On uniform stability of certain parallel server networks with no abandonment in the Halfin–Whitt regime

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Abstract. In this paper we show that a large class of parallel server networks, with \(\sqrt{n}\)-safety staffing, and no abandonment, in the Halfin–Whitt regime are exponentially ergodic and their invariant probability distributions are tight. This includes all networks with a single nonleaf server pool, such as the ‘N’ and ‘M’ models, as well as networks with class-dependent service rates.

We first give a simple algebraic characterization of a parameter that plays the role of the spare capacity (safety staffing) for the diffusion limit. We show that if the spare capacity parameter is negative, the controlled diffusion is transient under any stationary Markov control, and it cannot be positive recurrent when this parameter is zero. On the other hand, if this parameter is positive, then, for the aforementioned classes of networks, the limiting diffusion is uniformly exponentially ergodic over all Markov controls.

We establish the analogous stability properties for the prelimit diffusion-scaled queueing processes. We use a unified approach in which the Lyapunov function employed in the study of the diffusion limit paves the way. As well known, joint work conservation, that is, keeping all servers busy unless all queues are empty, cannot be always enforced in multiclass multi-pool networks, and as a result the diffusion limit and the prelimit do not “match” on the entire state space. For this reason, we introduce the concept of “system-wide work conserving policies”, which are defined as policies that minimize the number of idle servers at all times. This is a natural extension of work conservation for multiclass multi-pool networks. We show that, provided that the spare capacity parameter is positive, the diffusion-scaled processes are geometrically ergodic and the invariant distributions are tight, uniformly over all system-wide work conserving policies. This also results in an interchange of limits property.

1. Introduction

Large scale parallel server networks have been the subject of intense study, due to their use in modeling a variety of systems including telecommunications, data centers, customer services and manufacturing systems; see, e.g., [1–8]. Ensuring stability of these systems through allocating available resources by means of adjusting controller parameters is of great importance. In this paper, we focus on studying blanket stability properties of multiclass multi-pool queueing networks in the Halfin–Whitt regime (or Quality-and-Efficiency-Driven (QED) regime [9–11]), where the arrival rates and the numbers of servers become large as the scale of the system grows, while the service rates remain fixed in such a way that the system becomes critically loaded. The approach followed in this paper was motivated by the work in [12], which employs a common Lyapunov function in the study of of the diffusion limit and prelimit for multiclass single-pool “V” networks. It is shown in [12], that “V” networks with \(\sqrt{n}\) safety staffing are exponentially ergodic, and the invariant probability measures have uniform exponential tails both in the diffusion limit and in
the $n^{th}$ system (prelimit), uniformly over the scale of the network, and over all stationary (work-conserving) Markov controls.

The stability analysis of multiclass multi-pool networks in the Halfin–Whitt regime is considerably more challenging than the corresponding one for the ‘V’ network. Note however, that the question of ‘stabilizability’ of such networks has been addressed to some extent. In [13], a recursive leaf elimination algorithm is developed to derive the drift of the limiting diffusion, and, consequently, by using the structural properties of the drift, a static priority scheduling and routing control is identified which stabilizes the limiting diffusion, when at least one of the classes has a positive abandonment rate. In [14], in the study of the ‘N’ network, it is shown that a class of state-dependent policies, referred to as balanced saturation policies (BSP) are stabilizing for the prelimit diffusion-scaled queueing process, when at least one abandonment rate is strictly positive. In [15], this result is extended to general multiclass multi-pool networks.

From the existing literature of critically loaded parallel server networks, we wish to emphasize [16,17], not only because of the importance of the results, but also because they are the closest to this paper. We also refer the reader to [18, 19], even though these concern the underloaded case. Stolyar [16] studied a static priority scheduling policy for the ‘N’ network with safety staffing and no abandonment, and established the tightness of stationary distributions of the prelimit process (there is no analysis of the rate of convergence though). Stolyar and Yudovina [17] showed that the stationary distributions of the diffusion-scaled processes may not be tight in the underloaded and Halfin–Whitt regime under a natural load balancing scheduling policy, “Longest-queue freest-server” (LQFS-LB), but they are tight for the class of networks with pool-dependent service rates. Stolyar and Yudovina [19] and Stolyar [16] then proved tightness of the stationary distributions of a leaf-activity priority policy in the sub-diffusion and diffusion scales, respectively, in the underloaded regime.

