \gamma/2, 0 < \gamma < 1 \}$, the pricing and hedging problems of various
options (such as American and Asian options) are solved under the measure $\mathbb{Q}$ in the same way as in the standard Black-Scholes model. Thus, the Black-Scholes pricing formula is still valid under the long range dependence property of the driving mixed GFBM $Y$.

6.3. **Comments on Arbitrage.** In the theory of asset pricing, the “First Fundamental Asset Pricing Theorem” requires the existence of an equivalent martingale measure for no arbitrage, and works in the framework of semimartingales for pricing models. For stock price models with FBM $B^H$, since $B^H$ is a semimartingale if and only if $H = 1/2$ [34, 40], arbitrage strategies have been discussed in both fractional Bachelier and Black-Scholes models [40, 48, 44, 15]. In particular, Rogers [40] constructed arbitrage for the fractional Bachelier model: a market with a money account $\tilde{B}$ and a risky stock (no dividends or transaction costs) with price $\tilde{P}(t) = \tilde{P}(0) + vt + \sigma B^H(t)$, for $H \in (0, 1/2) \cup (1/2, 1)$ for $t \geq 0$.

For the GFBM $X$ in (1.1), the only parameter set that guarantees no-arbitrage is $\alpha = 0$ and $\gamma = 0$ (i.e., $H = 1/2$), that is, the standard Brownian motion case. Recall Remark 3.1, $H = 1/2$ corresponds to the line $\alpha = \gamma/2$ in Figure 1, but if $\gamma \neq 0$, it is a “fake” Brownian motion and it may lead to an arbitrage, because of the non-semimartingale property. Although we have shown that $X$ is a semimartingale with respect to the filtration $\mathcal{F}^H(\cdot)$ for $\alpha \in (1/2, 1/2 + \gamma/2)$ and $\gamma \in (0, 1)$ (resulting in $H \in (1/2, 1)$ in region (I)), it is a process of finite variation. As it is shown in [26], in a frictionless market with continuous trading, arbitrage opportunities are present without unbounded variation of the stock price process. Thus, the differentiable sample path in Proposition 3.1 leads an arbitrage opportunity in this parameter range. On the other hand, for the mixed BM and GFBM $Y$ in (5.1), because in the parameter range $\alpha \in (1/2, 1/2 + \gamma/2)$ and $\gamma \in (0, 1)$ (resulting in $H \in (1/2, 1)$ for $X$ in region (I)), the process $Y$ becomes a Brownian motion with a random drift of finite variation, there exists an equivalent martingale measure and hence, it forbids arbitrage.

We next discuss the construction of arbitrage opportunities. For the GFBM $X$, with $\alpha \in (\gamma/2, 1/2)$ and $\gamma \in (0, 1)$ (resulting in $H \in (1/2, 1)$), one can construct arbitrage strategy using an approach similar that of Shiryaev [48] since the $p$-variation of the process $X$ is finite for $p > 1/H$ for $H \in (1/2, 1)$ and well defined, so that pathwise Riemann-Stieltjes stochastic integral with respect to $X$ is well defined [48, 44]. (See further discussions on stochastic integrals with respect to the GFBM $X$ in Section 7.3.) Indeed, replacing the FBM $B^H$ in [48] by the GFBM $X$ in (1.1), we find that the self-financing, admissible strategy $\pi := (\beta, \gamma)$ in the Bachelier model $\tilde{P}(t) = \tilde{P}_0 + X(t), t \geq 0$ with a risk free bond of zero interest rate yields the portfolio value

$$V^\pi_t = \beta_t + \gamma_t \tilde{P}(t) = V^\pi_0 + \int_0^t \gamma_u d\tilde{P}(u), \quad t \geq 0,$$

where the stochastic integral is understood as the pathwise Riemann-Stieltjes integral with respect to $X$, and then with $\tilde{P}_0 := 1, \gamma_t := 2X(t), \beta_t := -X(t)^2 - 2X(t), t \geq 0$, the resulting portfolio value satisfies $\mathbb{P}(V^\pi_0 = 0, V^\pi_t > 0) = 1$ for every $t > 0$, because $V^\pi_t = \beta_t + \gamma_t \tilde{P}(t) = X(t)^2, t \geq 0$. 


Similarly, in the fractional Black-Scholes model, which is a market with a money market account \( \xi_t = e^{rt} \) and a risky stock (no dividends or transaction costs) \( \tilde{P}(t) = \tilde{P}_0 \exp \left( rt + B^H(t) \right) \), one can also adopt the self-finance arbitrage strategy for \( H \in (1/2, 1) \) for standard FBM in Shiryaev [48]. For the GFBM \( X \), with \( \tilde{P}(t) = \tilde{P}_0 \exp \left( rt + X(t) \right) \), we can analogously choose the portfolio \( \beta_t = 1 - e^{2X(t)} \) and \( \gamma_t = 2(e^{X(t)} - 1) \) for a self-financing arbitrage opportunity with the portfolio value process \( V_{\pi t} \), \( t \geq 0 \), defined by

\[
V_{\pi t} := \beta_t e^{rt} + \gamma_t \tilde{P}(t) = e^{rt}(e^{X(t)} - 1)^2 = \int_0^t \beta_s r ds + \gamma_s d\tilde{P}(s), \quad t \geq 0.
\]  

However, for \( \alpha \in (\gamma/2 - 1/2, \gamma/2) \) and \( \gamma \in (0, 1) \) (resulting in \( H \in (0, 1/2) \)), we cannot use the approach in Shiryaev [48] to construct arbitrage strategies. Potentially, one could try to construct arbitrage strategies by extending the approaches in [15]. We leave this to future work.

For the process \( Y \) in (5.1), with \( \alpha \in (\gamma/2, 1/2) \) and \( \gamma \in (0, 1) \) (resulting in \( H \in (1/2, 1) \) for \( X \) in region (II)-1), one can construct arbitrage strategy using an approach similar that of Shiryaev [48]. The construction of an arbitrage opportunity when the parameter is in the region (II)-2 is also an open problem.

### 7. Rough Fractional Stochastic Volatility

In a seminal paper Gatheral et al. [25] conducted an empirical study on stochastic volatility and discovered that the log-volatility behaves like a FBM with Hurst exponent \( H < 1/2 \) (mostly between 0.08 and 0.2), and thus proposed a “rough” volatility model. For the recent developments in the empirical studies of the Hurst parameter of financial data and the microstructure of leverage effects, see also [1, 17, 21] and papers listed on the webpage [51].

