Multi-patch multi-group epidemic model with varying infectivity

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Abstract. This paper presents a law of large numbers result, as the size of the population tends to infinity, of SIR stochastic epidemic models, for a population distributed over $L$ distinct patches (with migrations between them) and $K$ distinct groups (possibly age groups). The novelty is that the infectivity of an infected individual is infection age dependent. More precisely, to each infected individual is attached a random infection age dependent infectivity, such that the various random functions attached to distinct individuals are i.i.d. Our proof assumes less restriction on those random functions than our earlier proof for the homogeneous model in [8]. We first explain the new proof under the weak assumptions for the homogeneous model, and then describe the multipatch–multigroup model and prove the law of large numbers for that model.

1. Introduction

It is well–known that ODE epidemic models are law of large numbers limits, as the size of the population tends to infinity, of individual based stochastic Markov models, see e.g. [1] and Chapter 2 of Part I in [5]. The Markov property of such stochastic epidemic models requires in particular that the duration of the infectious period of the individuals follows an exponential distribution. In [15], the last two authors have shown that the large population law of large numbers limit of a non-Markov stochastic epidemic model is a system of Volterra type integral equations. An alternative formulation leads to a system of first order partial differential equations with an integral equation as boundary condition, see [14].

On the other hand, not only should the infectious period be non-exponentially distributed, but, as was advocated as early as in 1927 by Kermack and McKendrick in [11], the infectivity should be allowed to vary with the time elapsed since infection. In [8], the authors have obtained the Kermack–McKendrick model as the large population law of large numbers limit of a stochastic model, where to each infectious individual is attached an independent copy of a random infection age dependent random infectivity function. Note that the last two authors have established a central limit theorem in the same framework, see [13].

The present paper contains two novelties. First, the techniques employed in [8] for the SIR model with varying infectivity to prove the law of large numbers require that each infectivity function be uniformly bounded and satisfy a regularity condition as imposed in [8, Assumption 2.1]. Specifically, the infectivity function is assumed to have at most finitely many jumps, and to satisfy a uniform continuity assumption between the jumps. These conditions were necessary in the proof of convergence of the aggregate infectivity process in $D$, using standard tightness criteria via the conditions on the modulus of continuity as stated in [3] (see the specific criteria used in Lemmas 4.3–4.5 in [8]). In fact, the same technique fails to work for a model with both infection age varying infectivity and recovery age varying susceptibility in [10]. Rather, the only approach with which we were able to prove the result in that case was using a comparison of the model with a sequence of i.i.d. processes, which is easily shown to converge thanks to the law of large numbers for processes in $D$, cf. [17]. The construction of the law of those processes involves the solution of a McKean–Vlasov type Poisson driven SDE (see equation (2.5) for the homogeneous model and
equation (4.9) for the multi-patch multi-group model), as in the “propagation of chaos” theory, see Sznitman [18]. Note that our result can in fact be interpreted as a propagation of chaos result, and that the general approach of the proof is inspired by the work of Chevallier [7]. The advantage of this new approach is that the proof requires much fewer assumptions on the infection age dependent infectivity functions than those needed in our previous paper [8]: namely, we only assume that those functions belong a.s. to $D$ and are uniformly bounded. We first present this new proof of the law of large numbers result from [8] for the homogeneous model with varying infectivity in Section 2.

The main goal (and the second novelty) of the present paper is to adapt this approach to a multipatch–multigroup model. In this model, the population is divided into groups, which are mainly thought of being age groups, and into patches, which are geographical distinct areas. Individuals remain in the same group during the epidemic, while they may move from one patch to another. Infectious individuals infect susceptible ones from other groups, but also from other patches. The infection rate is assumed to take a very general form, as given in formula (3.2). It allows different infection rates from each group and patch to others, thus including non-local infections. The main reason for allowing non-local infections is that the propagation of an epidemic from one patch to another is partly due to movements of individuals going from home to work and back, as well as those who visit a given place during holidays or weekends, and then return home. These movements cannot be conveniently modeled as migrations, and their effect on the epidemic are infections at distance. Our model with multiple patches and groups captures both spatial and population heterogeneity. The multi-patch SEIR model with a homogeneous population in each patch but with a constant infectivity rate was recently considered in [12], where both the law of large numbers and functional central limit theorems (FCLTs) are established. Note that the techniques for the proof of convergence in [12] also use the standard tightness criteria as discussed above, and as in [15] do not require any condition on the exposed and infectious period distributions (except that the distributions for the initially infected individuals are continuous for the FCLT). In the present paper, since the infectious periods are induced from the random infectivity functions, for which no regularity conditions are imposed, the distributions of the infectious periods are also general. A multipatch-multigroup epidemic model was recently studied in [2], which focuses on the ODEs for the Markovian SEIRS model and its global stability property. Our work is also motivated by the study in Britton et al. [4] on the influence of population heterogeneity on herd immunity in the recent Covid-19 pandemic.

The main result for the multipatch–multigroup model is stated in Theorem 3.1, where the law of large numbers limit is given by a system of Volterra type integral equations. Besides the complication of the notation (a double index for the group and the patch), the main difficulty of adapting the new proof to the multipatch–multigroup model is the need for a formula for the proportion (in the large population limit) of susceptibles from group $k$ located in patch $\ell$ at time $t$. In the homogeneous model, this formula is the well-known formula for the solution of a linear one-dimensional ODE (see equation (2.6) and its use in the proof of Lemma 2.1). In the multipatch–multigroup model, this formula is replaced by formula (4.8) in Proposition 4.1. This is the formula for the solution at time $t$ of a forward ODE, which is the law of the location of a susceptible, weighted by an exponential factor taking into account the patches visited between time 0 and time $t$. That exponential factor is the conditional probability, given the various positions of the individual during the time interval $[0, t]$, of not having been infected by time $t$. The proof of that formula relies upon the Feynman–Kac formula for an adjoint backward ODE, which is established in Lemma 4.1. This formula for the proportion (in the large population limit) of susceptibles from group $k$ located in patch $\ell$ plays a crucial role in the subsequent proofs. First of all, it is used to establish the existence of a unique solution to the system of McKean-Vlasov Poisson-driven SDEs (equation (4.9)) in Lemma 4.2. Then it is used repeatedly in the proof of Theorem 3.1, see the proofs of Lemmas 4.5–4.7. Notably, the proof of the law of large numbers for the multipatch–multigroup model is much more sophisticated than the same proof for the homogeneous model.
The paper is organized as follows. In Section 2, we present the new proof of the law of large numbers for the homogeneous model, and in Section 3, we describe our multipatch–multigroup model, and state the law of large numbers for this model. In Section 4, we provide the proof for that main result. Specifically, in Section 4.1 we prove the existence and uniqueness of a solution to the limiting system of integral equations, in Section 4.2, we derive an expression for the limiting proportion of susceptibles in each patch and group via a Feynman-Kac formula for the associated backward ODEs, in Section 4.3, we propose an auxiliary system of Poisson-driven McKean-Vlasov SDEs and prove that it has a unique solution, in Section 4.4, we construct a sequence of i.i.d. processes from the solution to the Poisson-driven SDEs and establish estimates for the differences between the original processes and the i.i.d. processes for various quantities, and finally in Section 4.5, we complete the proof of the law of large numbers using the constructed i.i.d. processes and the estimates from previous sections.

2. The homogeneous model

We reformulate the SIR epidemic model with varying infectivity, and obtain the LLN result under weaker assumptions than in the authors’ previous paper [8].

Let \( \{\lambda_{-j}, j \geq 1\} \) and \( \{\lambda_j, j \geq 1\} \) be two mutually independent sequences of i.i.d. random elements of \( D \), where \( \lambda_{-1} \) has the law of the infectivity of initially infectious individuals, and \( \lambda_1 \) has the law of the infectivity of an individual infected after time 0. We only assume that there exists a deterministic \( \lambda^* > 0 \) such that \( 0 \leq \lambda_j(t) \leq \lambda^* \), for all \( j \in \mathbb{Z}\setminus\{0\} \) and \( t \geq 0 \). We extend \( \lambda_j(t) \) for \( j \geq 1 \) to all \( t \in \mathbb{R} \), assuming that \( \lambda_j(t) = 0 \) for \( t < 0 \). We next define, for each \( j \in \mathbb{Z}\setminus\{0\} \),

\[
\eta_j := \inf\{t > 0, \lambda_j(t + r) = 0, \forall r > 0\}. 
\]

We denote by \( F(t) := \mathbb{P}(\eta_1 \leq t) \) and \( F_0(t) := \mathbb{P}(\eta_{-1} \leq t) \) the distribution functions of \( \eta_j \) for \( j \geq 1 \) and for \( j \leq -1 \) respectively. Also, let \( F^0 = 1 - F_0 \) and \( F^c = 1 - F \). Finally we define \( \lambda^0(t) = \mathbb{E}[\lambda_{-1}(t)], \lambda(t) = \mathbb{E}[\lambda_1(t)] \).

2.1. Model and Results. We split our population in two subsets: those who were infected at time \( t = 0 \), there are \( I^N(0) \) of them, and those who were susceptible at time \( t = 0 \), there are \( S^N(0) \) of them. We assume that \( I^N(0) := N^{-1}I^N(0) \to \bar{I}(0) \) and \( S^N(0) := N^{-1}S^N(0) \to \bar{S}(0) \) in probability, where \( (I^N(0), \bar{S}(0)) \in (0,1)^2 \) are deterministic and such that \( \bar{I}(0) + \bar{S}(0) \leq 1 \).

For \( 1 \leq j \leq S^N(0) \), we define \( A_j^N(t) \) to be the \{0,1\}–valued counting process which is zero if the individual \( j \) has not been infected by time \( t \), and 1 if he/she has been infected by time \( t \). We also define \( \tau_j^N := \inf\{t > 0, A_j^N(t) = 1\} \).

The total force of infection in the population at time \( t \) is

\[
\bar{\mathfrak{F}}^N(t) = \sum_{j=1}^{I^N(0)} \lambda_{-j}(t) + \sum_{j=1}^{S^N(0)} \lambda_j(t - \tau_j^N). \tag{2.1}
\]

Moreover, noting \( \bar{\mathfrak{F}}^N(t) := N^{-1}\mathfrak{F}^N(t) \), we define the \( A_j^N \)'s as follows:

\[
A_j^N(t) = \int_0^t \int_0^\infty \mathbf{1}_{A_j^N(s^-) = 0} \mathbf{1}_{u \leq \bar{\mathfrak{F}}^N(s^-)} Q_j(ds, du), \tag{2.2}
\]

where \( \{Q_j, j \geq 1\} \) are mutually independent standard Poisson random measures (PRMs) on \( \mathbb{R}_+^2 \). Denoting by \( S^N(t) \) the number of susceptible individuals in the population at time \( t \), we clearly have

\[
S^N(t) = S^N(0) - \sum_{j=1}^{S^N(0)} A_j^N(t). \tag{2.3}
\]

We define \( \bar{S}^N(t) := N^{-1}S^N(t) \).
In addition, the processes $I^N(t)$ and $R^N(t)$ can be written as

\[
I^N(t) = \sum_{j=1}^{N(0)} 1_{\eta_j > t} + \sum_{j=1}^{S(0)} 1_{t \geq \tau^N_j > t - \eta_j},
\]

\[
R^N(t) = \sum_{j=1}^{N(0)} 1_{\eta_j \leq t} + \sum_{j=1}^{S(0)} 1_{\tau^N_j + \eta_j \leq t},
\]

Let $\bar{I}^N(t) := N^{-1}I^N(t)$ and $\bar{R}^N(t) := N^{-1}R^N(t)$.

We prove the following LLN result.

**Theorem 2.1.** As $N \to \infty$, $(\bar{S}^N, \bar{\bar{S}}^N, \bar{I}^N(t), \bar{R}^N(t)) \to (\bar{S}, \bar{\bar{S}}, \bar{I}(t), \bar{R}(t))$ in $D^4$ in probability, where for $t \geq 0$, the limits \((\bar{S}, \bar{\bar{S}})\) are the unique solution to the following two integral equations

\[
\bar{S}(t) = \bar{S}(0) - \int_0^t \bar{S}(s) \bar{\bar{S}}(s) ds,
\]

\[
\bar{\bar{S}}(t) = \bar{I}(0) \bar{\lambda}^0(t) + \int_0^t \bar{\lambda}(t - s) \bar{S}(s) \bar{\bar{S}}(s) ds,
\]

and then given the solution \((\bar{S}, \bar{\bar{S}})\), the limits \((\bar{I}(t), \bar{R}(t))\) are given by the following integral expressions

\[
\bar{I}(t) = \bar{I}(0) F_0^c(t) + \int_0^t F_c(t - s) \bar{S}(s) \bar{\bar{S}}(s) ds,
\]

\[
\bar{R}(t) = \bar{I}(0) F_0(t) + \int_0^t F(t - s) \bar{S}(s) \bar{\bar{S}}(s) ds.
\]

### 2.2. The new idea.

The new idea is to associate to a standard PRM $Q$ on $\mathbb{R}_+^2$, the process $A(t)$ which solves

\[
A(t) = \int_0^t \int_0^\infty 1_{A(s) = 0} 1_{u \leq \bar{\lambda}(s - \tau)} Q(ds, du),
\]

where

\[
\bar{\lambda}(t) = \bar{I}(0) \bar{\lambda}^0_1(t) + \bar{S}(0) \mathbb{E}[\lambda(t - \tau)], \quad \tau = \inf\{t > 0, A(t) = 1\}.
\]

Here $\bar{\lambda}^0_1(t) = \mathbb{E}[\lambda_{-1}(t)]$, and $\lambda$ is a random element of $\mathbb{D}$ which is independent of $Q$ (hence also of $\tau$), and has the same law as $\lambda_1$. Note that clearly $\lambda(t - \tau) = \int_0^t \lambda(t - s)dA(s)$.