Despite all these important results, the ergodic properties of multiclass multi-pool networks in the Halfin–Whitt regime are far from being well understood. In this paper, we contribute to the following important aspects of this problem: (i) we identify and characterize the spare capacity (safety staffing) for the networks without abandonment in the diffusion limit; and (ii) we establish geometric ergodicity for certain classes of networks, uniformly over all stationary Markov controls and determine an upper bound of the rate of convergence.

We first provide a simple algebraic representation of the spare capacity (safety staffing) parameter for the limiting diffusion. We recall the form of the drift derived via the leaf elimination algorithm in [13] given in (2.26). The spare capacity $\varrho$ is given by (3.1). We show that for any general network topology, if the spare capacity is negative, then the limiting diffusion is transient under any stationary Markov control, and if it is zero, the diffusion cannot be positive recurrent. To prove these properties, we establish a key property on the drift parameters in Lemma 3.1). This requires an in-depth look into the leaf elimination algorithm, including a sensitivity analysis of adding a class or server pool. We also define the corresponding spare capacity parameter for the prelimit (the $n^{th}$ system), and show that without abandonment, if the spare capacity is negative, the prelimit diffusion-scaled queueing processes is transient under any stationary Markov scheduling policy. Lastly, we provide a characterization of the spare capacity parameter for the limiting diffusion when the latter it is positive recurrent. We show that the spare capacity is equal to an average ‘idleness’ weighted by the critical quantity studied in Lemma 3.1.

In establishing uniform exponential ergodicity for certain network topologies, we focus on (i) networks with one dominating server pool, that is, a single non-leaf server pool, which include the ‘N’, ‘M’ and generalized ‘N’, ‘M’ networks with diameters equal to three or four, and (ii) networks with class-dependent service rates. These networks share an important structural property in their drift, that is, the matrix $B_1$ in (2.26) is diagonal. For these networks, we use a common Lyapunov function given in Definition 4.1. As in [12] for the ‘V’ network, this Lyapunov function consists of two components that treat the positive and negative half spaces of the state space in
a delicate manner. The main difference is that for multi-pool networks we use the sum instead of
the product of the two components corresponding to the positive and negative half state spaces.
An important ‘tilting’ parameter must be carefully chosen to account for not only the different
effects of queueing and idleness (positive and negative half state space), but also the second order
derivatives of the extended generator of the diffusion. Note that these Lyapunov functions differ
from the quadratic Lyapunov functions used in [13–15, 20, 21] employed for the study of stability
under constant controls.

The same Lyapunov function is used to prove the uniform exponential ergodicity for the prelimit
diffusion-scaled processes. However, unlike the ‘V’ network studied in [12], the Foster-Lyapunov
equations for the limiting diffusion do not carry over to the analogous equations for the diffusion-
scaled queueing processes over the entire state space. The reason lies in the jointly work conserving
(JWC) condition (that is, all the queues have to be empty when there are idle servers) which is
essential in establishing the weak convergence to the controlled limiting diffusion (see [22, 23]).
To tackle this difficulty, we first provide an explicit ‘drift’ representation of the diffusion-scaled
processes which differs from the drift of the diffusion by an extra term that accounts for the
deviation from the JWC condition in the \(n\)th system, and which vanishes in the limit. A natural
extension of the concept of work conservation for multiclass multi-pool networks is minimization of
the idle servers at all times. This defines an action space which we call system-wide work conserving
(SWC). Establishing blanket geometric ergodicity over all SWC Markov policies when the spare
capacity is positive, is accomplished by first proving a useful upper bound for the minimum of
idle servers and cumulative queue size for the \(n\)th system, and then using this to derive the Foster–
Lyapunov drift inequalities in the region of the state space where the drifts of the diffusion limit and
the \(n\)th system do not match. This facilitates establishing the drift inequalities for the diffusion-
scaled processes. As a consequence of the Foster-Lyapunov equations, the invariant probability
measures of the diffusion-scaled queueing processes have uniform exponential tails.