For a given asset with log-price taking the form

\[
\frac{dP(t)}{P(t)} = \mu(t) dt + \sigma(t) dW(t), \quad t \geq 0,
\]  

where \( \mu(t) \) is a drift term and \( W(t) \) is a one-dimensional BM, the stochastic volatility \( \sigma(t) \) is modeled as

\[
\sigma(t) = \sigma(0) \exp(Z(t)), \quad t \geq 0,
\]

where the process \( Z(t) \) is a stationary fractional Ornsten-Uhlenbeck (fOU) process, given by the stationary solution of the SDE:

\[
dZ(t) = -a(Z(t) - m)dt + \nu dB^H(t)
\]

with \( m \in \mathbb{R} \) and \( a \) and \( \nu \) being positive constant parameters. Here the fOU process \( Z \) has an explicit solution given by

\[
Z(t) = Z(0)e^{-at} + m(1 - e^{-at}) + \nu \int_{-\infty}^{t} e^{-a(t-s)} dB^H(s),
\]

where the stochastic integral with respect to the FBM \( B^H \) is a pathwise Reimann-Stieltjes integral (see [16]). This model has received great attention in the community of stochastic volatility (see, e.g., [7] and an extensive relevant literature in [51]).
In addition to the estimation of the Hurst parameter being \( H < 1/2 \), there are several important findings in the empirical study of [25]. First, the distribution of the increments of the log-volatility is approximately normal. Second, although the log-volatility estimations are smooth over most intervals, there is observable non-smoothness for some stock prices and indices and for certain time windows. It is also argued that the reason why the above model of stochastic volatility is used, instead of \( \sigma(t) = \sigma \exp(\nu B^H(t)) \), is because the above model is stationary. Although mathematical tractability is desirable with the stationary model, it is evident that non-stationarity (in terms of increments) is prominent in financial data (see also [6, 35]).

In this section we first study a generalization of the rough Bergomi model introduced in [7], and then discuss other extensions.

### 7.1. A generalized rough Bergomi model

In [7], the rough Bergomi (rBergomi) model was introduced as a non-Markovian generalization of Bergomi model with FBM \( B^H \). Specifically, the stock price process \( P(u) \) and the instantaneous volatility \( v(u) \), \( u \geq t \) under the physical measure are defined by

\[
\frac{dP(u)}{P(u)} = \mu(u)du + \sqrt{v(u)}dB(u),
\]

\[
v(u) = \xi(t) \exp \left( \eta \sqrt{2H} \int_t^u (u-s)^{-1/2}dB(s) - \frac{1}{2} \eta^2 (u-t)^{2H} \right),
\]

for \( u \geq t \), where \( \mu \) is an expected log return process, \( \eta \) is a constant, \( \xi(t) = \mathbb{E} [ v(u) \mid \mathcal{F}(t) ] \), \( u \geq t \) is the forward variance curve, \( H \in (0, 1/2) \), and \( B = \rho B + \sqrt{1-\rho^2} B^\perp \) for two independent standard Brownian motions \( B, B^\perp \) with correlation coefficient \( \rho \in (-1, 1) \). Here, the filtration \( \mathcal{F}(t), t \geq 0 \) is generated by the price process \( P(t), t \geq 0 \), and the process \( \int_t^u (u-s)^{-1/2}dB(s) \) is the so-called Riemann-Liouville FBM or Volterra fractional Brownian motion. In [7], it was discussed as a first approximation that \( P(u) \) becomes a true martingale by the deterministic change of measure under a fixed time horizon \( t \leq u \leq T \) in the rough Bergomi model under the equivalent martingale measure. For the details of the martingale property of the rough Bergomi model, see [23]. Recently, the rough Bergomi model is studied in the limiting case \( H \to 0 \) in [19].

Using the generalized Riemann-Liouville FBM \( X(t) \) in (5.12), we may modify the above model with replacement of \( v \) in (7.2) by

\[
v(u) = \xi(t) \exp \left( \eta (X(u) - X(t)) - \frac{1}{2} \eta^2 \mathbb{E}[|X(u) - X(t)|^2] \right),
\]

\[
= \xi(t) \exp \left( \eta c \int_t^u (u-s)^{\alpha} s^{-\gamma/2}B(ds) - \frac{1}{2} \eta^2 c^2 \int_t^u (u-s)^{2\alpha} s^{-\gamma}ds \right),
\]

for \( u \geq t \), where \( \alpha \in \left( -(1-\gamma)/2, \gamma/2 \right) \), \( \gamma \in (0, 1) \) and \( H = \alpha - \gamma/2 + 1/2 \in (0, 1/2) \). Here, \( c \) is the normalizing constant in (2.1).

We conduct the following analysis on the approximation of variance of VIX futures as in Section 6 of [7].
Let $\sqrt{\varsigma(T)}$ be the terminal value of the VIX futures, where

$$\varsigma(T) = \frac{1}{\Delta} \int_T^{T+\Delta} \mathbb{E}[v(u) \mid \mathcal{F}(T)] du,$$  \hfill (7.4)

where $\{\mathcal{F}(t), t \geq 0\}$ is the filtration generated by the process $P(t)$ with $v(u)$ in (7.3) using the process $X(t)$ in (5.12), and $\Delta$ is the period under consideration (e.g., a month).

Similar to Section 3.1 in [28], we obtain the following formula of forward variance curve: for $u \geq T > t$,

$$\mathbb{E}[v(u) \mid \mathcal{F}(T)] = \xi(t) \exp \left( \eta c \int_T^u (u-s)^{\alpha} s^{-\gamma/2} B(ds) \right)$$

$$\times \mathbb{E} \left[ \exp \left( \eta c \int_T^u (u-s)^{\alpha} s^{-\gamma/2} B(ds) - \frac{1}{2} \eta^2 c^2 \int_T^u (u-s)^{2\alpha} s^{-\gamma} ds \right) \mid \mathcal{F}(T) \right]$$

$$= \xi(t) \exp \left( \eta c \int_T^u (u-s)^{\alpha} s^{-\gamma/2} B(ds) \right)$$

$$\times \exp \left( \frac{1}{2} \eta^2 c^2 \int_T^u (u-s)^{2\alpha} s^{-\gamma} ds - \frac{1}{2} \eta^2 c^2 \int_T^u (u-s)^{2\alpha} s^{-\gamma} ds \right).$$

Using this and plugging into (7.4), we obtain an expression of $\varsigma(T)$. Direct evaluation of $\text{Var}(\varsigma(T))$ would require a good simulation scheme for the generalized R-L FBM, which we leave for future work. (The hybrid scheme for the Brownian semistationary processes in [9] cannot be adapted for our model.) We next discuss an approximation scheme for $\text{Var}(\varsigma(T))$.