**Remark 2.1.** In this SDE, the second equation of $\bar{\lambda}(t)$ depends on the law of $\lambda$ with the unknown $\tau$ as a functional of $A(t)$. We call an SDE where the law of the unknown function enters the coefficients a McKean–Vlasov SDE. The equation above can be regarded as a Poisson-driven McKean–Vlasov SDE.

McKean–Vlasov SDE equations appear naturally in the theory of propagation of chaos, see Sznitman [18]. In fact, this new approach can be considered as establishing a propagation of chaos result for the times of infection of the initially susceptible individuals.

Observe that by the first equation in (2.4), we have the formula

\[
\bar{S}(t) = \bar{S}(0) \exp \left( - \int_0^t \bar{\bar{S}}(s) ds \right), \quad t \geq 0.
\]

We first establish
Lemma 2.1. Equation (2.5) has a unique solution \((A, \bar{\Phi})\), which is such that \(\bar{\Phi} \equiv \bar{\Phi}\), where \((\bar{S}, \bar{\Phi})\) is the unique solution of (2.4).

Proof. Let \(m \in D\) be an arbitrary measurable function such that \(0 \leq m(t) \leq \lambda^*\) for all \(t \geq 0\). We consider the increasing \(\{0, 1\}\)-valued process \(A^{(m)}\) defined by

\[
A^{(m)}(t) = \int_0^t \int_0^\infty 1_{A^{(m)}(s^-)=0} 1_{u \leq m(s^-)} Q(ds, du),
\]

and define \(\tau^{(m)} = \inf\{t > 0, A^{(m)}(t) = 1\}\). The result will follow from the existence and uniqueness of \(m^*\) such that \(m^* = \bar{\Phi}^{(m^*)}\), where

\[
\bar{\Phi}^{(m)}(t) = \bar{I}(0)\bar{\lambda}^0(t) + \bar{S}(0)\mathbb{E}[\lambda(t - \tau^{(m)})] = \int_0^t \bar{\lambda}(t-s)dA^{(m)}(s)
\]

\[
= I(0)\bar{\lambda}^0(t) + S(0)\int_0^t \bar{\lambda}(t-s)\mathbb{P}(A^{(m)}(s) = 0)m(s)ds
\]

\[
= \bar{I}(0)\bar{\lambda}^0(t) + \bar{S}(0)\int_0^t \bar{\lambda}(t-s)m(s)e^{-\int_0^s m(r)dr}ds.
\]

The third equation follows from the definition of \(A^{(m)}(t)\). Define \(\bar{S}^{(m)}(t) := \bar{S}(0)e^{-\int_0^t m(r)dr}\) (compare with equation (2.6)). We see that \(m = \bar{\Phi}^{(m)}\) if and only if the pair \((\bar{S}^{(m)}, \bar{\Phi}^{(m)})\) solves the system of integral equations (2.4). Since that system has a unique solution, the equation \(m = \bar{\Phi}^{(m)}\) has a unique solution \(m^*\), and moreover \(m^* \equiv \bar{\Phi}\), which establishes the Lemma.

We next define \(\{(A_j(t), \tau_j), j \geq 1\}\) as the solution of (2.5) with \((Q, \lambda)\) replaced by \((Q_j, \lambda_j)\). This yields an i.i.d. sequence \(\{(A_j(\cdot), \tau_j), j \geq 1\}\) of random elements of \(D \times \mathbb{R}_+\).

We prove the following estimate when using the i.i.d. sequence \(\{(A_j(t), \tau_j), j \geq 1\}\) to approximate \(\{(A_j^N(t), \tau_j), j \geq 1\}\). This is in a similar flavor as that established for the model with varying infectivity and susceptibility in [10], while we note the clear distinctions in the model and proof; see also Lemma 6.2 in [9].

Lemma 2.2. There exists a constant \(C_{T, \lambda^*}\) such that for all \(N \geq 1, 0 \leq t \leq T\),

\[
\frac{1}{N} \mathbb{E}\left[\sum_{j=1}^{SN(0)} \sup_{0 \leq t \leq T} |A_j^N(t) - A_j(t)|\right] \leq C_{T, \lambda^*}(\varepsilon_N + 2N^{-1/2}),
\]

where \(\varepsilon_N := \mathbb{E}\left[|I^N(0) - \bar{I}(0)| + |\bar{S}^N(0) - \bar{S}(0)|\right]\).

Proof. Define

\[
\tilde{A}_j^N(t) = \int_0^t \int_0^\infty 1_{u \leq \tilde{S}^N(s^-)} Q_j(ds, du), \quad \tilde{A}_j(t) = \int_0^t \int_0^\infty 1_{u \leq \tilde{S}(s^-)} Q_j(ds, du).
\]

We have \(A_j^N(t) = A_j^N(t) \wedge 1, A_j(t) = \tilde{A}_j(t) \wedge 1\). Consequently,

\[
|A_j^N(t) - A_j(t)| \leq |\tilde{A}_j^N(t) - \tilde{A}_j(t)| \leq \int_0^t \int_{\tilde{S}^N(s^-) \vee \tilde{S}(s^-)} \tilde{S}_j^N(ds, du).
\]

Since the right hand side is non-decreasing,

\[
\sup_{0 \leq r \leq t} |A_j^N(r) - A_j(r)| \leq \int_0^t \int_{\tilde{S}^N(s^-) \vee \tilde{S}(s^-)} \tilde{S}_j^N(ds, du).
\]
Taking expectations on both sides then yields
\[
\mathbb{E} \left[ \sup_{0 \leq r \leq t} \left| A_j^N(r) - A_j(r) \right| \right] \leq \mathbb{E} \int_0^t |\tilde{S}^N(s) - \tilde{S}(s)| ds.
\] (2.8)

Next, using (2.5),
\[
\mathbb{E} \left[ |\tilde{S}^N(t) - \tilde{S}(t)| \right] \leq \mathbb{E} \left[ \frac{1}{N} \sum_{j=1}^{I_N(0)} (\lambda_j(t) - \bar{\lambda}(t)) \right] + \mathbb{E} \left[ \frac{1}{N} \sum_{j=1}^{S_N(0)} (\lambda_j(t) - \bar{\lambda}(t)) \right]
\]
\[
+ \lambda^* \mathbb{E} \left[ |\tilde{I}^N(0) - \tilde{I}(0)| + |\tilde{S}^N(0) - \tilde{S}(0)| \right].
\] (2.9)

Since the \(\lambda_j\)'s are mutually independent, and globally independent of \(I_N(0)\),
\[
\mathbb{E} \left[ \frac{1}{N} \sum_{j=1}^{I_N(0)} (\lambda_j(t) - \bar{\lambda}(t)) \right] \leq \mathbb{E} \left[ \frac{1}{N} \sum_{j=1}^{I_N(0)} |\lambda_j(t) - \bar{\lambda}(t)| \right] \leq \frac{\lambda^*}{\sqrt{N}}.
\] (2.10)

Moreover,
\[
\mathbb{E} \left[ \frac{1}{N} \sum_{j=1}^{S_N(0)} (\lambda_j(t) - \bar{\lambda}(t)) \right]
\]
\[
\leq \mathbb{E} \left[ \frac{1}{N} \sum_{j=1}^{S_N(0)} |\lambda_j(t) - \bar{\lambda}(t)| \right] + \mathbb{E} \left[ \frac{1}{N} \sum_{j=1}^{S_N(0)} (\lambda_j(t) - \bar{\lambda}(t)) \right]
\] (2.11)

Since the sequence \(\{(A_j(\cdot), \tau_j), j \geq 1\}\) is i.i.d. and \(\lambda_j \leq \lambda^*\) almost surely,
\[
\mathbb{E} \left[ \frac{1}{N} \sum_{j=1}^{S_N(0)} (\lambda_j(t) - \bar{\lambda}(t)) \right] \leq \frac{\lambda^*}{\sqrt{N}}.
\]

On the other hand, using Markov's inequality and the fact that \(A_j\) and \(A_j^N\) are integer-valued,
\[
\mathbb{E} \left[ \lambda_j(t) - \bar{\lambda}(t) \right] \leq \lambda^* \mathbb{P}(\tau_j \neq \tau_j \land t)
\]
\[
\leq \lambda^* \mathbb{E} \left[ \sup_{0 \leq r \leq t} |A_j^N(r) - A_j(r)| \right].
\]

Combining the above and returning to (2.11), we obtain, using the fact that \(\lambda_j, A_j^N\) and \(A_j\) are independent of \(S_N(0)\),
\[
\mathbb{E} \left[ \frac{1}{N} \sum_{j=1}^{S_N(0)} (\lambda_j(t) - \bar{\lambda}(t)) \right] \leq \lambda^* \mathbb{E} \left[ \sup_{0 \leq r \leq t} |A_j^N(r) - A_j(r)| \right] + \frac{\lambda^*}{\sqrt{N}}.
\] (2.12)

where we have exploited the exchangeability of the sequence \(\{A_j^N(\cdot) - A_j(\cdot), 1 \leq j \leq S_N(0)\}\) for the last inequality. It now follows from (2.8), (2.9), (2.10) and (2.12) that
\[
\mathbb{E} \left[ \sup_{0 \leq r \leq t} |A_j^N(r) - A_j(r)| \right] \leq \lambda^* (\varepsilon_N + 2N^{-1/2}) + \lambda^* \int_0^t \mathbb{E} \left[ \sup_{0 \leq r \leq s} |A_j^N(r) - A_j(r)| \right] ds.
\]

The result, with \(C_{T,\lambda^*} := \lambda^* \exp(\lambda^* T)\) now follows from Gronwall's Lemma. \(\square\)
Note that since \( \sup_{0 \leq t \leq T} |A_j^N(t) - A_j(t)| \) is either zero or else \( \geq 1 \), this Lemma implies that
\[
\mathbb{P}\left( \sup_{0 \leq t \leq T} |A_j^N(t) - A_j(t)| \neq 0 \right) \leq C(\varepsilon_N + N^{-1/2}),
\]
and also
\[
\mathbb{P}\left( \tau_j^N \wedge T \neq \tau_j \wedge T \right) \leq C(\varepsilon_N + N^{-1/2}).
\]

Completing the proof of Theorem 2.1. Now let us go back to (2.1). It follows from [17] that as \( N \to \infty \), the first term converges a.s. in \( \mathcal{D} \),
\[
\frac{1}{N} \sum_{j=1}^{I_N(0)} \lambda_{-j}(t) \to \bar{I}(0) \lambda^0(t),
\]
and the second term, replacing \( \tau_j^N \) by \( \tau_j \), converges a.s. in \( \mathcal{D} \):
\[
\frac{1}{N} \sum_{j=1}^{S_N(0)} \lambda_j(t - \tau_j) \to \bar{S}(0) \mathbb{E}[\lambda_1(t - \tau)].
\]

It remains to consider the error term
\[
\frac{1}{N} \sum_{j=1}^{S_N(0)} (\lambda_j(t - \tau_j^N) - \lambda_j(t - \tau_j)),
\]
which tends to 0 locally uniformly in \( t \) in probability, thanks to Lemma 2.2. Indeed,
\[
\mathbb{E}\left[ \frac{1}{N} \sum_{j=1}^{S_N(0)} (\lambda_j(t - \tau_j^N) - \lambda_j(t - \tau_j)) \right] \leq \mathbb{E}\left[ \frac{\lambda^*}{N} \sum_{j=1}^{S_N(0)} \mathbb{P}(\tau_j^N \wedge t \neq \tau_j \wedge t) \right] \\
= \lambda^* C(\varepsilon_N + N^{-1/2}),
\]
which tends to zero as \( N \to \infty \). Thus we have shown that \( (\bar{S}^N, \bar{F}^N) \to (\bar{S}, \bar{F}) \) in \( \mathcal{D}^2 \) in probability as \( N \to \infty \).