1.1. Organization of the paper. In the next subsection, we summarize the notation used in
the paper. In Subsection 2.1, we describe the model and state informally the assumptions used.
We define the diffusion scaled processes, and characterize the corresponding controlled generator in
Subsection 2.2. In Subsection 2.3, the notion of system-wide work conserving policies is introduced,
and this is used in Subsection 2.4 to take limits and establish the diffusion approximation. In
Section 3, we define the parameter of spare capacity \(\varrho\) for multiclass multi-pool networks and
show that whenever \(\varrho < 0\), then, for any network topology, the process is transient under any
stationary Markov control both for the diffusion limit and the \(n\)th system. In the same subsection,
we establish the relation between the spare capacity and average idleness. In Section 4 we first
provide equivalent characterizations of uniform exponential ergodicity of controlled diffusions, and then proceed to establish that the diffusion limits of the aforementioned classes of networks are uniformly exponentially ergodic and their invariant probability measures have uniform exponential tails. Finally, Section 5 is devoted to the study of uniform exponential ergodicity of the $n^{th}$ system of networks under consideration.

1.2. Notation. We use $\mathbb{R}^m$ (and $\mathbb{R}_+^m$), $m \geq 1$, to denote real-valued $m$-dimensional (nonnegative) vectors, and write $\mathbb{R}$ for the real line. We use $z^T$ to denote the transpose of a vector $z \in \mathbb{R}^m$. Throughout the paper $e \in \mathbb{R}^m$ stands for the vector whose elements are equal to 1, that is, $e = (1, \ldots, 1)^T$, and $e_i \in \mathbb{R}^m$ denotes the vector whose elements are all 0 except for the $i^{th}$ element which is equal to 1. For $x, y \in \mathbb{R}$, $x \vee y = \max\{x, y\}$, $x \wedge y = \min\{x, y\}$, $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$.

For a set $A \subseteq \mathbb{R}^m$, we use $A^c$, $\partial A$, and $1_A$ to denote the complement, the boundary, and the indicator function of $A$, respectively. A ball of radius $r > 0$ in $\mathbb{R}^m$ around a point $x$ is denoted by $B_r(x)$, or simply as $B_r$, if $x = 0$. We also let $\mathbb{B} \equiv B_1$. The Euclidean norm on $\mathbb{R}^m$ is denoted by $|·|$, and $(·, ·)$ stands for the inner product. For $x \in \mathbb{R}^m$, we let $||x|| := \sum_i |x_i|$, and by $K_r$, or $K(r)$, for $r > 0$, we denote the closed cube

$$K_r := \{x \in \mathbb{R}^m : ||x||_1 \leq r\}. \quad (1.1)$$

Also, we define $x_{\max} := \max_i x_i$, and $x_{\min} := \min_i x_i$, and $x^\pm := (x_1^+, \ldots, x_m^+)$. For a finite signed measure $\nu$ on $\mathbb{R}^m$, and a Borel measurable $f: \mathbb{R}^m \to [1, \infty)$, the $f$-norm of $\nu$ is defined by

$$||\nu||_f := \sup_{g \in \mathcal{B}(\mathbb{R}^m), |g| \leq f} \left| \int_{\mathbb{R}^m} g(x) \nu(dx) \right|, \quad (1.2)$$

where $\mathcal{B}(\mathbb{R}^m)$ denotes the class of Borel measurable functions on $\mathbb{R}^m$.

2. The queueing network model and the diffusion limit

In this section, we consider a sequence of parallel server networks whose processes, parameters, and variables are indexed by $n$. We recall some of the definitions and notations used in [13, 15].