To estimate the conditional variance of $\varsigma(T)$, we use the approximation:

$$\varsigma(T) \approx \exp \left( \frac{1}{\Delta} \int_T^{T+\Delta} \mathbb{E}[\log v(u) \mid \mathcal{F}(T)] du \right).$$  \hfill (7.5)

We then mimic the calculations in Appendix 3 of [7]. By the second expression of (7.3), we obtain

$$\int_T^{T+\Delta} \mathbb{E}[\log v(u) \mid \mathcal{F}(T)] du$$

$$= \eta c \int_T^{T+\Delta} \int_T^u (u-s)^{\alpha} s^{-\gamma/2} B(ds) du - \frac{1}{2} \eta^2 c^2 \int_T^{T+\Delta} \int_T^u (u-s)^{2\alpha} s^{-\gamma} ds du$$

$$= \eta c \int_T^{T+\Delta} \left( \int_T^u (u-s)^{\alpha} du \right) s^{-\gamma/2} B(ds)$$

$$- \frac{1}{2} \eta^2 c^2 \int_T^{T+\Delta} \left( \int_T^u (u-s)^{2\alpha} du \right) s^{-\gamma} ds$$

$$= \frac{\eta c}{\alpha + 1} \int_T^{T+\Delta} ((T + \Delta - s)^{\alpha+1} - (T - s)^{\alpha+1}) s^{-\gamma/2} B(ds)$$

$$- \frac{\eta^2 c^2}{2(2\alpha + 1)} \int_T^{T+\Delta} ((T + \Delta - s)^{2\alpha+1} - (T - s)^{2\alpha+1}) s^{-\gamma} ds.$$
By changing variables with $u = \frac{t - T}{T - t}$, we obtain that the integral is equal to
\[
\int_0^1 \left( (T-t)(1-u) + \Delta \right)^{a+1} - \left( (T-t)(1-u) \right)^{a+1} \right] (t + (T-t)u)^{-\gamma}(T-t)du
\]
\[
= (T-t)^{2a+3-\gamma} \int_0^1 \left[ \left( (1-u) + \frac{\Delta}{T-t} \right)^{a+1} - (1-u)^{a+1} \right]^2 \left( \frac{t}{T-t} + u \right)^{-\gamma} du.
\]
Thus, we have
\[
\text{Var} \left( \frac{1}{\Delta} \int_T^{T+\Delta} E[\log v(u) | F(T)]du \bigg| F(t) \right)
\]
\[
= \eta^2 (T-t)^{2a+1-\gamma} f_{a,\gamma} \left( \frac{\Delta}{T-t}, \frac{t}{T-t} \right),
\]
where $f_{a,\gamma}$ is a function defined by
\[
f_{a,\gamma}(a,b) = \left( \frac{c}{\alpha+1} \right)^2 \frac{1}{a^2} \int_0^1 \left[ \left( (1-u) + a \right)^{\alpha+1} - (1-u)^{\alpha+1} \right]^2 (b+u)^{-\gamma} du.
\]  
(7.6)
Recall that $H = \alpha + 1/2 - \gamma/2$; hence, the component $(T-t)^{2a+1-\gamma} = (T-t)^{2H}$, which is the same as that in [7]. When $\gamma = 0$, this function $f_{a,\gamma}(a,b)$ reduces to the function $f^H(\theta)$ given in [7], that is,
\[
f^H(\theta) = \frac{D_H}{\theta^2} \int_0^1 \left( (1+\theta-x)^{1/2+H} - (1-x)^{1/2+H} \right)^2 du.
\]
Indeed, note that $c = c(\alpha, \gamma)$ actually depends on $\alpha$ and $\gamma$, and is given by the formula
\[
c = c(\alpha, \gamma) = \text{Beta}(1-\gamma, 2\alpha+1)^{-1/2}.
\]  
(7.7)
(This is stated in Section 2.2, following from a similar argument in the proof of Lemma 2.1 in [27].) In the special case $\gamma = 0$, we have $c = (2\alpha+1)^{1/2} = (2H)^{1/2}$, and $\alpha + 1 = H + 1/2$, so the constant $c/(\alpha+1)$ matches $D_H = \sqrt{\frac{2H}{H+1/2}}$ in (6.1) of [7].

Therefore, with $f_{a,\gamma}$ in (7.6), the VIX variance swaps are then approximated by
\[
\text{VVIX}^2(t,T) \approx \text{Var} \left( \log \sqrt{\zeta(T)} \bigg| F(t) \right)
\]
\[
\approx \frac{1}{4} \eta^2 (T-t)^{2a+1-\gamma} f_{a,\gamma} \left( \frac{\Delta}{T-t}, \frac{t}{T-t} \right) =: \nu.
\]  
(7.8)
Note that unlike [7], we cannot simply regard $\text{VVIX}^2(t,T)$ as a function of $T-t$. Also note that the VIX$^2$ process measures the variance of the process of interest whereas VVIX$^2$ measures the variance of the VIX$^2$, that is, the variance of the variance.

To illustrate the impact of the parameters $(\alpha, \gamma)$, we provide Table 1 to show the different values $\nu$ of VVIX$^2$ approximations for a fixed $H = 0.05$, with the above parameter values $\eta = 2$, $T = 1$, $\Delta = 1/12$ and $t = 0.5, 0.75$. (Note that the value of $\eta = 2$ is chosen according to the estimates from SPX fits in Section
5.2.1 and Section 6.1 of [7]. We observe that for the same value of $H = 0.05$, different values of $(\alpha, \gamma)$ can lead to dramatically different values of VVIX$^2$ approximations in (7.8).

<table>
<thead>
<tr>
<th>$(\alpha, \gamma)$</th>
<th>$H$</th>
<th>$f_{\alpha,\gamma}$</th>
<th>VVIX$^2 \approx \nu$ in (7.8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(-0.45, 0)$</td>
<td>0.05</td>
<td>0.2413</td>
<td>0.2251</td>
</tr>
<tr>
<td>$(-0.4, 0.1)$</td>
<td>0.05</td>
<td>0.3930</td>
<td>0.3666</td>
</tr>
<tr>
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<td>0.05</td>
<td>0.4856</td>
<td>0.4531</td>
</tr>
<tr>
<td>$(-0.2, 0.5)$</td>
<td>0.05</td>
<td>0.4043</td>
<td>0.3772</td>
</tr>
<tr>
<td>$(0, 0.9)$</td>
<td>0.05</td>
<td>0.0718</td>
<td>0.0670</td>
</tr>
<tr>
<td>$(0.03, 0.96)$</td>
<td>0.05</td>
<td>0.0272</td>
<td>0.0254</td>
</tr>
</tbody>
</table>

Table 1. Illustration of the impact of $\gamma$ on the approximation $\nu$ of VVIX$^2$ in (7.8) for (a) $t = 0.50$ and for (b) $t = 0.75$.