By the law of large numbers,
\[
N^{-1} \sum_{j=1}^{I_N(0)} 1_{\eta_{-j} > t} \to \bar{I}(0) F_0^c(t) \text{ in } \mathcal{D}
\]
in probability as \( N \to \infty \). By the above proof, we also have
\[
N^{-1} \sum_{j=1}^{S_N(0)} A_j^N(t) \to \int_t^\infty \bar{S}(s) \bar{F}(s)ds \text{ in } \mathcal{D}
\]
in probability as \( N \to \infty \). By a similar argument as the derivation in (2.7), we obtain
\[
\bar{S}(0) \mathbb{E}[1_{\tau + \eta_1 \leq t}] = \bar{S}(0) \mathbb{E}\left[ \int_0^t F(t-s)dA(s) \right] \\
= \bar{S}(0) \int_0^t F(t-s) \bar{S}(s)e^{-\int_0^s \bar{F}(r)dr}ds \\
= \int_0^t F(t-s) \bar{S}(s) \bar{F}(s)ds.
\]
Then by Lemma 2.2 and LLN of i.i.d. random elements in $D$, we obtain

$$N^{-1} \sum_{j=1}^{S_{N}(0)} 1_{t_j^N + \eta_j \leq t} \to \int_0^t F(t-s)\bar{S}(s)\bar{S}(s)\,ds \quad \text{in} \quad D$$

in probability as $N \to \infty$. Thus we have shown that jointly with $(\bar{S}_N, \bar{S}_N^N) \to (\bar{S}, \bar{S})$, $(\bar{I}_N, \bar{R}_N) \to (\bar{I}, \bar{R})$ in $D^2$ in probability as $N \to \infty$ (note that at most one term in either $\bar{S}$, $\bar{S}$, $\bar{I}$ or $\bar{R}$ is discontinuous, so the sum also converges in $D$). This completes the proof. \hfill $\square$

3. The multipatch–multiple group model

We assume that the population of size $N$ is split into $K$ groups, and distributed into $L$ distinct patches. Let $S_{N,k}^N(t)$, $I_{N,k}^N(t)$ and $R_{N,k}^N(t)$ denote the numbers of susceptible, infected and recovered individuals from group $k$ who are in patch $\ell$ at time $t$, respectively. Let $B_{N,k}^N(t)$ be the number of individuals in group $k$ and in patch $\ell$ at time $t$, i.e.,

$$B_{N,k}^N(t) = S_{N,k}^N(t) + I_{N,k}^N(t) + R_{N,k}^N(t).$$

Note that in our model, the total population size, $\sum_{\ell=1}^{L} \sum_{k=1}^{K} B_{N,k}^N(t)$ is fixed, and we can, without loss of generality, assume that it is equal to $N$. Let $S_k^N(0) = \sum_{\ell=1}^{L} S_{N,k}^N(0)$, $S_N^N(0) = \sum_{k=1}^{K} S_k^N(0)$, $I_k^N(0) = \sum_{\ell=1}^{L} I_{N,k}^N(0)$, $I_N^N(0) = \sum_k I_k^N(0)$, and $R_k^N(0) = \sum_{\ell=1}^{L} R_{N,k}^N(0)$, $R_N^N(0) = \sum_k R_k^N(0)$. Let $K = \{1, \ldots, K\}$ and $\ell = \{1, \ldots, L\}$. Recall that $\sum_k (I_k^N(0) + S_k^N(0) + R_k^N(0)) = N$, the total population size.

We assume that $\inf_{N\geq 1} \inf_{\ell,k} N^{-1} B_{N,k}^N(0) > 0$, almost surely. While susceptible, an individual from group $k$ moves from patch to patch according to a time–inhomogeneous Markov process $X_k(t)$, with transition function $p_{\ell,\ell'}^k(s,t) = P(X_k(t) = \ell' | X_k(s) = \ell)$, and while infectious, an individual from group $k$ moves from patch to patch according to a time–inhomogeneous Markov process $Y_k(t)$, with transition function $q_{\ell,\ell'}^k(s,t) = P(Y_k(t) = \ell' | Y_k(s) = \ell)$. Let $X_k(t)$ and $Y_k(t)$ be associated with the rates $\nu_{S_k}^\ell(t)$ and $\nu_{I_k}^\ell(t)$, $\ell, \ell' \in \ell$, respectively. Similarly, the recovered individuals migrate with rates $\nu_{R_k}^\ell(t)$. We assume that those movements are mutually independent. The inhomogeneity may be due to restrictions of movements imposed by the authorities during the epidemic. We shall write $X_{j,k}^s(t)$ (resp. $Y_{j,k}^s(t)$) for the position at time $t$ of the individual $j$ from group $k$ if it is susceptible (resp. infected) during the time interval $(s, t)$, and was in patch $\ell$ at time $s$. $X_{j,k}(t)$ (resp. $Y_{j,k}(t)$) will denote the position of the individual $j$ at time $t$, in case that individual is initially susceptible (resp. infected) and is still susceptible (resp. infected) at time $t$.

The initially infected individual $j$ from group $k$ has at time $t \geq 0$ the infectivity $\lambda_{j,k}(t)$ (recall that in this case $j \leq -1$), while an initially susceptible individual $j$ from group $k$ who gets infected at time $\tau_{j,k}^N$ has at time $t$ the infectivity $\lambda_{j,k}(t - \tau_{j,k}^N)$. The random functions $\{\lambda_{j,k}, j \in \mathbb{Z}, 1 \leq k \leq K\}$ are mutually independent. For each $1 \leq k \leq K$, $\{\lambda_{j,k}, j \leq -1\}$ have the same law, as well as $\{\lambda_{j,k}, j \geq 1\}$. But the laws of $\lambda_{-1,k}$ and $\lambda_{1,k}$ are different. We see that the infectivity of the initially infected individuals differ from those of the newly infected individuals. The law of the infectivity depends upon the group $k$, which is quite natural in the case of age groups, since the reaction to infection depends upon the age. On the contrary, the infectivity does not depend upon the patch where the infected individual finds itself. We assume that for any $j \in \mathbb{Z}$ and $1 \leq k \leq K$, $\lambda_{j,k}$ has trajectories in $D$, and moreover, $0 \leq \lambda_{j,k}(t) \leq \lambda^*$ for all $t \geq 0$ and a.s., where $\lambda^* > 0$ is a fixed constant.

We also define, for each $j \in \mathbb{Z} \setminus \{0\}$ and $k = 1, \ldots, K$, $\eta_{j,k} = \inf\{t > 0, \lambda_{j,k}(t + r) = 0, \forall r > 0\}$. We denote by $F_{k}(t) := \mathbb{P}(\eta_{j,k} \leq t)$ and $F_{k}^t(t) = \mathbb{P}(\eta_{-1,k} \leq t)$ the distribution functions of $\eta_{j,k}$ for $j \geq 1$ and for $j \leq -1$, respectively.
The total force of infection delivered by the individuals of group $k$ in patch $\ell$ is given by
\begin{equation}
\tilde{\mathcal{I}}^N_{k,\ell}(t) = \sum_{j=1}^{I^N_k(0)} \lambda_{j,k}(t) 1_{Y_{j,k}(t)=\ell} + \sum_{j=1}^{S^N_k(0)} \lambda_{j,k}(t-\tau_{j,k}^N) \sum_{\ell'} 1_{X_{j,k}(\tau_{j,k}^N)=\ell'} 1_{Y_{j,k}^N(t')=\ell}.
\end{equation}

What depends upon the patch is the contact rate. We assume that a susceptible of patch $\ell$ and group $k$ has contacts with possibly infectious individuals of patch $\ell'$ and group $k'$ at rate $\beta_{k,k'}^\ell(t)$ at time $t$, and that there exists a constant $\beta^*>0$ such that $\beta_{k,k'}^\ell(t) \leq \beta^*$ for all $\ell, \ell', k, k'$ and $t \geq 0$. The functions $\beta_{k,k'}^\ell(t)$ dictate the dependence upon the pairs $(k,k')$ and $(\ell,\ell')$ in the contact rates. This include the so-called “infection at distance”, where $\beta_{k,k'}^\ell(t)$ can take very small values. As we have discussed in the introduction, this is an approach to model movements during a weekend or holiday, which should not be regarded as migrations between different patches. We assume also that for a given parameter $\gamma \in [0,1]$, the rate at which new infections affect the individuals from group $k$ in patch $\ell$ at time $t$ is
\begin{equation}
\Upsilon^N_{k,\ell}(t) = \frac{S^N_k(t)}{N^{1-\gamma}(B^N_{k,\ell}(t))^{\gamma}} \sum_{k'} \sum_{\ell'} \beta_{k,k'}^\ell(t) \tilde{\mathcal{I}}^N_{k',\ell'}(t).
\end{equation}

Note that when $\gamma = 0$, the infectivity rate is proportional to the fraction of the susceptible individuals in all patches, while when $\gamma = 1$, the infectivity rate is proportional to the fraction of the susceptible individuals in its own patch. When $\gamma \in (0,1)$, the infectivity rate is proportional to a fraction in between, which may be thought of as more realistic. By convention, whenever $B^N_{k,\ell}(t) = 0$, we set $\Upsilon^N_{k,\ell}(t) = 0$.

Also, let
\begin{equation}
\bar{\Upsilon}^N_{k,\ell}(t) := \frac{1}{N^{1-\gamma}(B^N_{k,\ell}(t))^{\gamma}} \sum_{k'} \sum_{\ell'} \beta_{k,k'}^\ell(t) \tilde{\mathcal{I}}^N_{k',\ell'}(t),
\end{equation}

which is the force of infectivity to each susceptible of group $k$ in patch $\ell$. We define $B^N_{k,\ell}(t) := N^{-1}B^N_{k,\ell}(t)$, $\tilde{\mathcal{I}}^N_{k,\ell}(t) := N^{-1}\tilde{\mathcal{I}}^N_{k,\ell}(t)$. Then we have
\begin{equation}
\bar{\Upsilon}^N_{k,\ell}(t) = \frac{1}{(B^N_{k,\ell}(t))^{\gamma}} \sum_{k'} \sum_{\ell'} \beta_{k,k'}^\ell(t) \tilde{\mathcal{I}}^N_{k',\ell'}(t).
\end{equation}

Now we model the infection of each initially susceptible individual. For each $1 \leq k \leq K$, $1 \leq j \leq S^N_k(0)$ and $0 \leq \ell \leq L$, let
\begin{equation}
A^{N,\ell}_{j,k}(t) = \int_0^t \int_0^\infty 1_{A^N_{j,k}(s')=0} 1_{X_{j,k}(s)=\ell} 1_{u \leq \bar{\Upsilon}^N_{k,\ell}(s'-u)} Q^\ell_{j,k}(ds,du),
\end{equation}

where $\{Q^\ell_{j,k}, k \in K, \ell \in L, j \geq 1\}$ are mutually independent standard PRMs on $\mathbb{R}^2_+$ and $A^{N,\ell}_{j,k}(t) = \sum_{\ell'} A^{N,\ell'}_{j,k}(t)$. $A^{N,\ell}_{j,k}(t)$ is the force of infection. $A^{N,\ell}_{j,k}(t) = 0$ if the individual $j$ from group $k$ has been infected on the time interval $(0,t]$. Otherwise, $A^{N,\ell}_{j,k}(t) = 0$. Recall that $\tau^N_{j,k}$ denotes the time at which the initially susceptible individual $j$ from group $k$ gets infected. We have $\tau^N_{j,k} = \inf\{t > 0, A^{N,\ell}_{j,k}(t) = 1\}$. If $A^{N,\ell}_{j,k}(t) = 1$, the unique $\ell$ such that $A^{N,\ell}_{j,k}(t) = 1$ is the patch where the individual $j$ from group $k$ has been infected.

We define
\begin{align*}
\eta_{i,k} &= \inf\{t > 0 : \lambda_{i,k}(t+r) = 0, \forall r \geq 0\}, \\
\eta^0_{i,k} &= \inf\{t > 0 : \lambda^0_{i,k}(t+r) = 0, \forall r \geq 0\}.
\end{align*}

Note that the random functions $\lambda_{i,k}(t) = 0$ and $\lambda^0_{i,k}(t) = 0$ for $t < 0$. The random times $\eta_{i,k}, \eta^0_{i,k}$ represent the infected periods of newly and initially infected individuals, respectively. Note that the infected periods may include exposed and infectious periods together. The exposed periods can
be defined by \( \zeta_{j,k} = \inf \{ t > 0 : \lambda_{j,k}(t) > 0 \} \), and \( \zeta_{j,k}^0 = \inf \{ t > 0 : \lambda_{j,k}^0(t) > 0 \} \) for the newly and initially infected individuals, respectively, which can be either equal to zero (no exposed periods) or strictly positive.

Under the i.i.d. assumptions of the random functions \( \{\lambda_{j,k}(\cdot)\}_j \), the sequence of variables \( \{\eta_{j,k}\}_j \) is i.i.d. for each type \( k \), and so are the sequences \( \{\eta_{j,k}^0\}_j \). Let \( F_k \) and \( F_k^0 \) be the c.d.f.’s for the variables \( \eta_k \) and \( \eta_k^0 \), respectively. We moreover define \( \bar{\lambda}_k(t) := \mathbb{E}[\lambda_{1,k}(t)] \) and \( \bar{\lambda}_k^0(t) := \mathbb{E}[\lambda_{-1,k}(t)] \)

Besides (3.1), the evolution of the epidemic is characterized by the dynamics of \( S_k \), \( I_k \) and \( R_k \):

\[
S_k(t) = S_k(0) - \sum_{j=1}^{N} A_{j,k}(t) - \sum_{\ell' = 1, \ell' \neq \ell}^{L} P_{S,k}^{\ell',\ell}(0) \int_0^t \nu_{S,k}(s)S_k(t) \, ds + \sum_{\ell' = 1, \ell' \neq \ell}^{L} P_{S,k}^{\ell',\ell}(0) \int_0^t \nu_{S,k}(s)S_k(t) \, ds
\]

(3.5)

\[
I_k(t) = I_k(0) + \sum_{j=1}^{N} A_{j,k}(t) - \sum_{\ell' = 1}^{L} P_{I,k}^{\ell'}(0) \int_0^t \nu_{I,k}(s)I_k(t) \, ds + \sum_{\ell' = 1, \ell' \neq \ell}^{L} P_{I,k}^{\ell',\ell}(0) \int_0^t \nu_{I,k}(s)I_k(t) \, ds
\]

(3.6)

\[
R_k(t) = R_k(0) + \sum_{\ell' = 1}^{L} P_{R,k}^{\ell'}(0) \int_0^t \nu_{R,k}(s)R_k(t) \, ds + \sum_{\ell' = 1, \ell' \neq \ell}^{L} P_{R,k}^{\ell',\ell}(0) \int_0^t \nu_{R,k}(s)R_k(t) \, ds
\]

(3.7)

where \( P_{S,k}^{\ell',\ell}, P_{I,k}^{\ell'}, P_{R,k}^{\ell'}, k \in \mathcal{K}, \ell, \ell' \in \mathcal{L} \) are mutually independent standard Poisson processes.