2.1. Model and assumptions. Consider a general Markovian parallel server (multiclass multipool) network with $m$ classes of customers and $J$ server pools. Customer classes take values in $I = \{1, \ldots, m\}$ and server pools in $J = \{1, \ldots, J\}$. Forming their own queue, customers of each class are served according to a First-Come-First-Served (FCFS) service discipline. We assume throughout the paper that customers do not abandon. For all $i \in I$, let $J(i)$ denote the subset of server pools that can serve customer class $i$. On the other hand, for all $j \in J$, let $I(j)$ be the subset of customer classes that can be served by server pool $j$.

We form a bipartite graph $\mathcal{G} = (I \cup J, \mathcal{E})$ with a set of edges defined by $\mathcal{E} = \{(i, j) \in I \times J : j \in J(i)\}$, and use the notation $i \sim j$, if $(i, j) \in \mathcal{E}$, and $i \sim j$, otherwise. We assume that the graph $\mathcal{G}$ is a tree. We define

$$\mathbb{R}_+^G := \{\xi = [\xi_{ij}] \in \mathbb{R}_+^{m \times J} : \xi_{ij} = 0 \text{ for } i \sim j\}, \quad (2.1)$$

and analogously define $\mathbb{Z}_+^G$, $\mathbb{Z}_+^G$, and $\mathbb{Z}^G$.

In each server pool $j$, we let $N_j^n$ be the number of servers, which are assumed to be statistically identical. For each $i \in I$, customer class $i$ arrives according to a Poisson process with arrival rate $\lambda_i^n > 0$. These customers are served at an exponential rate $\mu_{ij}^n > 0$ at server pool $j$ if $j \in J(i)$, and $\mu_{ij}^n = 0$ otherwise. Finally, we assume that the arrival and service processes of all classes are mutually independent. We study these networks in the Halfin–Whitt regime, which involves the following assumption on the parameters. There exist positive constants $\lambda_i$ and $\nu_j$, nonnegative
And the action space

2.2. Diffusion scaling.

Note that this space consists of work-conserving actions only.

\[ n \lambda_i - n \lambda_i \sqrt{n} \rightarrow \dot{\lambda}_i, \quad \sqrt{n} (\mu_{ij} - \mu_{ij}) \rightarrow \dot{\mu}_{ij}, \quad \text{and} \quad \frac{N_j - n \nu_j}{\sqrt{n}} \rightarrow \nu_j. \]  

(2.2)

An additional standard assumption referred to as the complete resource pooling condition [23,33] concerns the fluid scale equilibrium, and is stated as follows. The linear program (LP) given by

\[
\begin{align*}
\text{Minimize} & \quad \max_{j \in \mathcal{J}} \sum_{i \in \mathcal{I}(j)} \xi_{ij}, \\
\text{subject to} & \quad \sum_{j \in \mathcal{J}(i)} \mu_{ij} \nu_{ij} \xi_{ij} = \lambda_i \quad \forall i \in \mathcal{I},
\end{align*}
\]

has a unique solution \( \xi^* = [\xi^*_{ij}] \in \mathbb{R}^G_+ \) satisfying

\[
\sum_{i \in \mathcal{I}} \xi^*_{ij} = 1, \quad \forall j \in \mathcal{J}, \quad \text{and} \quad \xi^*_{ij} > 0 \quad \text{for all } i \sim j.
\]

(2.4)

We define \( x^* \in \mathbb{R}^m \), and \( z^* \in \mathbb{R}^G_+ \) by

\[
x^*_i = \sum_{j \in \mathcal{J}} \xi^*_{ij} \nu_j, \quad \text{and} \quad z^*_ij = \xi^*_{ij} \nu_j.
\]

(2.5)