To further illustrate the significance of the impact of the non-stationarity parameter $\gamma$, in Figure 2 we show $\nu$ in (7.8) for the same $H = 0.05$ with two pairs of $(\alpha, \gamma)$ in two perspectives: (a) as a function of $t$ for a fixed $T = 1$ and (b) as a function of $T \geq t$ for a fixed $t = 0.25$. Comparing with the case of standard FBM ($\alpha = -0.45, \gamma = 0$), we observe that the values of $\nu$ are larger for the pair $(\alpha, \gamma) = (-0.2, 0.5)$ for $t$ less than a certain threshold (as shown in Table 1 for the two times $t = 0.5, 0.75$, and then becomes smaller (but relatively close) for fixed $T = 1$, and they are larger over all $T$ for the fixed $t = 0.25$. More importantly, for a fixed $T = 1$, as $t$ is far from $T$, the difference in the values of $\nu$ can be substantial and as $t$ gets close to $T$, the difference shrinks to zero; on the other hand, for a fixed $t = 0.25$, as $T$ increases the difference in the values of $\nu$ remain somewhat constant.

Figure 3 shows the values in (7.8) with respect to $\gamma \in (0, 1)$ for each $H = 0.05, 0.1, 0.15$ and $t = 0.5, 0.75$ with $\alpha = \gamma/2 + H - 1/2$. Note that as $\gamma \uparrow 1$, the normalizing constant $c(\alpha, \gamma)$ in (7.7) converges $\lim_{\gamma \uparrow 1} c(\alpha, \gamma) = 0$, and so does $f_{\alpha,\gamma}$ in (7.6). Hence, the value $\nu$ in (7.8) converges to 0, as $\gamma \uparrow 1$. Moreover, for different values of $H$, we observe that when $\gamma$ is relatively small (close to zero), the differences of the VVIX$^2$ values in (7.8) are relatively large, while they become smaller as the value of $\gamma$ increases to 1. This implies that if there is some slight non-stationarity in the increments that is ignored in the analysis and model calibration, it can lead to substantial bias in the estimates of the VIX variance swap. For
\( \alpha = -0.2, \gamma = 0.5 \) \( \alpha = -0.45, \gamma = 0 \) \( \alpha = -0.2, \gamma = 0.5 \) \( \alpha = -0.45, \gamma = 0 \)

**Figure 2.** (a) The values \( v \) in (7.8) for different values of \( t \) with \( H = 0.05 \) and \( T = 1 \). (b) The values \( v \) in (7.8) for different values of \( T \geq t = 0.25 \) with \( H = 0.05 \).

Different times \( t \), the shapes of curves in Figures 3(a) and (b) look similar with slightly different values for each \( \gamma \).

**Figure 3.** (a) The values \( v \) in (7.8) for different values of \( \gamma \) for each \( H = 0.05, 0.1, 0.15 \) with \( \alpha = \gamma / 2 + H - (1/2) \), \( t = 0.5 \). (b) The values \( v \) in (7.8) for different values of \( \gamma \) for each \( H = 0.05, 0.1, 0.15 \) with \( \alpha = \gamma / 2 + H - (1/2) \), \( t = 0.75 \).

We also plot \( v \) in (7.8) as a surface of the parameters \( (\alpha, \gamma) \), again fixing the same parameter values \( \eta = 2, T = 1, t = 0.5, \Delta = 1/12 \) and \( 0 \leq \gamma < 1 \), for (a) \( t = 0.5 \) and (b) \( t = 0.75 \), in Figure 4(a) and Figure 4(b), respectively.

**Figure 4.** (a) The values \( v \) of the VVIX\(^2\) approximations in (7.8) for different values of \( (\alpha, \gamma) \) with \( T = 1, \Delta = 1/12, \eta = 2 \), (a) \( t = 0.5 \) and (b) \( t = 0.75 \).
Remark 7.1. We remark that in Remark 2.1 [7], the authors stated that the Riemann-Liouville FBM is an example of a Brownian semistationary (BSS) process [4], which is of the form \( X(t) = \int_{-\infty}^{t} g(t-s)\sigma(s)dB(s) \) for some deterministic function \( g \) and an adapted intermittency process \( \sigma(s) \). However, our generalized Riemann-Liouville FBM (5.12) does not belong to this class (BSS) of processes, since the process \( u^{-\gamma/2} \) in the definition of the \( X(t) \) in (5.12) violates all the "(semi)stationarity" conditions imposed upon \( \sigma(t) \) (see, e.g., [3]).

7.2. On other stochastic volatility models. In addition to the generalized rough Bergomi model above, we propose the following candidates to model the volatility process \( \sigma(t) \) in (7.1).

(i) The GFBM and its mixture models. The log-volatility \( \ln \sigma(t) \) takes the form \( \ln(\sigma(t)/\sigma(0)) = X(t) \) or \( Y(t) = \tilde{B}(t) + X(t), \ t \geq 0 \), where \( X(\cdot) \) is the GFBM given in (1.1), and \( Y(\cdot) \) is the mixture process in (5.1), of a standard BM \( \tilde{B}(\cdot) \) and the process \( X(\cdot) \) in (1.1). We recall the semimartingale properties of \( X \) and \( Y \) in Propositions 3.1 and 4.2, and Proposition 5.1, respectively. Although in the conventional stochastic volatility models, the volatility \( \sigma_t \) is often modeled as a continuous Brownian semi-martingale, for example, the Heston model and the SABR (stochastic alpha-beta-rho) model, one may also model the log-volatility process as a semimartingale in certain scenarios using the GFBM and its mixtures; see, e.g., the recent development of the corresponding rough volatility models using FBM in [29, 18, 22]. For instance, in the recent work on rough SABR model by Fukasawa and Gatheral [22], the log-volatility is modeled as a Riemann-Liouville (R-L) FBM, whose driving BM is correlated with the BM driving the stock price. There could be potential extensions of that rough SABR model using the generalized R-L FBM in (5.12) and its mixture.