We define \( S_k(t) := N^{-1} S_k(t), I_k(t) := N^{-1} I_k(t) \) and \( R_k(t) := N^{-1} R_k(t) \). We want to show that under the above assumptions, we have the following Theorem.

**Theorem 3.1.** As \( N \to \infty \), \( (\bar{S}_k^{\ell'}, \bar{I}_k^{\ell'}, \bar{R}_k^{\ell'}, k \in \mathcal{K}, \ell, \ell' \in \mathcal{L}) \to (\bar{S}_k, \bar{I}_k, \bar{R}_k, k \in \mathcal{K}, \ell \in \mathcal{L}) \) in \( D^{4KL} \) in probability, where the limits are the unique solution of the following system of integral equations:
\[ S_k^\ell(t) = S_k^\ell(0) - \int_0^t S_k^\ell(s)\tilde{\Gamma}_k^\ell(s)ds + \sum_{\ell' = 1}^L \int_0^t \nu_{S,k}^\ell(s)S_k^\ell(s)ds, \]
\[ \tilde{S}_k^\ell(t) = \tilde{S}_k^\ell(0) + \int_0^t \tilde{S}_k^\ell(s)\tilde{\Gamma}_k^\ell(s)ds + \sum_{\ell' = 1}^L \int_0^t \nu_{I,k}^\ell(s)\tilde{I}_k^\ell(s)ds, \]
\[ I_k^\ell(t) = I_k^\ell(0) + \int_0^t I_k^\ell(s)F_k^0(ds) + \int_0^t \tilde{S}_k^\ell(s)\tilde{\Gamma}_k^\ell(s)ds + \sum_{\ell' = 1}^L \int_0^t \nu_{I,k}^\ell(s)I_k^\ell(s)ds, \]
\[ \tilde{I}_k^\ell(t) = \tilde{I}_k^\ell(0) + \int_0^t \tilde{I}_k^\ell(s)F_k^0(ds) + \sum_{\ell' = 1}^L \int_0^t \nu_{I,k}^\ell(s)I_k^\ell(s)ds, \]
\[ \tilde{R}_k^\ell(t) = \tilde{R}_k^\ell(0) + \int_0^t \tilde{R}_k^\ell(s)F_k^0(ds) + \sum_{\ell' = 1}^L \int_0^t \nu_{I,k}^\ell(s)\tilde{R}_k^\ell(s)ds, \]

where
\[ \tilde{\Gamma}_k^\ell(t) = \langle \tilde{B}_k^\ell(t) \rangle - \gamma \sum_{k'} \sum_{\ell''} \beta_{k,k'}^\ell(t)\tilde{S}_{k'}^\ell(t), \quad \tilde{B}_k^\ell(t) = \tilde{S}_k^\ell(t) + \tilde{I}_k^\ell(t) + \tilde{R}_k^\ell(t). \]
If \( 0 < \gamma \leq 1 \), \( \bar{S}_k^{\ell}(t) \bar{I}_k^{\ell}(t) = \bar{S}_k^{\ell}(t)(B_k^{\ell}(t))^{-\gamma} \sum_{k'} \nu_{k,k'}^{\ell,\ell'}(t) \bar{S}_k^{\ell'}(t) \), which involves the map \( (x, y, z) \mapsto \frac{x}{(x+y+z)^{\gamma}} \). For any \( 0 < \varepsilon < 1 \), this map is globally Lipschitz on the subset \( \{(x, y, z) \in [0,1]^3, x+y+z \geq \varepsilon \} \). We will show that for any \( T > 0 \), \( \inf_{0 \leq t \leq T} \inf_{t,k} B_k^{\ell}(t) > 0 \). Then the existence and uniqueness follows from a standard argument of Volterra integral equations with Lipschitz coefficients.

By the expressions of \( \bar{S}_k^{\ell}(t), \bar{I}_k^{\ell}(t) \) and \( \bar{R}_k^{\ell}(t) \) in (3.8), we have

\[
\bar{B}_k^{\ell}(0) + \sum_{\ell' = 1}^L \int_0^t \nu_{S,k,k'}^{\ell,\ell'}(s) \bar{S}_k^{\ell'}(s) ds + \sum_{\ell' = 1}^L \int_0^t \nu_{I,k,k'}^{\ell,\ell'}(s) \bar{I}_k^{\ell'}(s) ds + \sum_{\ell' = 1}^L \int_0^t \nu_{R,k,k'}^{\ell,\ell'}(s) \bar{R}_k^{\ell'}(s) ds
\]

where \( \nu_{S,k,k'}^{\ell,\ell'}(s), \nu_{I,k,k'}^{\ell,\ell'}(s), \nu_{R,k,k'}^{\ell,\ell'}(s) \) are the Lipschitz coefficients.

Thus, we obtain

\[
\bar{B}_k^{\ell}(t) \geq \bar{B}_k^{\ell}(0) e^{-\nu_{k,k'}^{\ell,\ell'} t} \geq \bar{B}_k^{\ell}(0) e^{-\nu_{k,k'}^{\ell,\ell'} t} := C_k^{\ell,T} \quad \text{for all } t \in [0, T],
\]

and we have shown that for any \( \ell, k \) and \( T > 0 \), there exists a constant \( C_k^{\ell,T} > 0 \) such that \( \bar{B}_k^{\ell}(t) \geq C_k^{\ell,T} \) for any \( t \in [0, T] \).

4.2. An expression of \( \bar{S}_k^{\ell}(t) \) via a Feynman-Kac formula for the associated backward ODEs. The following estimate will be useful below. From the above equations and Gronwall’s Lemma, we deduce that for all \( t \geq 0 \),

\[
\sum_{\ell,k} \bar{S}_k^{\ell}(t) \leq \lambda^* L K e^{\lambda^* \beta^* L K t}.
\]

From now on, for each \( 1 \leq k \leq K \), we shall consider the \( \{0, 1, \ldots, L\} \)-valued process \( X_k(t) \), which starts with the initial distribution \( P(X_k(0) = \ell) = \bar{S}_k^{\ell}(0), 1 \leq \ell \leq L \), and \( P(X_k(0) = 0) = 1 - \sum_{\ell} \bar{S}_k^{\ell}(0) \). The process \( X_k(t) \) is a non homogeneous Markov process, whose jumps are specified by the rates \( \nu_{S,k}^{\ell,\ell'}(t) \). \( \nu_{S,k}^{\ell,\ell'}(t) \) has already been defined for \( 1 \leq \ell, \ell' \leq L \), and moreover we assume that \( \nu_{S,k}^{0,\ell} = \nu_{S,k}^{\ell,0} = 0 \) for any \( 0 \leq \ell \leq L \). Note also that \( \nu_{S,k}^{\ell,\ell'}(t) = -\sum_{\ell' \neq \ell} \nu_{S,k}^{\ell,\ell'}(t) \) for each \( \ell \in \mathcal{L} \).

It is clear that the explicit formula \( \bar{S}(t) = \bar{S}(0) \exp \left(-\int_0^t \bar{S}(s) ds\right) \) in (2.6) was crucial in the proofs of Lemma 2.1 and the convergence of \( \bar{I}_k^{\ell} \) in (2.13) above. We need an extension of this formula in the context of the present multipatch–multigroup model. Such a formula will be provided by the next Proposition.

Let us first rewrite the first line of (3.8). Denoting by \( Q_k^{\ell}(t) \) the infinitesimal generator of the process \( X_k(t) \), i.e., the matrix whose \( (\ell, \ell') \) entry equals \( \nu_{k,k'}^{\ell,\ell'}(t) \), this equation can be rewritten as...
of the following backward system of ODEs:

\[ \frac{d\bar{S}_k(t)}{dt} = -D_k(t)\bar{S}_k(t) + Q_k(t)S_k(t), \]  

(4.5)

with \( D_k(t) \) denoting the diagonal matrix whose \((\ell, \ell)\) entry equals \( \Gamma_k^{\ell}(t) \), and \( Q_k(t) \) denoting the transpose of the matrix \( Q_k(t) \). But we first need to establish a Feynman–Kac formula for a system of backward ODEs adjoint to (4.5).

**Lemma 4.1.** For any \( t > 0 \), \( 1 \leq \ell \leq L \), let \( \{u_{k,t,\ell}(s), \ 0 \leq s \leq t\} \) be the unique \( \mathbb{R}^L \)-valued solution of the following backward system of ODEs:

\[ \frac{du_{k,t,\ell}(s)}{ds} - D_k(s)u_{k,t,\ell}(s) + Q_k(s)u_{k,t,\ell}(s) = 0, \quad 0 \leq s \leq t, \]  

(4.6)

whose final value \( u_{k,t,\ell}(t) \) equals the vector whose \( \ell \)-th coordinate equals 1, all others being 0. Then for any \( 0 \leq s < t \), \( 1 \leq \ell' \leq L \),

\[ u_{k,t,\ell}(s) = \mathbb{E} \left[ 1_{X_k(t)=t} \exp \left( -\int_s^t \Gamma_k(\nu) \, dr \right) \bigg| X_k(s) = \ell' \right], \]  

(4.7)

where \( u_{k,t,\ell}(s) \) stands for the \( \ell' \)-th coordinate of the vector \( u_{k,t,\ell}(s) \).

**Proof.** In order to simplify our notation, we delete the subindices \( k, t, \ell \) and \( k \) in the proof. We fix \( 0 \leq s < t \). For \( s < r < t \), let \( \{u(r), \ s \leq r \leq t\} \) be the solution of (4.6). Note that \( u \in C^1((0,t);\mathbb{R}^L) \). Let

\[ V_r = \exp \left( -\int_s^r \Gamma X(v) \, dv \right), \quad W_r = u X(r)V_r. \]

The jump Markov process \( X(s) \) has the same law as the solution of the following SDE:

\[ X(t) = X(s) + \sum_{\ell, \ell'} \nu_{X(r), \ell'}(v) \int_s^t 1_{X(r')=t} 1_{v \leq \nu_{X(r'), \ell'}(r)} Q_{\ell', \ell}(dr, dv) \]

\[ = X(s) + \sum_{\ell} \int_s^t (\ell' - X(v)) \nu_{X(v), \ell'}(v) \, dv + \sum_{\ell, \ell'} (\ell' - \ell) \int_s^t \int_0^\infty 1_{X(r')=t} 1_{v \leq \nu_{X(r'), \ell'}(r)} Q_{\ell', \ell}(dr, dv), \]

where \( \{Q_{\ell', \ell}, \ 1 \leq \ell, \ell' \leq L\} \) are mutually independent standard PRMs on \( \mathbb{R}^2 \), and \( Q_{\ell', \ell}(dr, dv) = Q_{\ell', \ell}(dr, dv) - dr \, dv \). For this it follows that

\[ W_t = W_s + \int_s^t \left[ \frac{du X(r)}{dr} - \Gamma X(v)(r)u X(v)(r) \right] V_r \, dr \]

\[ + \sum_{\ell} \int_s^t \nu_{X(r), \ell}(r) [u^X(r) - u X(r)] V_r \, dr + M_t, \]

where \( M_t \) is a martingale such that \( M_s = 0 \). We further note that

\[ \sum_{\ell} (u^X - u^X) \nu_{X, \ell}(v) = \sum_{\ell \neq x} (u^X - u^X) \nu_{X, \ell}(v) = (Qu)^x. \]

From the above formulas, we deduce that

\[ W_t = W_s + \int_s^t \left[ \frac{du X(r)}{dr} - D(r)u X(r)(r) + (Qu) X(r)(r) \right] V_r \, dr + M_t \]

\[ = W_s + M_t, \]
where we have used the fact that \( u \) solves (4.6). Taking the conditional expectation given that \( X(s) = \ell' \) in the last identity yields the formula (4.7), since

\[
W_t = 1_{X(t) = \ell} \exp \left( - \int_s^t \bar{\Gamma}^X(v)dv \right), \quad \text{and}
\]
\[
W_s = u^{X(s)}(s).
\]

\[
\square
\]

**Remark 4.1.** Would the Markov process \( X_k \) be a diffusion, then the system of ODEs (4.6) would be replaced by a parabolic PDE, and a similar Feynman–Kac formula in such a situation is well–known, see, e.g., Chapter 3.8 in Pardoux and Răşcanu [16].

We can now derive an explicit formula for \( \bar{S}^\ell_k(t) \).

**Proposition 4.1.** The solution of equation (4.5) is given by the following formula: for any \( 1 \leq k \leq K, 1 \leq \ell \leq L, t > 0 \),

\[
\bar{S}^\ell_k(t) = \mathbb{E}\left[ 1_{X_k(t) = \ell} \bar{S}^{X_k(0)}(0) \exp \left( - \int_0^t \bar{\Gamma}^{X_k(s)}(s)ds \right) \right],
\]

where \( X_k \) is initialized as indicated above.