For each \( i \in \mathcal{I} \) and \( j \in \mathcal{J} \), we let \( X^n_i = \{X^n_i(t) : t \geq 0\} \) denote the total number of class \( i \) customers in the system (both in service and in queue), \( Z^n_{ij} = \{Z^n_{ij}(t), t \geq 0\} \) the number of class \( i \) customers served in pool \( j \), \( Q^n_i = \{Q^n_i(t), t \geq 0\} \) the number of class \( i \) customers in the queue, and \( Y^n_j = \{Y^n_j(t), t \geq 0\} \) the number of idle servers in server pool \( j \). Let \( X^n = (X^n_i)_{i \in \mathcal{I}}, Y^n = (Y^n_j)_{j \in \mathcal{J}}, Q^n = (Q^n_i)_{i \in \mathcal{I}}, \text{and} \ Z^n = (Z^n_{ij})_{i \in \mathcal{I}, j \in \mathcal{J}} \). The process \( Z^n \) is the scheduling control. Dropping the explicit dependence on \( n \) for simplicity, let \( (x,z) \in Z^\mathbb{R}_+ \) denote a state-action pair. We define

\[
q_i(x,z) := x_i - \sum_{j \in \mathcal{J}} z_{ij}, \quad i \in \mathcal{I},
\]

\[
y^*_j(z) := N^n_j - \sum_{i \in \mathcal{J}} z_{ij}, \quad j \in \mathcal{J},
\]

(2.6)

and the action space \( Z^n(x) \) by

\[ Z^n(x) := \{z \in \mathbb{Z}^\mathbb{R}_+ : q_i(x,z) \wedge y^*_j(z) = 0, \quad q_i(x,z) \geq 0, \quad y^*_j(z) \geq 0 \quad \forall (i,j) \in \mathcal{E} \}. \]

Note that this space consists of work-conserving actions only.

2.2. Diffusion scaling. With \( \xi^* \in \mathbb{R}^G_+ \) the solution of the (LP), we define \( \tilde{z}^n \in \mathbb{R}^\mathbb{R}_+ \) and \( \tilde{x}^n \in \mathbb{R}^m \) by

\[
\tilde{z}^n_{ij} := \frac{1}{n} \xi^*_{ij} N^n_j, \quad \tilde{x}^n_i := \sum_{j \in \mathcal{J}} \tilde{z}^n_{ij},
\]

(2.7)

and

\[
\hat{X}^n_i(t) := \frac{1}{\sqrt{n}} (X^n_i(t) - n \tilde{x}^n_i), \quad \hat{Z}^n_{ij}(t) := \frac{1}{\sqrt{n}} (Z^n_{ij}(t) - n \tilde{z}^n_{ij}),
\]

\[ \hat{Q}^n_i(t) := \frac{1}{\sqrt{n}} Q^n_i(t), \quad \hat{Y}^n_j(t) := \frac{1}{\sqrt{n}} Y^n_j(t). \]

(2.8)

These obey the balance equations

\[
\dot{\hat{X}}^n_i(t) = \hat{Q}^n_i(t) + \sum_{j \in \mathcal{J}(i)} \hat{Z}^n_{ij}(t) \quad \forall i \in \mathcal{I},
\]

\[
\dot{\hat{Y}}^n_j(t) + \sum_{i \in \mathcal{I}(j)} \hat{Z}^n_{ij}(t) = 0 \quad \forall j \in \mathcal{J}.
\]

(2.9)
We introduce suitable notation in the diffusion scale as follows (see [15, Definition 2.3]).

**Definition 2.1.** For \( x \in \mathbb{Z}^m_+ \) and \( z \in \mathbb{Z}^n(x) \), we define

\[
\hat{x}_n := \frac{x - n \bar{x}^n}{\sqrt{n}} , \quad \hat{z}_n := \frac{z - n \bar{z}^n}{\sqrt{n}} ,
\]

and let \( S^n \) denote the state space in the diffusion scale, that is,

\[
S^n := \{ \hat{x} \in \mathbb{R}^m : \sqrt{n} \hat{x} + n \bar{z}^n \in \mathbb{Z}^m_+ \} .
\]