(ii) The generalized fractional Ornstein-Uhlenbeck (fOU) processes driven by the GFBM \( X(\cdot) \) and its mixture \( Y(\cdot) \). The log-volatility \( \log \sigma(\cdot) \) takes the form

\[
\log(\sigma(t)/\sigma(0)) = Z(t), \ t \geq 0,
\]

where \( Z(\cdot) \) is the solution to the SDE driven by the GFBM \( X(\cdot) \) in (1.1):

\[
dZ(t) = -a(Z(t) - m)dt + \nu dX(t), \ t \geq 0,
\]

with \( m \in \mathbb{R} \) and \( a \) and \( \nu \) being positive constant parameters. Using pathwise Riemann-Stieltjes integral, we can also write the solution as

\[
Z(t) = Z(0)e^{-at} + m(1 - e^{-at}) + \nu \int_{0}^{t} e^{-a(t-s)}dX(s), \ t \geq 0.
\]

We refer to this as the generalized fOU process. In (7.10), one can also use the mixed GFBM process \( Y(\cdot) \) in (5.1). It is clear that in both models, the log-volatility process \( \log \sigma(\cdot) \) is not a semimartingale.

7.3. Comments on stochastic integrals with respect to the GFBM. Observe that we have used stochastic integrals with respect with the GFBM \( X(\cdot) \) in (6.8) and (6.9) in portfolio optimization and in (7.11) in rough
volatility models. In (6.8) and (6.9), the stochastic integral is of the type \( \int f(X) dX \), while in (7.11), the integrand is a deterministic function \( f(t - s) \).

Stochastic integrals with respect to FBM \( B^H \) have been extensively studied in the literature (see for example [39, Chapter 7], [36, Chapter 3] and [11, 37]). For \( H > 1/2 \), the stochastic integral \( \int f(B^H) dB^H \) can be defined pathwise using Young integral due to the regularity of the sample paths of \( B^H \), see, e.g., [36, Chapter 3.1]. Thanks to the variation property of the GFBM \( X \) in Proposition 4.1 when \( H > 1/2 \), the integral \( \int f(X) dX \) for the GFBM \( X \) can also be well defined pathwise using Young integral. On the other hand, for the standard FBM \( B^H \) with \( H < 1/2 \), the stochastic integral is studied using rough path theory [20] and/or Malliavin calculus [11, 37]. This study for the GFBM \( X \) with \( H < 1/2 \) is out of the scope of this paper, and will be investigated in the future.

When the integrand is a deterministic function of the form \( \int_0^t f(s) dB^H(s) \) or \( \int_0^t f(t - s) dB^H(s) \), the integral can be defined as a pathwise Riemann-Stieltjes integral, see Proposition A.1 in [16]. By a slight modification of the proof of that proposition, using self-similarity and the Hölder continuity properties of the GFBM \( X \), the same conclusions in Proposition A.1 of [16] also hold by replacing \( B^H \) with the GFBM \( X \). Therefore, the generalized fOU process \( Z \) in (7.11) is well defined, and so is the process driven by the mixed GFBM process \( Y \).

8. Appendix

8.1. Proof of Proposition 4.1. In this subsection, we prove Proposition 4.1. We first establish the following lemma.

**Lemma 8.1.** For every \( 0 < u < v \), the limits of the covariance of increments at \( u, v \) are given by

\[
\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^{2H}} \mathbb{E}[(X(u + \varepsilon) - X(u))^2] = c^2 \text{Beta}(1 + 2\alpha, 1 - \gamma) > 0, \tag{8.1}
\]

\[
\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^{2H}} \mathbb{E}[(X(u + \varepsilon) - X(u))(X(v + \varepsilon) - X(v))] = 0.
\]

**Proof of Lemma 8.1.** First, we consider the case \( u = v \). In this case, since the expectation in (8.1) becomes the second moment \( \Phi(u + \varepsilon, u) \) of the increment in (2.3), by a straightforward calculation (see, e.g., [27]) for \( s < t \),

\[
\mathbb{E}[|X(t) - X(s)|^2] = c^2(t - s)^{2H} \left( \text{Beta}(1 + 2\alpha, 1 - \gamma) \right.
\]

\[
+ \int_{s/(t-s)}^\infty [(1 + v)^{\alpha} - v^{\alpha}] \left( v - \frac{s}{t - s} \right)^{-\gamma} dv
\]

\[
+ \left[ \int_0^1 (1 - w)^{-\gamma}((1 + xw)^{\alpha} - (xw)^{\alpha})^2x^{1-\gamma}dw \right]_{\{x = s/(t-s)\}}
\]
and hence, substituting \( t = u + \varepsilon \) and \( s = u \) and dividing by \( \varepsilon^{2H} \) on both sides, we obtain the first claim in (8.1). Indeed, since \( \int_{0}^{\infty} [(1 + v)^{\alpha} - v^{\alpha}]^{2} v^{-\gamma} dv < \infty \) under (1.2) with \( \gamma \in (0, 1) \), we have

\[
\int_{u/\varepsilon}^{\infty} [(1 + v)^{\alpha} - v^{\alpha}]^{2} (v - \frac{u}{\varepsilon})^{-\gamma} dv = \int_{u/\varepsilon}^{\infty} \left[ (1 + v - \frac{u}{\varepsilon})^{\alpha} - (v - \frac{u}{\varepsilon})^{\alpha} \right]^{2} v^{-\gamma} dv \xrightarrow[\varepsilon \downarrow 0]{0}, \quad u > 0,
\]

for the second term, and for the third term, we apply the dominated convergence theorem to obtain

\[
\int_{0}^{1} (1 - w)^{-\gamma}((1 + xw)^{\alpha} - (xw)^{\alpha})^{2}x^{1-\gamma}dw \xrightarrow[x \to \infty]{0},
\]

because for every \( x \geq 1 \), by the monotonicity of \( x \mapsto (1 + xw)^{\alpha} - (xw)^{\alpha}x^{1-\gamma} \),

\[
\int_{0}^{1} (1 - w)^{-\gamma}((1 + xw)^{\alpha} - (xw)^{\alpha})^{2}x^{1-\gamma}dw \leq \int_{0}^{1} (1 - w)^{-\gamma}((1 + w)^{\alpha} - w^{\alpha})^{2}dw < \infty
\]

in the case of \( \alpha > 0 \), and similarly for every \( x \geq 1 \),

\[
\int_{0}^{1} (1 - w)^{-\gamma}((1 + xw)^{\alpha} - (xw)^{\alpha})^{2}x^{1-\gamma}dw \leq \int_{0}^{1} (1 - w)^{-\gamma}w^{2\alpha}dw = \text{Beta}(1 - \gamma, 1 + 2\alpha) < \infty
\]

in the case of \( \alpha < 0 \). See the proof of Theorem 3.1 in [27] for the derivation of these upper bounds.