**Proof.** The duality between equations (4.5) and (4.6) is expressed by the obvious fact that

\[
\frac{d}{ds} (\bar{S}_k(s), u_{k,t,\ell}(s)) = 0, \quad 0 \leq s \leq t,
\]

hence \( (\bar{S}_k(t), u_{k,t,\ell}(t)) = (\bar{S}_k(0), u_{k,t,\ell}(0)) \). Recall that by our choice of \( u_{k,t,\ell}(t) \), \( (\bar{S}_k(t), u_{k,t,\ell}(t)) = \bar{S}^\ell_k(t) \). Now we deduce form (4.7) with \( s = 0 \) that

\[
(\bar{S}_k(0), u_{k,t,\ell}(0)) = \sum_{\ell'} \bar{S}^{\ell'}_k(0) \mathbb{E}\left[ 1_{X_k(t) = \ell} \exp \left( - \int_0^t \bar{\Gamma}^{X_k(r)}(r)dr \right) \right| X_k(0) = \ell']
\]
\[
= \mathbb{E}\left[ 1_{X_k(t) = \ell} \bar{S}^{X_k(0)}(0) \exp \left( - \int_0^t \bar{\Gamma}^{X_k(r)}(r)dr \right) \right].
\]

The result follows from the last three identities. \( \square \)

4.3. **An auxiliary system of Poisson-driven SDEs.** We want to associate to a collection of mutually independent standard PRMs \( \{Q^\ell_k, k \in \mathcal{K}, \ell \in \mathcal{L}\} \) on \( \mathbb{R}_+^2 \) and a family of mutually independent processes \( (X_k(t), t \geq 0, k \in \mathcal{K}) \), also independent from the \( \{Q^\ell_k, k \in \mathcal{K}, \ell \in \mathcal{L}\} \), the processes \( \{A^\ell_k(t), k \in \mathcal{K}, \ell \in \mathcal{L}\} \), which is the solution of the following system of SDEs:
The proofs below can be simplified in this case, see also Remark 4.4.

We can thus regard these equations as a McKean–Vlasov driven SDEs:

\[ Q_k(t) = \lambda_k^0(t) \sum_{\ell'=1}^L I_{k-\ell'}(0) \varphi_k^\ell(0, t) + \mathbb{E} \left[ S_k X_k(0) \lambda_k(t - \tau_k) q_k X_k(\tau_k, t) \right], \]

\[ S_k(t) = S_k(0) - \int_0^t S_k(s) \Gamma_k(s) ds + \sum_{\ell'=1}^L \int_0^t \nu_{S,k}(s) S_k(s) ds, \]

\[ I_k(t) = I_k(0) - \sum_{\ell'=1}^L I_{k-\ell'}(0) \int_0^t q_k^\ell(0, s) F_k^0(ds) + \int_0^t S_k(s) \Gamma_k(s) ds, \]

\[ \bar{R}_k(t) = \sum_{\ell'=1}^L I_{k-\ell'}(0) \int_0^t q_k^\ell(0, s) F_k^0(ds) + \sum_{\ell'=1}^L \int_0^t \int_0^{t-s} q_k^\ell(0, s + u) F_k(du) S_k(s) \bar{\Gamma}_k(s) ds, \]

where

\[ \bar{\Gamma}_k(t) = (\bar{B}_k(t))^{-\gamma} \sum_{k'} \sum_{\ell'} \beta_{k,k'}^{\ell,\ell'}(t) \bar{\Theta}_{k'}^{\ell}(t), \]

\[ \tau_k = \inf\{t > 0, A_k(t) = 1\}, \]

\[ \bar{B}_k(t) = S_k(t) + \bar{I}_k(t) + \bar{R}_k(t). \]

**Remark 4.2.** In this system of SDEs, the laws of the random functions \( X_k \) and \( \lambda_k \) as well as that of the PRMs \( Q_k^\ell \) are given. However, \( \tau_k \) is an unknown of this equation, whose law enters the coefficient on the second line. Recall Remark 2.1. We can thus regard these equation as a McKean–Vlasov SDE, and also the following as a propagation of chaos result for the times of infection of the various initially susceptible individuals.

**Remark 4.3.** In the case of \( \gamma = 0 \), one can instead define the following simpler system of Poisson-driven SDEs:

\[ A_k^\ell(t) = \int_0^t \int_0^\infty 1_{A_k(s^\ell)=0} 1_{X_k(s)=\ell} 1_{u \leq \Gamma_k^\ell(s^\ell)} Q_k^\ell(ds, du), \]

\[ \Theta_k^\ell(t) = \lambda_k^0(t) \sum_{\ell'=1}^L I_{k-\ell'}(0) \varphi_k^\ell(0, t) + \mathbb{E} \left[ S_k X_k(0) \lambda_k(t - \tau_k) q_k X_k(\tau_k, t) \right], \]

\[ S_k^\ell(t) = S_k^\ell(0) - \int_0^t S_k(s) \bar{\Gamma}_k(s) ds + \sum_{\ell'=1}^L \int_0^t \nu_{S,k}(s) S_k(s) ds, \]

where

\[ \bar{\Gamma}_k^\ell(t) = \sum_{k'} \sum_{\ell'} \beta_{k,k'}^{\ell,\ell'}(t) \bar{\Theta}_{k'}^{\ell}(t), \]

\[ \tau_k = \inf\{t > 0, A_k(t) = 1\}. \]

The proofs below can be simplified in this case, see also Remark 4.4.
We first need to show:

**Lemma 4.2.** The system of equations (4.9) has a unique solution, which is such that \( \bar{S}^\ell_k \equiv \tilde{S}^\ell_k \), for all \( k \in K, \ell \in L \).

**Proof.** To any \( \{m^\ell_k, k \in K, \ell \in L\} \in D^{LK} \) which satisfies \( \inf_{\ell,t} m^\ell_k(t) \geq 0 \) and \( \sup_{0 \leq t \leq T, \ell,k} m^\ell_k(t) < \infty \) for all \( T \), we associate \( \{A^m_k, 1 \leq k \leq K\} \), which solves

\[
A^{(m)}_{\ell,k}(t) = \int_0^t \int_0^\infty 1_{A^m_k(s^\ell)}(s) 1_{u \leq \Gamma^{(m)}_{\ell,k}(s)} Q^\ell_k(ds, du), \quad \text{with} \quad A^{(m)}_k(t) = \sum_\ell A^{(m)}_{\ell,k}(t),
\]

where \( X_k(s) \) is as above a \( \{0,1,\ldots,L\} \)-valued Markov jump process which is such that \( \mathbb{P}(X_k(0) = \ell) = \bar{S}^\ell_k(0), 1 \leq \ell \leq L, \mathbb{P}(X_k(0) = 0) = 1 - \sum_\ell \bar{S}^\ell_k(0) \), and starting from 0, \( X_k(t) = 0 \) for all \( t > 0 \), and

\[
\bar{\Gamma}^{(m)}_{\ell,k}(t) = (\bar{B}^{(m)}_{\ell,k}(t))^{-1} \sum_{k',\ell'} \bar{S}^{\ell'}_{k',k}(t)m^{\ell'}_{k'}(t),
\]

with \( \bar{B}^{(m)}_{\ell,k}(t) = \bar{S}^{(m)}_{\ell,k}(t) + \bar{\Gamma}^{(m)}_{\ell,k}(t) + \bar{R}^{(m)}_{\ell,k}(t) \), where \( (\bar{S}^{(m)}_{\ell,k}(t), \bar{\Gamma}^{(m)}_{\ell,k}(t), \bar{R}^{(m)}_{\ell,k}(t)) \) solves the last three lines of (4.9), with \( \bar{\Gamma}^{(m)}_\ell \) replaced by \( \bar{\Gamma}^{(m)}_{\ell,k} \). We moreover define \( \bar{\tau}^{(m)} = \inf\{t > 0, A^{(m)}_k(t) = 1\} \).

The result will follow from the existence and uniqueness of \( m^* := \{m^*_{k,\ell}, k \in K, \ell \in L\} \) such that \( m^* = \bar{S}^{(m^*)} \), where

\[
\bar{S}^{(m^*)}_{\ell,k}(t) = \lambda^0_k(t) \sum_{\ell' = 1}^L I^\ell_{\ell'}(0) q^\ell_{k}(0,t),
\]

where

\[
\bar{S}^{(m)}_{\ell,k}(t) = \lambda^0_k(t) \sum_{\ell' = 1}^L I^\ell_{\ell'}(0) q^\ell_{k}(0,t) + \mathbb{E} \left[ \int_0^t \bar{S}^{\ell'}_{k}(0) \lambda_k(t-s) q^\ell_{k}(s) \bar{A}^{(m)}_k(s) ds \right]
\]

and

\[
\bar{S}^{(m^*)}_{\ell,k}(t) = \lambda^0_k(t) \sum_{\ell' = 1}^L I^\ell_{\ell'}(0) q^\ell_{k}(0,t) + \mathbb{E} \left[ \int_0^t \bar{S}^{\ell'}_{k}(0) \lambda_k(t-s) q^\ell_{k}(s) \bar{A}^{(m)}_k(s) ds \right]
\]

and

\[
\bar{S}^{(m^*)}_{\ell,k}(t) = \lambda^0_k(t) \sum_{\ell' = 1}^L I^\ell_{\ell'}(0) q^\ell_{k}(0,t) + \mathbb{E} \left[ \int_0^t \bar{S}^{\ell'}_{k}(0) \lambda_k(t-s) q^\ell_{k}(s) \bar{A}^{(m)}_k(s) ds \right]
\]

and

\[
\bar{S}^{(m^*)}_{\ell,k}(t) = \lambda^0_k(t) \sum_{\ell' = 1}^L I^\ell_{\ell'}(0) q^\ell_{k}(0,t) + \mathbb{E} \left[ \int_0^t \bar{S}^{\ell'}_{k}(0) \lambda_k(t-s) q^\ell_{k}(s) \bar{A}^{(m)}_k(s) ds \right]
\]

where in the last equality we have defined

\[
\bar{S}^{(m^*)}_{\ell,k}(t) := \mathbb{E} \left[ 1_{X_k(t) = \ell} \bar{S}^{(m^*)}_{k}(0) \exp \left( -\int_0^t \bar{\Gamma}^{(m^*)}_{\ell,k}(s) ds \right) \right].
\]
It follows from (3.8) and (4.8) that $m = \mathcal{G}^{(m)}$ iff $(\tilde{S}^{(m)}, \tilde{\mathcal{G}}^{(m)}, \tilde{f}^{(m)}, \tilde{R}^{(m)})$ solves (3.8). Since that system of integral equations has a unique solution, the equation $m = \mathcal{G}^{(m)}$ has a unique solution $m^*$, and moreover $n_{k,\ell}^* = \overline{\mathcal{G}}_{k,\ell}$ for all $\ell, k$.

4.4. Estimates using the i.i.d. processes constructed from solution of the Poisson-driven SDEs (4.9). Recall $A_{j,k}^{N,\ell}(t)$ in (3.4). Let $\{A_{j,k}^{\ell}(t) : j \geq 1\}$ be the solution of (4.9) with $(Q_k^\ell, X_k, \lambda_k)$ being replaced by $(Q_{j,k}^\ell, X_{j,k}, \Lambda_{j,k})$ for each $j \geq 1$. We need the following lemma on the approximation of $A_{j,k}^{N,\ell}$ by $A_{j,k}^{\ell}$ as in Lemma 2.2. In the sequel, we shall use the notation

$$\mathcal{E}_T^N := \left\{ t \in [0,T \wedge \sigma_T^N] : \inf_{0 \leq t \leq T \wedge \sigma_T^N} \bar{B}_{k,\ell}^N(t) \geq C_T^* \right\},$$

where $C_T^* := \frac{1}{2} \min_{k,\ell} C_{k,T}^\ell$, and $C_{k,T}^\ell$'s are the lower bounds which appear in formula (4.3). Lemma 4.4 below establishes that $\mathbb{P}(\mathcal{E}_T^N) \leq C/\sqrt{N}$ for some constant $C$. Moreover, we define for any $T > 0$ the stopping time

$$\sigma_T^N := \inf \left\{ t > 0 : \inf_{k,\ell} \bar{B}_{k,\ell}^N(t) < C_T^* \right\}. \quad (4.11)$$

Note that on the event $\mathcal{E}_T^N$, $\sigma_T^N \geq T$.

**Lemma 4.3.** For any $T > 0$, $k \in \mathcal{K}$, and $\ell \in \mathcal{L}$, as $N \to \infty$,

$$\frac{1}{N} \mathbb{E} \left[ \sum_{j=1}^{S_k^N(0)} \sup_{0 \leq t \leq T \wedge \sigma_T^N} \left| A_{j,k}^{N,\ell}(t) - A_{j,k}^\ell(t) \right| \right] \to 0. \quad (4.12)$$

Before we prove this Lemma, let us first understand how to bound $\bar{B}_{k,\ell}^N(t)$ from below.

**Lemma 4.4.** For any $T > 0$, there exists a constant $C_T > 0$ such that for any $k \in \mathcal{K}$ and $\ell \in \mathcal{L}$,

$$\mathbb{P} \left( \inf_{0 \leq t \leq T} \bar{B}_{k,\ell}^N(t) < C_{k,T}^\ell/2 \right) \leq C_T N^{-1/2},$$

where $C_{k,T}^\ell$ denotes the constant which appears on the right of (4.3).