It is clear that the diffusion-scaled work-conserving action space \( \hat{Z}^n(\hat{x}) \) takes the form

\[
\hat{Z}^n(\hat{x}) := \{ \hat{z} : \sqrt{n} \hat{z} + n \hat{z}^n \in \mathbb{Z}^n(\sqrt{n} \hat{x} + n \bar{z}^n) \} , \quad \hat{x} \in S^n .
\]

Recall that a scheduling policy is called stationary Markov if \( Z^n(t) = z(X^n(t)) \) for some function \( z : \mathbb{Z}^m_+ \to \mathbb{Z}^G_+ \), in which case we identify the policy with the function \( z \). Under a stationary Markov policy, \( X^n \) is Markov with controlled generator

\[
\mathcal{L}^n_z f(x) := \sum_{i \in I} \left( \lambda^n_i (f(x + e_i) - f(x)) + \sum_{j \in J(i)} \mu^n_{ij} \hat{z}_{ij} (f(x - e_i) - f(x)) \right)
\]

for \( f \in C(\mathbb{R}^m) \) and \( x \in \mathbb{Z}^m_+ \). Let \( \ell^n = (\ell^n_1, \ldots, \ell^n_m)^T \) be defined by

\[
\ell^n_i := \frac{1}{\sqrt{n}} \left( \lambda^n_i - \sum_{j \in J(i)} \mu^n_{ij} \hat{z}_{ij} N^n_j \right) .
\]

By (2.7), the assumptions on the parameters in (2.2) and (2.3), we have

\[
\ell^n_i \longrightarrow_{n \to \infty} \ell_i := \hat{\lambda}_i - \sum_{j \in J(i)} \hat{\mu}_{ij} \hat{z}_{ij}^* ,
\]

with \( z^* \) as in (2.5). Let \( \ell := (\ell_1, \ldots, \ell_m)^T \).

We drop the dependence on \( n \) in the diffusion-scaled variables in order to simplify the notation. A work-conserving stationary Markov policy \( z \), that is a map \( z : \mathbb{Z}^m_+ \to \mathbb{Z}^G_+ \) such that \( z(x) \in \mathbb{Z}^n(x) \) for all \( x \in \mathbb{Z}^m_+ \), gives rise to a policy \( \hat{z} : S^n \to \mathbb{R}^G_+ \) with \( \hat{z}(\hat{x}) \in \hat{Z}^n(\hat{x}) \) for all \( \hat{x} \in S^n \), via (2.10) (and vice-versa). Using (2.8), (2.11), and (2.12) and rearranging terms, the controlled generator of the corresponding diffusion-scaled process can be written as

\[
\hat{\mathcal{L}}^n_z f(\hat{x}) = \sum_{i \in I} \frac{\lambda^n_i}{n} \hat{d} f(\hat{x}; \frac{1}{\sqrt{n}} e_i) + \hat{d} f(\hat{x}; -\frac{1}{\sqrt{n}} e_i)
\]

\[
- \sum_{i \in I} b^n_i(\hat{x}, \hat{z}) \frac{\hat{d} f(\hat{x}; -\frac{1}{\sqrt{n}} e_i)}{n^{-1/2}} , \quad \hat{x} \in S^n , \quad \hat{z} \in \hat{Z}^n(\hat{x}) ,
\]

where \( \hat{d} f \) is given by

\[
\hat{d} f(x; y) := f(x + y) - f(x) , \quad x, y \in \mathbb{R}^m ,
\]

and the ‘drift’ \( b^n = (b^n_1, \ldots, b^n_m)^T \) is given by

\[
b^n_i(\hat{x}, \hat{z}) := \ell^n_i - \sum_{j \in J(i)} \mu^n_{ij} \hat{z}_{ij} , \quad \hat{z} \in \hat{Z}^n(\hat{x}) , \quad i \in I .
\]