Next, we consider the case \( u < v \) by evaluating the sum of four terms

\[
\frac{1}{c^{2}}E[(X(u + \varepsilon) - X(u))(X(v + \varepsilon) - X(v))] = C_{1,\varepsilon} + C_{2,\varepsilon} + C_{3,\varepsilon} + C_{4,\varepsilon}, \quad (8.2)
\]

where

\[
C_{1,\varepsilon} := \int_{0}^{u} ((u + \varepsilon - w)^{\alpha} - (u - w)^{\alpha})(v + \varepsilon - w)^{\alpha} - (v - w)^{\alpha})|w|^{-\gamma}dw,
\]

\[
C_{2,\varepsilon} := \int_{u}^{u+\varepsilon} (u + \varepsilon - w)^{\alpha}((v + \varepsilon - w)^{\alpha} - (v - w)^{\alpha})|w|^{-\gamma}dw,
\]

\[
C_{3,\varepsilon} := \int_{0}^{1} ((u + \varepsilon + w)^{\alpha} - (u + w)^{\alpha})(v + \varepsilon + w)^{\alpha} - (v + w)^{\alpha})|w|^{-\gamma}dw,
\]

\[
C_{4,\varepsilon} := \int_{1}^{\infty} ((u + \varepsilon + w)^{\alpha} - (u + w)^{\alpha})(v + \varepsilon + w)^{\alpha} - (v + w)^{\alpha})|w|^{-\gamma}dw.
\]

For the first term \( C_{1,\varepsilon} \), we consider the case \( \alpha > 0 \) first. Note that a simple application of the Hölder continuity, i.e., \( x^{\alpha} - y^{\alpha} \leq |x - y|^{\alpha} \), \( x, y > 0 \) implies \( C_{1,\varepsilon} \leq \int_{0}^{u} \varepsilon^{2\alpha}|w|^{-\gamma}dw = u^{1-\gamma}\varepsilon^{2\alpha} \) but then we may not use this inequality to show \( \lim_{\varepsilon \downarrow 0} C_{1,\varepsilon}/\varepsilon^{2H} = 0 \), because \( 2\alpha \leq 2H = 2\alpha - \gamma + 1 \). We shall use the estimates of the difference \( x^{\alpha} - y^{\alpha} \) for \( 0 < y < x \) outside the neighborhood of the origin. If \( \alpha > 0 \), using the inequality

\[
(v + \varepsilon - w)^{\alpha} - (v - w)^{\alpha} = (v - w)^{\alpha}\left(1 + \frac{\varepsilon}{v - w}\right)^{\alpha} - 1 \right) \leq \alpha(v - u)^{\alpha-1}\varepsilon \quad (8.3)
\]

for \( 0 < w < u < v \), we have

\[
C_{1,\varepsilon} \leq \alpha(v - u)^{\alpha-1}\varepsilon \cdot \int_{0}^{u} ((u + \varepsilon - w)^{\alpha} - (u - w)^{\alpha})w^{-\gamma}dw, \quad (8.4)
\]
where the integral on the right hand is evaluated by
\[
\int_{0}^{u} ((u + \varepsilon - w)^\alpha - (u - w)^\alpha) w^{-\gamma} dw \\
= \int_{0}^{u+\varepsilon} (u + \varepsilon - w)^\alpha w^{-\gamma} dw - \int_{u}^{u+\varepsilon} (u - w)^\alpha w^{-\gamma} dw \leq (u + \varepsilon)^{\alpha - \gamma + 1} \text{Beta}(1 + \alpha, 1 - \gamma) - \int_{u}^{u+\varepsilon} (u + \varepsilon - w)^\alpha w^{-\gamma} dw
\]
and
\[
0 \leq \int_{u}^{u+\varepsilon} (u + \varepsilon - w)^\alpha w^{-\gamma} dw \leq u^{-\gamma} \int_{u}^{u+\varepsilon} (u + \varepsilon - w)^\alpha dw = \frac{u^{-\gamma}}{1 + \alpha} \cdot \varepsilon^{\alpha + 1}.
\]
Here, \( \alpha - \gamma + 1 > 0 \). Thus, if \( \alpha > 0 \), combining these inequalities together with (8.4), we obtain
\[
\frac{1}{\varepsilon^{2H}} C_{1,\varepsilon} \leq \alpha (v - u)^{\alpha - 1} \varepsilon^{-2H} ((\alpha - \gamma + 1) u^{\alpha - \gamma} \text{Beta}(1 + \alpha, 1 - \gamma) + \frac{u^{-\gamma}}{1 + \alpha} \cdot \varepsilon^{\alpha + 1})
\]
\[
= \alpha (v - u)^{\alpha - 1} (\alpha - \gamma + 1) u^{\alpha - \gamma} \text{Beta}(1 + \alpha, 1 - \gamma) \varepsilon^{2(1-H)} + \alpha (v - u)^{\alpha - 1} \frac{u^{-\gamma}}{1 + \alpha} \cdot \varepsilon^{2(1-H)+\alpha}
\]
\[
\overset{\varepsilon \downarrow 0}{\longrightarrow} 0,
\]
because \( 0 < H < 1 \) and \( 2(1 - H) + \alpha = 1 - \alpha + \gamma > 0 \).