Note that clearly Lemma 4.3 and Lemma 4.4 imply that in probability, as $N \to \infty$,

$$\frac{1}{N} \sum_{j=1}^{S_k^N(0)} \sup_{0 \leq t \leq T \wedge \sigma_T^N} \left| A_{j,k}^{N,\ell}(t) - A_{j,k}^\ell(t) \right| \to 0.$$

**Proof of Lemma 4.4.** By (3.5), (3.6) and (3.7), we obtain, with $\tilde{P}_{S,k}^\ell, \tilde{P}_{I,k}^\ell, \tilde{P}_{R,k}^\ell$, $k \in \mathcal{K}, \ell, \ell' \in \mathcal{L}$ the corresponding compensated standard Poisson processes,

$$\bar{B}_{k,\ell}^N(t) = \bar{B}_{k,\ell}^N(0) + \sum_{\ell' = 1}^{L} \int_0^t \nu_{S,k}^\ell(s) \tilde{S}_{k,\ell'}^N(s) ds + \sum_{\ell' = 1}^{L} \int_0^t \nu_{I,k}^\ell(s) \tilde{I}_{k,\ell'}^N(s) ds + \sum_{\ell' = 1}^{L} \int_0^t \nu_{R,k}^\ell(s) \tilde{R}_{k,\ell'}^N(s) ds$$

$$- N^{-1} \sum_{\ell' = 1}^{L} \tilde{P}_{S,k}^\ell \left( \int_0^t \nu_{S,k}^\ell(s) \tilde{S}_{k,\ell'}^N(s) ds \right) - N^{-1} \sum_{\ell' = 1}^{L} \tilde{P}_{S,k}^\ell \left( \int_0^t \nu_{S,k}^\ell(s) S_{k,\ell'}^N(s) ds \right)$$

$$- N^{-1} \sum_{\ell' = 1}^{L} \tilde{P}_{I,k}^\ell \left( \int_0^t \nu_{I,k}^\ell(s) \tilde{I}_{k,\ell'}^N(s) ds \right) - N^{-1} \sum_{\ell' = 1}^{L} \tilde{P}_{I,k}^\ell \left( \int_0^t \nu_{I,k}^\ell(s) I_{k,\ell'}^N(s) ds \right)$$

$$- N^{-1} \sum_{\ell' = 1}^{L} \tilde{P}_{R,k}^\ell \left( \int_0^t \nu_{R,k}^\ell(s) \tilde{R}_{k,\ell'}^N(s) ds \right) - N^{-1} \sum_{\ell' = 1}^{L} \tilde{P}_{R,k}^\ell \left( \int_0^t \nu_{R,k}^\ell(s) R_{k,\ell'}^N(s) ds \right).$$
\[ \begin{align*}
&= \tilde{B}_{k}^{N,\ell}(0) + \int_{0}^{t} \nu_{k}^{\ell}(s) \tilde{B}_{k}^{N,\ell}(s)ds + \int_{0}^{t} Z_{k}^{N,\ell}(s)ds + \xi_{k}^{N,\ell}(t) \\
&\geq \tilde{B}_{k}^{N,\ell}(0) + \int_{0}^{t} \nu_{k}^{\ell}(s) \tilde{B}_{k}^{N,\ell}(s)ds + \xi_{k}^{N,\ell}(t),
\end{align*} \]

where \( \nu_{k}^{\ell}(s) = \min\{\nu_{S,k}^{\ell}(s), \nu_{I,k}^{\ell}(s), \nu_{R,k}^{\ell}(s)\} < 0 \), and

\[ Z_{k}^{N,\ell}(s) = \left( \sum_{\ell'=1}^{L} \nu_{S,k}^{\ell}(s) - \nu_{k}^{\ell}(s) \right) \tilde{Z}_{k}^{N,\ell}(s) + \left( \sum_{\ell'=1}^{L} \nu_{I,k}^{\ell}(s) - \nu_{k}^{\ell}(s) \right) \tilde{I}_{k}^{N,\ell}(s) \]

\[ + \left( \sum_{\ell' \neq \ell} \nu_{R,k}^{\ell}(s) - \nu_{k}^{\ell}(s) \right) \tilde{R}_{k}^{N,\ell}(s) \]

\[ = \left( \nu_{S,k}^{\ell}(s) - \nu_{k}^{\ell}(s) \right) \tilde{Z}_{k}^{N,\ell}(s) + \left( \nu_{I,k}^{\ell}(s) - \nu_{k}^{\ell}(s) \right) \tilde{I}_{k}^{N,\ell}(s) \]

\[ + \left( \nu_{R,k}^{\ell}(s) - \nu_{k}^{\ell}(s) \right) \tilde{R}_{k}^{N,\ell}(s) \]

\[ \geq 0, \]

and \( \xi_{k}^{N,\ell}(t) \) is the sum of the last six terms in the above expression.

Observe that \( \xi_{k}^{N,\ell}(t) \) is a square integrable martingale with a quadratic variation which is bounded by \( C/N \). We deduce from the above inequality that

\[ \tilde{B}_{k}^{N,\ell}(t) \geq \tilde{B}_{k}^{N,\ell}(0) \exp \left( \int_{0}^{t} \nu_{k}^{\ell}(s)ds \right) + \xi_{k}^{N,\ell}(t) + \int_{0}^{t} \nu_{k}^{\ell}(s) \exp \left( \int_{s}^{t} \nu_{k}^{\ell}(r)dr \right) \xi_{k}^{N,\ell}(s)ds . \]

Recall that \( \nu_{k}^{\ell}(s) < 0 \). We fix an arbitrary \( T > 0 \) and define the events

\[ A_{k}^{N,\ell} := \left\{ \xi_{k}^{N,\ell}(t) \geq \frac{-1}{4} \tilde{B}_{k}^{N,\ell}(0) \exp \left( \int_{0}^{t} \nu_{k}^{\ell}(s)ds \right), \forall 0 \leq t \leq T \right\}, \]

\[ B_{k}^{N,\ell} := \left\{ \xi_{k}^{N,\ell}(t) \leq \frac{1}{4} \tilde{B}_{k}^{N,\ell}(0) \exp \left( 2 \int_{0}^{T} \nu_{k}^{\ell}(s)ds \right), \forall 0 \leq t \leq T \right\} . \]

It is easy to verify that on the event \( A_{k}^{N,\ell} \cap B_{k}^{N,\ell} \),

\[ \tilde{B}_{k}^{N,\ell}(t) \geq \frac{1}{2} \tilde{B}_{k}^{N,\ell}(0) \exp \left( \int_{0}^{t} \nu_{k}^{\ell}(s)ds \right) , \]

i.e., \( \cap_{k,\ell}[A_{k}^{N,\ell} \cap B_{k}^{N,\ell}] \subset \mathcal{E}_{T}^{N} \), while \( \mathbb{P}(A_{k}^{N,\ell} \cap B_{k}^{N,\ell}) \geq 1 - C/\sqrt{N} \). The result follows. \( \square \)

**Proof of Lemma 4.3.** In this proof, \( C \) will denote an arbitrary positive constant, and \( \varepsilon_{N} \) an arbitrary sequence of positive numbers which converges to 0 as \( N \to \infty \). Both \( C \) and \( \varepsilon_{N} \) may vary from one line to another. Then,

\[ \left| A_{j,k}^{\ell}(t) - A_{j,k}(t) \right| \leq \int_{0}^{t} \int_{\Gamma_{k}^{N,\ell}(s) \Delta \Gamma_{k}^{\ell}(s)} Q_{j,k}(ds, du) , \]

and

\[ \sup_{0 \leq r \leq t} \left| A_{j,k}^{\ell}(r) - A_{j,k}(r) \right| \leq \int_{0}^{t} \int_{\Gamma_{k}^{N,\ell}(s) \Delta \Gamma_{k}^{\ell}(s)} Q_{j,k}(ds, du) , \]
from which we obtain

$$
\mathbb{E} \left[ \sup_{0 \leq r \leq t \wedge \sigma^N_T} \left| A_{j,k}^{N,t}(r) - A_{j,k}^{t}(r) \right| \right] \leq \mathbb{E} \left[ \int_0^{t \wedge \sigma^N_T} \left| \Gamma_k^{N,t}(s) - \Gamma_k^t(s) \right| ds \right].
$$

(4.13)

We have

$$
\left| \Gamma_k^{N,t}(t) - \Gamma_k^{t}(t) \right| = \frac{1}{(B_k^{N,t}(t))_{\gamma}} \sum_{k'} \sum_{k''} \beta_{k,k'}^{\ell,k''}(t) \tilde{S}_{k'}^{N,k'}(t) - \frac{1}{(B_k^{t}(t))_{\gamma}} \sum_{k'} \sum_{k''} \beta_{k,k'}^{\ell,k''}(t) \tilde{S}_{k'}^{k''}(t) \\
\leq \frac{1}{(B_k^{N,t}(t))_{\gamma}} \sum_{k'} \sum_{k''} \beta_{k,k'}^{\ell,k''}(t) \tilde{S}_{k'}^{N,k'}(t) - \frac{1}{(B_k^{t}(t))_{\gamma}} \sum_{k'} \sum_{k''} \beta_{k,k'}^{\ell,k''}(t) \tilde{S}_{k'}^{k''}(t) \\
+ \frac{1}{(B_k^{N,t}(t))_{\gamma}} - \frac{1}{(B_k^{t}(t))_{\gamma}} \sum_{k'} \sum_{k''} \beta_{k,k'}^{\ell,k''}(t) \tilde{S}_{k'}^{N,k'}(t) \\
\leq C_T^{-\gamma} \sum_{k'} \sum_{k''} \left| \tilde{S}_{k'}^{N,k'}(t) - \tilde{S}_{k'}^{k''}(t) \right| + \lambda^* \beta^* L K \frac{1}{(B_k^{N,t}(t))_{\gamma}} - \frac{1}{(B_k^{t}(t))_{\gamma}},
$$

(4.14)

where we have used for the last inequality both the lower bound (4.3), and the fact that $\tilde{S}_{k}^{N,k'}(t) \leq \lambda^*(\bar{I}_k^N(0) + \bar{S}_k^N(0)) \leq \lambda^*$ for all $t \geq 0$ and all $k, \ell$, which follows from (3.1).

By (3.1) and (3.8), we obtain

$$
\chi_{t < \sigma^N_T} \left| \tilde{S}_{k}^{N,k'}(t) - \tilde{S}_{k}^{t}(t) \right| \leq \left( N^{-1} \sum_{j=1}^{L} I_k^N(0) \sum_{\ell'} \lambda_{-j,k}(t) \mathbf{1}_{Y_{j,k}(t) = \ell} - \bar{\lambda}_k^0(t) \sum_{\ell'} I_k^t(0) q_{k,\ell'}(0,t) \right) \\
+ \chi_{t < \sigma^N_T} \left( N^{-1} \sum_{j=1}^{L} \lambda_{j,k}(t - \tau_{j,k}) \sum_{\ell'} \mathbf{1}_{X_{j,k}(\tau_{j,k}) = \ell'} \mathbf{1}_{Y_{j,k,\ell'}(t) = \ell} \right) \\
- \sum_{\ell'} \int_0^t \tilde{\lambda}_k(t - s) \tilde{S}_{k}(s) q_{k,\ell'}(s)_{k,\ell'}(s,t) ds.
$$

(4.15)

The convergence to 0 of the first term follows from the law of large numbers. Concerning the second term, we first note that

$$
\frac{1}{N} \sum_{j=1}^{L} I_k^N(0) \sum_{\ell'} \lambda_{j,k}(t - \tau_{j,k}) \sum_{\ell'} \mathbf{1}_{X_{j,k}(\tau_{j,k}) = \ell} \mathbf{1}_{Y_{j,k,\ell'}(t) = \ell} \\
= \frac{1}{N} \sum_{j=1}^{L} \int_0^t \lambda_{j,k}(t - s) \sum_{\ell'} \mathbf{1}_{X_{j,k}(s) = \ell} \mathbf{1}_{Y_{j,k,\ell'}(t) = \ell} ds A_{j,k}^{N,\ell}(s) \\
= \frac{1}{N} \sum_{j=1}^{L} \int_0^t \lambda_{j,k}(t - s) \sum_{\ell'} \mathbf{1}_{A_{j,k}(s) = \ell} \mathbf{1}_{Y_{j,k,\ell'}(t) = \ell} ds A_{j,k}^{N,\ell}(s) \\
= \frac{1}{N} \sum_{j=1}^{L} \int_0^t \lambda_{j,k}(t - s) \sum_{\ell'} \mathbf{1}_{A_{j,k}(s) = \ell} ds Q_{j,k}^{\ell'}(ds, du) \\
- \mathbf{1}_{A_{j,k}(s) = \ell} ds Q_{j,k}^{\ell'}(ds, du)
$$

(4.16)
\begin{equation}
+ \frac{1}{N} \sum_{j=1}^{S_{N}(0)} \int_{0}^{t} \int_{0}^{\infty} \lambda_{j,k}(t-s) \mathbf{1}_{A_{j,k}(s)} \mathbf{1}_{x_{j,k}(s) = \ell} \mathbf{1}_{u \in \Gamma_{j,k}(s)} \mathbf{1}_{X_{j,k}(s) = \ell} \mathbf{1}_{Y_{j,k}^{\ell}((s) = \ell} Q_{j,k}^{\ell}(ds, du).
\end{equation}