**Definition 2.2.** For with \( \hat{x} \in S^n \) and \( \hat{z} \in \hat{Z}^n(\hat{x}) \), we define (compare with (2.9))

\[
\hat{q}^n_i(\hat{x}, \hat{z}) := \hat{x}_i - \sum_{j \in J(i)} \hat{z}_{ij} , \quad i \in I , \quad \hat{y}^n_j(\hat{z}) := - \sum_{i \in I(j)} \hat{z}_{ij} , \quad j \in J ,
\]

and

\[
\hat{\theta}^n(\hat{x}, \hat{z}) := \langle e, \hat{q}^n(\hat{x}, \hat{z}) \rangle \land \langle e, \hat{y}^n(\hat{z}) \rangle .
\]
By (2.15), we have
\[ \langle e, \hat{q}^n(\hat{x}, \hat{z}) \rangle = \hat{\vartheta}^n(\hat{x}, \hat{z}) + \langle e, \hat{\varphi}^n(\hat{z}) \rangle = \hat{\vartheta}^n(\hat{x}, \hat{z}) + \langle e, \hat{x} \rangle^+ \] for all \( \hat{x} \in S^n \) and \( \hat{z} \in \hat{Z}^n(\hat{x}) \). Define the \((m - 1)\) and \((J - 1)\) simplexes
\[ \Delta_c := \{ u \in \mathbb{R}^m : u \geq 0, \langle e, u \rangle = 1 \}, \quad \text{and} \quad \Delta_s := \{ u \in \mathbb{R}^J : u \geq 0, \langle e, u \rangle = 1 \}, \]
and let \( \Delta := \Delta_c \times \Delta_s \). By (2.16), there exists \( u = (u^c, u^s) \in \Delta \) such that
\[ \hat{q}^n(\hat{x}, \hat{z}) = (\hat{\vartheta}^n(\hat{x}, \hat{z}) + \langle e, \hat{x} \rangle^+) u^c, \quad \text{and} \quad \hat{y}^n(\hat{z}) = (\hat{\vartheta}^n(\hat{x}, \hat{z}) + \langle e, \hat{x} \rangle^-) u^s. \] Let
\[ \mathcal{D} := \left\{ (\alpha, \beta) \in \mathbb{R}^m \times \mathbb{R}^J : \sum_{i=1}^m \alpha_i = \sum_{j=1}^J \beta_j \right\}. \]
As shown in [22, Proposition A.2], there exists a unique linear map \( \Phi = [\Phi_{ij}] : \mathcal{D} \rightarrow \mathbb{R}^G \) solving
\[ \sum_{j \in J(i)} \Phi_{ij}(\alpha, \beta) = \alpha_i \quad \forall i \in \mathcal{I}, \quad \text{and} \quad \sum_{i \in I(j)} \Phi_{ij}(\alpha, \beta) = \beta_j \quad \forall j \in \mathcal{J}. \] Since \((\hat{x} - \hat{q}^n(\hat{x}, \hat{z}), -\hat{y}^n(\hat{z})) \in \mathcal{D} \) by (2.6) and (2.15), using the linearity of the map \( \Phi \) and (2.17) and (2.18), it follows that
\[ \hat{z} = \Phi(\hat{x} - \hat{q}^n(\hat{x}, \hat{z}), -\hat{y}^n(\hat{z})) = \Phi(\hat{x} - \langle e, \hat{x} \rangle^+ u^c, -\langle e, \hat{x} \rangle^- u^s) - \hat{\vartheta}^n(\hat{x}, \hat{z})(B^c u^c + B^s u^s). \] We describe an important property of the linear map \( \Phi \) which we need later. Consider the matrices \( B^n_1 \in \mathbb{R}^{m \times J} \) and \( B^n_2 \in \mathbb{R}^{J \times J} \) defined by
\[ \sum_{j \in J(i)} \mu_{ij} \Phi_{ij}(\alpha, \beta) = (B^n_1 \alpha + B^n_2 \beta)_i, \quad \forall i \in \mathcal{I}, \forall (\alpha, \beta) \in \mathcal{D}. \] It is clear that for \( B^n_1 \) to be a nonsingular matrix the basis used in the representation of the linear map \( \Phi \) should be of the form \( \mathcal{D} = (\alpha, (\beta)_{-j}) \), \( j \in \mathcal{J} \), where \((\beta)_{-j} = \{ \beta_\ell, \ell \neq j \} \). Since \( \Phi \) has a unique representation in terms of such a basis, and since \( B^n_i, i = 1, 2 \), are determined uniquely from \( \Phi \) by (2.20), abusing the terminology, we refer to such an \( \mathcal{D} \) as a basis for \( B^n_i, i = 1, 2 \). In [13, Lemma 4.3], the following property is asserted. For more details, we refer the reader to [13, Section 4.1].