Still for the first term \( C_{1,\varepsilon} \), we consider the case \( \alpha < 0 \). Let \( \tilde{\alpha} := -\alpha > 0 \). Then using (8.3) with the power \( \tilde{\alpha} > 0 \), instead of the power \( \alpha \), we have
\[
| (v + \varepsilon - w)^\alpha - (v - w)^\alpha | = \left| \frac{(v + \varepsilon - w)^{\tilde{\alpha}} - (v - w)^{\tilde{\alpha}}}{(v + \varepsilon - w)^{\tilde{\alpha}} (v - w)^{\tilde{\alpha}}} \right| \leq \frac{\frac{\tilde{\alpha}(v - u)^{\tilde{\alpha}-1}}{(v + \varepsilon - w)^{\tilde{\alpha}} (v - w)^{\tilde{\alpha}}} | \varepsilon
\]
for \( 0 < w < u \). Using this inequality, we obtain
\[
|C_{1,\varepsilon}| \leq \frac{\tilde{\alpha}(v - u)^{\tilde{\alpha}-1}}{(v + \varepsilon - u)^{\tilde{\alpha}} (v - u)^{\tilde{\alpha}}} | \varepsilon \cdot \int_{0}^{u} |(u + \varepsilon - w)^\alpha - (u - w)^\alpha| w^{-\gamma} dw,
\]
where the integral on the right hand side is now evaluated by
\[
\int_{0}^{u} |(u + \varepsilon - w)^\alpha - (u - w)^\alpha| w^{-\gamma} dw \\
= -\int_{0}^{u+\varepsilon} (u + \varepsilon - w)^\alpha w^{-\gamma} dw + \int_{0}^{u} (u - w)^\alpha w^{-\gamma} dw \leq \frac{\tilde{\alpha}(v - u)^{\tilde{\alpha}-1}}{(v + \varepsilon - u)^{\tilde{\alpha}} (v - u)^{\tilde{\alpha}}} | \varepsilon \cdot \int_{0}^{u+\varepsilon} (u + \varepsilon - w)^\alpha w^{-\gamma} dw
\]
\[
\leq |(u + \varepsilon)^{\alpha - \gamma + 1} - u^{\alpha - \gamma + 1}| \text{Beta}(1 + \alpha, 1 - \gamma) + \int_{u}^{u+\varepsilon} (u + \varepsilon - w)^\alpha w^{-\gamma} dw.
\]
Thus, from here, we may follow the same steps after the inequality (8.5) and we obtain \( \lim_{\varepsilon \downarrow 0} |C_{1,\varepsilon}| / \varepsilon^{2H} = 0 \) under the case \( \alpha < 0 \) as well.

With similar reasoning, we obtain \( \lim_{\varepsilon \downarrow 0} |C_{i,\varepsilon}| / \varepsilon^{2H} = 0 \) for \( i = 2, 3, 4 \). Therefore, combining these estimate with (8.2) we conclude the second claim in (8.1).

We are now ready to prove Proposition 4.1.

**Proposition 4.1.** Recall that if \( \tilde{Z} \) is a normal random variable with mean 0 and variance \( \sigma^2 > 0 \), then \( \mathbb{E}[|\tilde{Z}|^p] = \sigma^p \sqrt{2\pi p} \Gamma((1 + p)/2) \) for \( p \geq 1 \) by the Gamma function formula. Thus, combining this fact with the consequence

\[
\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^{2H}} \operatorname{Var}(X(u + \varepsilon) - X(u)) = c^2 \text{Beta}(1 + 2\alpha, 1 - \gamma),
\]

from (8.1), we obtain

\[
\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \mathbb{E}[|X(u + \varepsilon) - X(u)|^{1/H}]
= \left(c^2 \text{Beta}(1 + 2\alpha, 1 - \gamma)\right)^{1/(2H)} \cdot \sqrt{\frac{2^{1/H}}{\pi} \Gamma\left(\frac{1 + (1/H)}{2}\right)},
\]

as the convergence of the moments of normal random variables. Thus it suffices to show

\[
\lim_{\varepsilon \downarrow 0} \mathbb{E}\left[\left(\frac{1}{\varepsilon} \int_0^t |X(s + \varepsilon) - X(s)|^{1/H} \, ds\right)^2\right] = (\rho_{\alpha, \gamma} t)^2, \quad t > 0. \tag{8.6}
\]

To show (8.6), we follow the proofs of Proposition 1 of [42] and Proposition 3.14 of [43], and use our Lemma 8.1. By the Fubini theorem, we rewrite

\[
\mathbb{E}\left[\left(\frac{1}{\varepsilon} \int_0^t |X(s + \varepsilon) - X(s)|^{1/H} \, ds\right)^2\right] = 2 \int_{[0,t]^2} m_\varepsilon \cdot 1_{\{u < v\}} \, du \, dv, \tag{8.7}
\]

where we define

\[
m_\varepsilon := \frac{1}{\varepsilon^2} \mathbb{E}\left[|(X(u + \varepsilon) - X(u))(X(v + \varepsilon) - X(v))|^{1/H}\right], \quad \varepsilon > 0. \tag{8.8}
\]

The joint distribution of of the increments \( X(u + \varepsilon) - X(u) \) and \( X(v + \varepsilon) - X(v) \) is normal with zero mean and variances defined by

\[
\sigma_{1,\varepsilon}^2 := \mathbb{E}[(X(u + \varepsilon) - X(u))^2] = \Phi(u + \varepsilon, u),
\]

\[
\sigma_{2,\varepsilon}^2 := \mathbb{E}[(X(v + \varepsilon) - X(v))^2] = \Phi(v + \varepsilon, v)
\]

and the covariance defined by

\[
\vartheta_\varepsilon := \mathbb{E}[(X(u + \varepsilon) - X(u))(X(v + \varepsilon) - X(v))].
\]
By the the conditional distribution of \( X(v + \varepsilon) - X(v) \), given \( X(u + \varepsilon) - X(u) \), we compute \( m_\varepsilon \) in (8.8) as

\[
m_\varepsilon = \frac{1}{\varepsilon^2} E \left[ \sigma_{1,\varepsilon} \cdot Z_1 + \frac{\partial^2 \Psi}{\partial u \partial v}(u, v) \sigma_{1,\varepsilon} \cdot Z_1 + \sqrt{1 - \frac{\partial^2 \Psi}{\partial u \partial v} \sigma_{2,\varepsilon}^2} Z_1 \right]^{1/H}.
\]

Here, \( Z_i, i = 1, 2 \) are i.i.d. standard normal random variables with mean 0 and variance 1. Then it follows from Lemma 8.1 that

\[
\lim_{\varepsilon \downarrow 0} \sigma_{i,\varepsilon}^2 / \varepsilon^{2H} = c^2 \text{Beta}(1 + 2\alpha, 1 - \gamma) \quad \text{for} \quad i = 1, 2,
\]

and

\[
\lim_{\varepsilon \downarrow 0} \frac{\partial^2 \Psi}{\partial u \partial v}(u, v) = 0, \quad \lim_{\varepsilon \downarrow 0} \frac{\partial^2 \Psi}{\partial u \partial v}(u, v) = 0, \quad \lim_{\varepsilon \downarrow 0} \frac{\sigma_{1,\varepsilon} \sigma_{2,\varepsilon}}{\varepsilon^{2H}} = c^2 \text{Beta}(1 + 2\alpha, 1 - \gamma).
\]