In order to bound the first term on the right of (4.16), we first deduce from (4.4) and (4.3) that there exists a constant $C_{T}$ such that for all $0 \leq t \leq T$, $k \in K$ and $\ell \in L$, $\Gamma_{k}(t) \leq C_{T}$. It then follows that, denoting by $\kappa_{N,\ell}^{k}(t)$ the first term on the right of (4.16),

\[ \mathbb{E} \left[ \sup_{0 \leq s \leq t \wedge \sigma_{T}^{N}} |\kappa_{N,\ell}^{k}(s)| \right] \leq \lambda^{*} \frac{C_{T}}{N} \mathbb{E} \left[ \sum_{j=1}^{S_{N}(0)} \sup_{0 \leq s \leq t \wedge \sigma_{T}^{N}} |A_{j,k}(s) - A_{j,k}^{N}(s)| \right] + \lambda^{*} \sum_{\ell'} \mathbb{E} \left[ \int_{0}^{t \wedge \sigma_{T}^{N}} |\Gamma_{k}^{N,\ell'}(s) - \Gamma_{k}^{\ell'}(s)| ds \right], \tag{4.17} \]

while, thanks to the law of large numbers, the second term on the right of (4.16) converges, as $N \to \infty$, towards

\[ \int_{0}^{t} \tilde{\lambda}_{k}(t-s) \sum_{\ell'} S_{k}(s) \Gamma_{k}^{\ell'}(s) q_{k}^{\ell'}(s,t) ds. \]

Combining the last estimates with (4.15) yields

\[ \mathbb{E} \left[ 1_{t < \sigma_{T}^{N}} \tilde{\mathcal{E}}_{k}^{N,\ell}(t) \right] \leq \varepsilon_{N} + C \sum_{\ell'} \mathbb{E} \left[ \int_{0}^{t \wedge \sigma_{T}^{N}} |\Gamma_{k}^{N,\ell'}(s) - \Gamma_{k}^{\ell'}(s)| ds \right] + \frac{1}{N} \mathbb{E} \left[ \sum_{j=1}^{S_{N}(0)} \sup_{0 \leq r \leq t \wedge \sigma_{T}^{N}} |A_{j,k}^{N,\ell'}(r) - A_{j,k}^{\ell'}(r)| \right]. \tag{4.18} \]

Remark 4.4. In the case $\gamma = 0$, instead of (4.14), we have the simpler bound

\[ |\Gamma_{k}^{N,\ell}(t) - \Gamma_{k}^{\ell}(t)| \leq \beta^{*} \sum_{\ell',k'} |\tilde{\mathcal{E}}_{k'}^{N,\ell'}(t) - \tilde{\mathcal{E}}_{k'}^{\ell'}(t)|. \]

Hence combining this estimate and (4.18), by Gronwall’s Lemma, we obtain

\[ \sum_{\ell,k} \mathbb{E} \left[ 1_{t < \sigma_{T}^{N}} |\Gamma_{k}^{N,\ell}(t) - \Gamma_{k}^{\ell}(t)| \right] \leq \varepsilon_{N} + C \sum_{\ell,k} \frac{1}{N} \mathbb{E} \left[ \sum_{j=1}^{S_{N}(0)} \sup_{0 \leq r \leq t \wedge \sigma_{T}^{N}} |A_{j,k}^{N,\ell}(r) - A_{j,k}^{\ell}(r)| \right]. \]

In that case, we do not need the estimate on $\tilde{D}_{k}^{N,\ell}$ in Lemma 4.4, nor the stopping time $\sigma_{T}^{N}$, nor the estimates in the next three Lemmas to complete the proof.

It remains to consider the second term on the right of (4.14). Observe that

\[ \frac{1}{(B_{k}^{N,\ell}(t))^\gamma} - \frac{1}{(B_{k}^{\ell}(t))^\gamma} = -\gamma \left( \int_{0}^{1} \left( u B_{k}^{N,\ell}(t) + (1-u) B_{k}^{\ell}(t) \right)^{-\gamma-1} du \right) \left( B_{k}^{N,\ell}(t) - B_{k}^{\ell}(t) \right). \tag{4.19} \]

It is clear that on the event $\{ t < \sigma_{T}^{N} \}$, the integral on the right hand side is bounded by $(C_{T}^{*})^{-\gamma-1}$. Hence it follows from (4.14), (4.18), (4.19) that, for all $t \in [0, T]$,

\[ \sum_{\ell,k} \mathbb{E} \left[ 1_{t < \sigma_{T}^{N}} |\Gamma_{k}^{N,\ell}(t) - \Gamma_{k}^{\ell}(t)| \right] \leq \varepsilon_{N} + C \sum_{\ell,k} \frac{1}{N} \mathbb{E} \left[ \sum_{j=1}^{S_{N}(0)} \sup_{0 \leq r \leq t \wedge \sigma_{T}^{N}} |A_{j,k}^{N,\ell}(r) - A_{j,k}^{\ell}(r)| \right]. \]
We first note that \(\text{Lemma 4.5.}\)

It will follow from the next three Lemmas that (4.20) holds without the last term. Hence from Gronwall’s Lemma

\[
\sum_{\ell,k} E \left[ 1_{t<\sigma^T_N} \left| N_k^\ell(t) - \Gamma_k^\ell(t) \right| \right] \leq \varepsilon_N + C \sum_{\ell,k} \frac{1}{N} \sum_{j=1}^{S_N^0} \sup_{0 \leq r \leq t \land \sigma^T_N} \left| A_{j,k}^N(r) - A_{j,k}^\ell(r) \right| .
\]

The result follows by combining this last estimate with (4.13) and Gronwall’s Lemma.

**Lemma 4.5.** For any \(T > 0\), there exists \(C > 0\) and a sequence \(\varepsilon_N\) of positive numbers which tends to 0 as \(N \to \infty\), and such that for any \(0 \leq t \leq T\) and \(k \in K\),

\[
\sum_{\ell} E \left[ 1_{t<\sigma_T^N} \left| \tilde{S}_k^\ell(t) - \tilde{S}_k^\ell(t) \right| \right] \leq \varepsilon_N + C \sum_\ell \frac{1}{N} \sum_{j=1}^{S_N^0} \left| A_{j,k}^N(t) - A_{j,k}^\ell(t) \right| .
\]

**Proof.** It follows from Gronwall’s Lemma that it suffices to show

\[
E \left[ 1_{t<\sigma_T^N} \left| \tilde{S}_k^\ell(t) - \tilde{S}_k^\ell(t) \right| \right] \leq \frac{1}{N} E \left[ 1_{t<\sigma_T^N} \sum_{j=1}^{S_N^0} \left| A_{j,k}^N(t) - A_{j,k}^\ell(t) \right| \right] + C \sum_\ell \frac{1}{N} \int_0^{t \wedge \sigma_T^N} \left| \tilde{S}_k^\ell(s) - \tilde{S}_k^\ell(s) \right| ds + \varepsilon_N .
\]

We first note that

\[
\tilde{S}_k^N(t) = \tilde{S}_k^N(0) - \frac{1}{N} \sum_{j=1}^{S_N^0} A_{j,k}^N(t) - \frac{1}{N} \sum_{\ell' \neq \ell} P_{\ell',k}^\ell \left( N \int_0^t v_{S_k^\ell(s)^0} \tilde{S}_k^N(t) ds \right) + \frac{1}{N} \sum_{\ell' \neq \ell} P_{S_{\ell',k}} \left( N \int_0^t v_{S_k^\ell(s)^0} \tilde{S}_k^\ell(s) \right) ,
\]

\[
\tilde{S}_k^\ell(t) = \tilde{S}_k^\ell(0) - \int_0^t \tilde{S}_k^\ell(s) \Gamma_k^\ell(s) ds + \sum_{\ell'=1}^L \int_0^t v_{S_{\ell',k}} \tilde{S}_k^\ell(s) ds .
\]

It is clear that as \(N \to \infty\), the following convergences hold in \(L^1(\Omega)\):

\[
\tilde{S}_k^N(0) \to \tilde{S}_k^0(0) ,
\]

\[
\sum_{\ell'=1}^L \frac{1}{N} P_{S_{\ell',k}} \left( N \int_0^t v_{S_k^\ell(s)^0} \tilde{S}_k^\ell(s) ds \right) \to \sum_{\ell'=1}^L \int_0^t v_{S_{\ell',k}}(s) \tilde{S}_k^\ell(s) ds ,
\]

\[
\sum_{\ell'=1}^L \frac{1}{N} P_{S_{\ell',k}} \left( N \int_0^t v_{S_k^\ell(s)^0} \tilde{S}_k^\ell(s) ds \right) \to \sum_{\ell'=1}^L \int_0^t v_{S_{\ell',k}}(s) \tilde{S}_k^\ell(s) ds ,
\]
and the following identity holds:

\[ - \sum_{\ell' \neq \ell} \int_0^t \nu_{S,k}(s)S_\ell^\ell(s)ds + \sum_{\ell' \neq \ell} \int_0^t \nu_{S,k}(s)S_{\ell'}^\ell(s)ds = \sum_{\ell = 1}^L \int_0^t \nu_{S,k}(s)S^\ell_k(s)ds. \]

Moreover, for all \( \ell' \neq \ell, \)

\[
\mathbb{E}\left[ \left| 1/N P_{S,k} \left( N \int_0^t \nu_{S,k}(s)S_{\ell'}^N(s)ds - \frac{1}{N} P_{S,k} \left( N \int_0^t \nu_{S,k}(s)\tilde{S}_N^\ell(s)ds \right) \right) \right| \right] \\
+ \left| 1/N P_{S,k} \left( N \int_0^t \nu_{S,k}(s)S_{\ell'}^N(s)ds - \frac{1}{N} P_{S,k} \left( N \int_0^t \nu_{S,k}(s)S^\ell_k(s)ds \right) \right) \right] \\
\leq C \mathbb{E} \left[ \int_0^t \left| S_{\ell'}^N(s) - S^\ell_k(s) \right| ds + \int_0^t \left| S_{\ell'}^N(s) - S^\ell_k(s) \right| ds \right].
\]

Now consider the difference

\[ \Delta_k^{N,\ell}(t) := - \frac{1}{N} \sum_{j=1}^{S_k^N(0)} A_{j,k}^{N,\ell}(t) + \int_0^t \tilde{S}^\ell_k(s)\tilde{\Gamma}_k(s)ds. \]

We rewrite it as

\[ \Delta_k^{N,\ell}(t) = \frac{1}{N} \sum_{j=1}^{S_k^N(0)} \left( A_{j,k}^{\ell}(t) - A_{j,k}^{N,\ell}(t) \right) + \int_0^t \tilde{S}^\ell_k(s)\tilde{\Gamma}_k(s)ds - \frac{1}{N} \sum_{j=1}^{S_k^N(0)} A_{j,k}^{\ell}(t). \]

It remains to show that, as \( N \to \infty, \)

\[ \frac{1}{N} \sum_{j=1}^{S_k^N(0)} A_{j,k}^{\ell}(t) \to \int_0^t \tilde{S}^\ell_k(s)\tilde{\Gamma}_k(s)ds \quad (4.21) \]

in \( L^1(\Omega). \) We note that

\[
\frac{1}{N} \sum_{j=1}^{S_k^N(0)} A_{j,k}^{\ell}(t) \to \mathbb{E} \left[ S_k^{X_k(0)} A_{\ell}^{\ell}(t) \right] \\
= \int_0^t \mathbb{E} \left[ S_k^{X_k(0)} 1_{X_k(s)=\ell} 1_{A_k(s)=0} \right] \tilde{\Gamma}_k(s)ds \\
= \int_0^t \mathbb{E} \left[ 1_{X_k(s)=\ell} S_k^{X_k(0)} \exp \left( - \int_0^s \tilde{\Gamma}_k(r)dr \right) \right] \tilde{\Gamma}_k(s)ds \\
= \int_0^t \tilde{S}^\ell_k(s)\tilde{\Gamma}_k(s)ds,
\]

where we have used successively the strong law of large numbers, the first line of (4.9), the fact that \( \mathbb{P}(A_k(s) = 0|X_k(\cdot)) = \exp \left( - \int_0^s \tilde{\Gamma}_k(r)dr \right), \)

and formula (4.8) from Proposition 4.1. \( \square \)

**Lemma 4.6.** For any \( T > 0, \) there exists a sequence \( \varepsilon_N \) of positive numbers which tends to 0 as \( N \to \infty, \) and such that for any \( 0 \leq t \leq T, \) \( k \in \mathcal{K}, \)

\[
\sum_{\ell} \mathbb{E} \left[ 1_{t < \sigma_T} \left| \tilde{I}_k^{N,\ell}(t) - \tilde{I}_k^{\ell}(t) \right| \right] \leq \varepsilon_N + \frac{C}{N} \sum_{\ell} \mathbb{E} \left[ \sup_{s \leq t \wedge \sigma_T} \sum_{j=1}^{S_k^N(0)} \left| A_{j,k}^{N,\ell}(s) - A_{j,k}^{\ell}(s) \right| \right].
\]
+ C \sum_{\ell} \mathbb{E} \left[ \int_0^{t \wedge \sigma N_T^N} |\bar{F}_k^{N,\ell}(r) - \bar{F}_k^{\ell}(r)| dr \right].

Proof. Again, it suffices to show that

\[ \mathbb{E} \left[ 1_{t < \sigma N_T^N} \left| \bar{I}_k^{N,\ell}(t) - \bar{I}_k^{\ell}(t) \right| \right] \leq \varepsilon_N + \frac{C}{N} \mathbb{E} \left[ 1_{t < \sigma N_T^N} \sum_{j=1}^{S_k^N(0)} \sum_{\ell' = 1}^L A_j^{N,\ell'}(t) - A_{j,k}^{\ell}(t) \right] + C \sum_{\ell'} \mathbb{E} \left[ \int_0^{t \wedge \sigma N_T^N} \left| \bar{I}_k^{N,\ell'}(s) - \bar{I}_k^{\ell}(s) \right| ds \right] + C \sum_{\ell} \mathbb{E} \left[ \int_0^{t \wedge \sigma N_T^N} |\bar{F}_k^{N,\ell}(r) - \bar{F}_k^{\ell}(r)| dr \right]. \]

Four of the terms in the equation for \( \bar{I}_k^{N,\ell}(t) \) (see (3.6)) are treated exactly as in the previous Lemma. Moreover, by the strong law of large numbers,

\[ \mathbb{E} \left[ \int_0^{t \wedge \sigma N_T^N} |\bar{F}_k^{N,\ell}(r) - \bar{F}_k^{\ell}(r)| dr \right] \to 0. \]

Again, it suffices to show that

\[ \mathbb{E} \left[ 1_{t < \sigma N_T^N} \left| \bar{I}_k^{N,\ell}(t) - \bar{I}_k^{\ell}(t) \right| \right] \leq \varepsilon_N + \frac{C}{N} \mathbb{E} \left[ 1_{t < \sigma N_T^N} \sum_{j=1}^{S_k^N(0)} \sum_{\ell' = 1}^L A_j^{N,\ell'}(t) - A_{j,k}^{\ell}(t) \right] + C \sum_{\ell'} \mathbb{E} \left[ \int_0^{t \wedge \sigma N_T^N} \left| \bar{I}_k^{N,\ell'}(s) - \bar{I}_k^{\ell}(s) \right| ds \right] + C \sum_{\ell} \mathbb{E} \left[ \int_0^{t \wedge \sigma N_T^N} |\bar{F}_k^{N,\ell}(r) - \bar{F}_k^{\ell}(r)| dr \right]. \]

For that sake, we introduce a new collection of i.i.d. PRMs \( \bar{Q}_{j,k}^\ell \) on \( \mathbb{R}^3 \), for \( k \in \mathcal{K}, \ell \in \mathcal{L} \) and \( j \geq 1 \) with mean measure \( dsduF_k(dr) \). The PRM \( \bar{Q}_{j,k}^\ell \) is the point measure associated to the collection of random points \((\tau_{j,k}^N, u_{j,k}, \eta_{j,k})\), \( j \geq 1 \), where \((\tau_{j,k}^N, u_{j,k})\) are the point of the PRM \( Q_{j,k}^\ell \) which are such that \( A_{j,k}^N((\tau_{j,k}^N)^-) = 0, X_{j,k}(\tau_{j,k}^N) = \ell \) and \( u_{j,k} \leq \bar{F}_k^{N,\ell}((\tau_{j,k}^N)^-) \), and the i.i.d. r.v.’s \( \eta_{j,k}, j \geq 1 \) are globally independent of \( Q_{j,k}^\ell \) and have the distribution function \( F_k(dr) \). With this new PRM, we have

\[ T_{j,k}^{N,\ell}(t) := \frac{1}{N} \sum_{j=1}^{S_k^N(0)} \sum_{\ell'} \int_0^{t} \int_0^{t-s} \int_0^{t-s} 1_{A_{j,k}^{N}(s^-) = \epsilon} 1_{X_{j,k}(s) = \epsilon} 1_{Y_{j,k}^{\epsilon}(s+r) = \epsilon} 1_{u_{j,k} \leq \bar{F}_k^{N,\ell}(s^-)} \bar{Q}_{j,k}^\ell(ds, du, dr). \]

Note that for each \( j \) and \( \ell' \), the integral is either 0 or 1, which will allow us to simplify the difference with a similar integral. We have

\[ T_{j,k}^{N,\ell}(t) = \frac{1}{N} \sum_{j=1}^{S_k^N(0)} \sum_{\ell'} \int_0^{t} \int_0^{t-s} \int_0^{t-s} 1_{X_{j,k}(s) = \epsilon} 1_{Y_{j,k}^{\epsilon}(s+r) = \epsilon} 1_{u_{j,k} \leq \bar{F}_k^{N,\ell}(s^-)} 1_{A_{j,k}^{N}(s^-) = 0} \bar{Q}_{j,k}^\ell(ds, du, dr) \]

\[ + \frac{1}{N} \sum_{j=1}^{S_k^N(0)} \sum_{\ell'} \int_0^{t} \int_0^{t-s} \int_0^{t-s} 1_{A_{j,k}(s^-) = 0} 1_{Y_{j,k}^{\epsilon}(s+r) = \epsilon} 1_{u_{j,k} \leq \bar{F}_k^{N,\ell}(s^-)} 1_{A_{j,k}^{N}(s^-) = 0} \bar{Q}_{j,k}^\ell(ds, du, dr). \]

(4.22)
The expectation of the absolute value of the first term on the right of (4.22) evaluated at time 
\( t \wedge \sigma_T^N \) is bounded by

\[
\frac{1}{N} \sum_{j=1}^{t \wedge \sigma_T^N} \mathbb{E} \left[ \sum_{\ell} \mathbb{E} \left[ \sup_{0 \leq s \leq t \wedge \sigma_T^N} \left| A_{j,k}^N(s) - A_{j,k}(s) \right| \right] \right].
\]

From the law of large numbers, the second term in (4.22) converges as \( N \to \infty \), towards

\[
\sum_{\ell} \int_0^t \int_0^{t-s} \mathbb{E} \left[ \tilde{S}_k^N(0) \right] ds ds,
\]

where we have used the fact that \( \mathbb{P}(A_k(s) = 0 \mid X_k) = \exp \left( -\int_0^s \bar{\Gamma}_k(r)(r)dr \right) \) and formula (4.8).

The result follows. \( \square \)

**Lemma 4.7.** For any \( T > 0 \), there exists a sequence \( \varepsilon_N \) of positive numbers which tends to 0 as \( N \to \infty \), and such that for any \( 0 \leq t \leq T \), and \( k \in \mathcal{K} \),

\[
\sum_{\ell} \mathbb{E} \left[ \mathbf{1}_{t < \sigma_T^N} \left| \tilde{R}_k^{N,\ell}(t) - \check{R}_k^{N,\ell}(t) \right| \right] \leq \varepsilon_N + C \sum_{\ell} \mathbb{E} \left[ \sup_{0 \leq s \leq t \wedge \sigma_T^N} \sum_{j=1}^{t \wedge \sigma_T^N} \left| A_{j,k}^N(s) - A_{j,k}(s) \right| \right]
\]

\[
+ C \sum_{\ell} \mathbb{E} \left[ \int_0^{t \wedge \sigma_T^N} \left| \bar{\Gamma}_k^{N,\ell}(r) - \bar{\Gamma}_k^{N,\ell}(r) \right| dr \right].
\]

The proof of this Lemma is very similar to that of the previous one, as the reader can easily verify.

### 4.5 Completing the proof of Theorem 3.1.

It follows from the above Lemmas that for each \( t > 0 \), \((\tilde{S}_k^{N,\ell}(t), \tilde{S}_k^{N,\ell}(t), \tilde{I}_k^{N,\ell}(t), \tilde{R}_k^{N,\ell}(t))\) converges in probability as \( N \to \infty \) towards \((\check{S}_k^{N,\ell}(t), \tilde{S}_k^{N,\ell}(t), \check{I}_k^{N,\ell}(t), \check{R}_k^{N,\ell}(t))\).

It remains to prove that the convergences hold in \( \mathbf{D} \).

We first consider \( \tilde{S}_k^{N,\ell} \). Consider the right hand side of (4.15). The first term tends to 0 a.s., locally uniformly in \( t \), thanks to Theorem 1 in [17]. Concerning the second term on the right of (4.15), its convergence follows from the decomposition (4.16). The first term on the right of (4.16) converges to 0 in probability locally uniformly in \( t \), thanks to (4.17) and the rest of the proof of Lemma 4.3, while the locally uniform convergence in \( t \) in probability of the second term follows again from Theorem 1 in [17].

We now establish the convergence in \( \mathbf{D} \) of the other quantities. We shall next use repeatedly Dini’s theorem, which implies that a sequence of increasing functions which converges pointwise to continuous function, converges in fact locally uniformly. This applies to random functions which converge in probability, since convergence in probability is equivalent to the fact that from any subsequence, one can extract a further subsequence which converges a.s. Also note that at most one limit term in each equation is discontinuous, so we will have no difficulty in adding convergences in \( \mathbf{D} \).

We next consider the process \( \check{S}_k^{N,\ell}(t) \). Let us first discuss the Poisson terms, which are of the form

\[
\frac{1}{N} \mathbb{P}_{S,k}^{N,\ell} \left( \int_0^t \nu_{S,k}(s) \check{S}_k^{N,\ell}(s) ds \right) = \frac{1}{N} \mathbb{P}_{S,k}^{N,\ell} \left( N \int_0^t \nu_{S,k}(s) \check{S}_k^{N,\ell}(s) ds \right),
\]

which is non-decreasing and from the LLN for Poisson processes converge in probability, towards the continuous function \( \int_0^t \nu_{S,k}(s) \check{S}_k(s) ds \). Hence the convergence is locally uniform in \( t \).
We finally need to consider the term from (3.5):
\[
\sum_{j=1}^{N} A_{j,k}^N(t) = \sum_{j=1}^{N} \int_0^t \int_0^\infty 1_{A_{j,k}^N(s^-) = \ell} 1_{X_{j,k}(s) = \ell} 1_{u \leq \tilde{N}_k^N(s^-)} Q_k^\ell(ds,du).
\]
There exists a standard PRM \(Q_k^\ell(ds,du)\) on \(\mathbb{R}^2_+\) such that
\[
\frac{1}{N} \sum_{j=1}^{N} A_{j,k}^N(t) = \frac{1}{N} \sum_{j=1}^{N} \int_0^t \int_0^\infty 1_{u \leq S_k^N(s^-) \leq \tilde{N}_k^N(s^-)} Q_k^\ell(ds,du)
\]
\[
= \frac{1}{N} \int_0^t \int_0^\infty 1_{u \leq S_k^N(s^-)} \tilde{N}_k^N(s^-) Q_k^\ell(ds,du)
\]
\[
= \int_0^t S_k^N(s) \tilde{N}_k^N(s) ds + \frac{1}{N} \int_0^t \int_0^\infty 1_{u \leq S_k^N(s^-)} \tilde{N}_k^N(s^-) Q_k^\ell(ds,du),
\]
where we have used the fact that \(S_k^N(s) = \sum_{j=1}^{N} 1_{A_{j,k}^N(s^-) = \ell} 1_{X_{j,k}(s) = \ell}\), and \(Q_k^\ell(ds,du) = Q_k^\ell(ds,du) - dsdu\). It follows from Lemmas 4.5, 4.3 and 4.4, that the first term on the right converges locally uniformly in \(t\) in probability towards \(\int_0^t S_k^N(s) \tilde{N}_k^N(s) ds\), while the second term is a martingale which converges locally uniformly in \(t\) towards \(0\) in probability.

We next consider the process \(\bar{I}_{k}^{N,\ell}(t)\). There are two new terms in (3.6), compared to (3.5). The first one is
\[
\sum_{\ell' = 1}^{L} \sum_{i=1}^{L} 1_{\eta_{i,k} \leq \ell} 1_{Y_{i,k}(\eta_{i,k}) = \ell},
\]
and the second one
\[
\sum_{j=1}^{N} 1_{\tau_{j,k} + \eta_{j,k} \leq \ell} \sum_{\ell'} 1_{X_{j,k}(\tau_{j,k}^N) = \ell'} 1_{Y_{j,k}(\tau_{j,k}^N + \eta_{j,k}) = \ell}.\]
It follows from the law of large numbers in \(D\) (see, e.g., [17]) that the first term converges in probability in \(D\) towards
\[
\sum_{\ell' = 1}^{L} \bar{I}_{k}^{\ell'}(0) \int_0^t q_k^{\ell',\ell}(0,s) F_k^0(ds).
\]
It remains to reconsider the argument used to treat \(\bar{I}_{k}^{N,\ell}(t)\) in the proof of Lemma 4.6. The uniformity in \(t\) of the convergence in probability to \(0\) of the first term on the right hand side of (4.22) is rather obvious. Concerning the second term, the uniformity in \(t\) of the convergence will follow from Dini’s theorem, if we show that the mapping
\[
t \mapsto G(t) := \int_0^t \int_0^{t-s} \bar{S}_k(s) \tilde{G}_k^\ell(s) q_k^{\ell,k}(s,s+u) F_k(du)ds
\]
is continuous. But for \(t' < t\), if \(g(s,u) := \bar{S}_k(s) \tilde{G}_k^\ell(s) q_k^{\ell,k}(s,s+u)\), we have with \(C := KL\lambda^s \beta^s\),
\[
G(t) - G(t') = \int_{\mathbb{R}^2_+} 1_{t' < s+u \leq t} g(s,u) F_k(du)ds
\]
\[
\leq C \int_{\mathbb{R}^2_+} 1_{t' < s+u \leq t} F_k(du)ds
\]
\[
\leq C (t - t'),
\]
where the last inequality follows by integrating first with respect to \(s\).
References


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