Substituting these upper bounds of both terms and using again the inequality \((x + y)^2 \leq 2(x^2 + y^2)\) for \( x, y > 0 \) and the symmetry of integral region, we obtain

\[
\int_0^T \int_0^T \left[ \frac{\partial^2 \Psi}{\partial u \partial v}(u, v) \right]^2 \, du \, dv \leq 4c^4 \int_0^T \left( \int_0^v (f_1(u, v))^2 + (f_2(u, v))^2 \, du \right) \, dv. \tag{8.9}
\]

Here, the first term \( f_1(u, v) \) is bounded by

\[
f_1(u, v) \leq \int_0^u v^{\alpha-1}(u - \theta)^{\alpha-1} \, d\theta = \text{Beta}(\alpha, 1 - \gamma) u^{\alpha-\gamma} v^{\alpha-1},
\]

for \( u < v \). Since \( 0 < \alpha < 1 \) and \((uw)^{\alpha-1}(vw)^{\alpha-1} \geq (1 + uw)^{\alpha-1}(1 + vw)^{\alpha-1}\) for \( u, v > 0 \), we have a bound for the second term \( f_2(u, v) \),

\[
f_2(u, v) = \int_0^{\infty} (uv)^{\alpha-\gamma+1}(1 + uw)^{\alpha-1}(1 + vw)^{\alpha-1} w^{-\gamma} \, dw
\]

\[
\leq (uv)^{2\alpha-\gamma} \int_1^{\infty} w^{2\alpha-2-\gamma} \, dw
\]

\[
+ (uv)^{\alpha-\gamma+1} \int_0^{1} (1 + uw)^{\alpha-1}(1 + vw)^{\alpha-1} w^{-\gamma} \, dw
\]

\[
\leq \frac{(uv)^{2H-1}}{2 - 2H} + \frac{(uv)^{\alpha-\gamma+1}}{1 - \gamma}.
\]

Substituting these upper bounds of both terms and using again the inequality \((x + y)^2 \leq 2(x^2 + y^2)\), \( x, y > 0 \) for the second term, we obtain the estimates

\[
\int_0^T \int_0^v (f_1(u, v))^2 \, du \, dv \leq \left[ \text{Beta}(\alpha, 1 - \gamma) \right]^2 \int_0^T \left( \int_0^v u^{2\alpha-2\gamma} v^{2\alpha-2} \, du \right) \, dv
\]

\[
= \left[ \text{Beta}(\alpha, 1 - \gamma) \right]^2 T^{4\alpha-2\gamma} \frac{2(2\alpha - \gamma)(2\alpha - 2\gamma + 1)}{2(2\alpha - \gamma)(2\alpha - 2\gamma + 1)} < \infty,
\]
and
\[
\int_0^T \int_0^v (f_2(u,v))^2 \, du \, dv
\leq \int_0^T \left[ \int_0^v \left( \frac{(uv)^{2H-1}}{2-2H} + \frac{(uv)^{\alpha-\gamma+1}}{1-\gamma} \right)^2 \, du \right] \, dv
\]
\[
\leq 2 \int_0^T \int_0^v \left( \frac{(uv)^{4H-2}}{(2-2H)^2} + \frac{(uv)^{2\alpha-2\gamma+2}}{(1-\gamma)^2} \right) \, du \, dv
\]
\[
= \frac{(2-2H)^2(4H-1)(4H+1) + (1-\gamma)^2(4H-2\gamma+3)(2H-\gamma+2)}{T^{4H-2\gamma+4}} < \infty.
\]
The right hand sides are finite when \(2\alpha > \gamma\) and \(0 < \gamma < 1\).

Therefore, combining these estimates with (8.9), we conclude the second derivative \(K(u,v)\) is square integrable in \((0,T) \times (0,T)\).

(ii) Suppose that \(-1/2 + \gamma/2 < \alpha \leq \gamma/2\) and \(0 < \gamma < 1\). Since \((x+y)^2 \geq x^2\) for \(x,y > 0\), we shall show
\[
\int_0^T \int_0^T [K(u,v)]^2 \, du \, dv \geq 2 \int_0^T \int_0^v [f_1(u,v)]^2 \, du \, dv = \infty.
\]
To do so, by the change-of-variable and by Jensen’s inequality, we observe that if \(u \leq v\),
\[
f_1(u,v) = u^{\alpha-\gamma} \int_0^{v(uw)^{\alpha-1}} (v-w)^{\alpha-1} w^{-\gamma} (1-w)^{\alpha-1} \, dw
\]
\[
= u^{\alpha-\gamma} \beta(a,1-\gamma) \int_0^{v(uw)^{\alpha-1}} (v-w)^{\alpha-1} w^{-\gamma} (1-w)^{\alpha-1} \, dw
\]
\[
\geq \beta(a,1-\gamma) u^{\alpha-\gamma} \left( v-u \cdot \frac{\alpha}{\alpha + 1-\gamma} \right)^{\alpha-1},
\]
because \(w \mapsto (v-w)^{\alpha-1}\), \(0 < w < 1\) is a convex function and the expectation of Beta distribution with parameters \((\alpha, 1-\gamma)\) is \(\alpha/(\alpha+1-\gamma)\). Thus, we have a lower bound for \(\int_0^v [f_1(u,v)]^2 \, du\), that is,
\[
\int_0^v [f_1(u,v)]^2 \, du
\geq \int_0^v \left[ \beta(a,1-\gamma) \right]^2 u^{2(\alpha-\gamma)} \left( v-u \cdot \frac{\alpha}{\alpha + 1-\gamma} \right)^{2(\alpha-1)} \, du
\]
\[
= [\beta(a,1-\gamma)]^2 v^{4\alpha-2\gamma-1} \int_0^{1} \left( 1-\theta \cdot \frac{\alpha}{\alpha + 1-\gamma} \right)^{2\alpha-2} \theta^{2\alpha-2\gamma} \, d\theta
\]
\[
= [\beta(a,1-\gamma)]^2 \cdot \frac{1}{v^{4\alpha-2\gamma-1}} \left( \frac{1-\gamma}{\alpha + 1-\gamma} \right)^{2\alpha-2} \frac{2\alpha-2\gamma+1}{2\alpha-2\gamma+1}
\]
for \(0 < v < T\). However, if \(2\alpha \leq \gamma\), this lower bound is not integrable over \((0,T)\), and hence, \(K(\cdot, \cdot)\) is not square integrable over \((0,T) \times (0,T)\).
\[
\square
\]

**References**