**Property A.** Given any \( i \in \mathcal{I} \), there exists an ordering of \( \{ \alpha_i, i \in \mathcal{I} \} \) with \( \alpha_i \) the last element, and \( j \in \mathcal{J} \), such that the matrix \( B^n_1 \) is lower diagonal with positive diagonal elements with respect to this ordered basis \( (\alpha, (\beta)_{-j}) \).

In view of (2.19) and (2.20), for any \( \hat{z} \in \hat{Z}^n(\hat{x}) \) with \( \hat{x} \in S^n \), there exists \( u = u(\hat{x}, \hat{z}) \in \Delta \) such that the drift \( b^n \) in (2.14) takes the form
\[ b^n(\hat{x}, \hat{z}) = \ell^n - B^n_1(\hat{x} - \langle e, \hat{x} \rangle^+ u^c) + B^n_2 u^c (\langle e, \hat{x} \rangle^- + \hat{\vartheta}^n(\hat{x}, \hat{z})) (B^n_1 u^c + B^n_2 u^s). \] 2.3. **Joint and system-wide work conservation.** We start with the following definition.

**Definition 2.3.** We say that an action \( \hat{z} \in \hat{Z}^n(\hat{x}) \) is jointly work conserving (JWC), if \( \hat{\vartheta}^n(\hat{x}, \hat{z}) = 0 \). Let
\[ \hat{\vartheta}^n_*(\hat{x}) := \min_{\hat{z} \in \hat{Z}^n(\hat{x})} \hat{\vartheta}^n(\hat{x}, \hat{z}), \quad \hat{x} \in S^n, \]
and
\[ \hat{Z}^n(\hat{x}) := \{ \hat{z} \in \hat{Z}^n(\hat{x}) : \hat{\vartheta}^n(\hat{x}, \hat{z}) = \hat{\vartheta}^n_*(\hat{x}) \}, \quad \hat{x} \in S^n. \] We refer to \( \hat{Z}^n(\hat{x}) \) as the system-wide work conserving (SWC) action set at \( \hat{x} \). A stationary Markov scheduling policy \( \hat{z} \) is called SWC if \( \hat{z}(\hat{x}) \in \hat{Z}^n(\hat{x}) \) for all \( \hat{x} \in S^n \). We let \( \hat{S}^n \) denote the class of
all such policies. Since \( z \) and \( \hat{z} \) are related by (2.10), abusing this terminology, we also refer to a Markov policy \( z : \mathbb{Z}_+^m \to \mathbb{Z}_+^\theta \) as SWC, if it satisfies

\[
\frac{z(x) - n\bar{z}^n}{\sqrt{n}} \in \mathbb{Z}^n\left(\frac{x - n\bar{x}^n}{\sqrt{n}}\right),
\]

and we write \( z \in \hat{\mathbb{X}}^n \).

We recall [23, Lemma 3] which states that there exists \( M_0 > 0 \) such that the collection of sets \( \hat{\mathbb{X}}^n \) defined by

\[
\hat{\mathbb{X}}^n := \{ \hat{x} \in \mathbb{R}^n : \|\hat{x}\|_1 \leq M_0 \sqrt{n} \},
\]

has the following property. If \( \hat{x} \in \hat{\mathbb{X}}^n \), then for any pair \((\hat{q}, \hat{y})\) such that \( \sqrt{n}\hat{q} \in \mathbb{Z}_+^m \), \( \sqrt{n}\hat{y} \in \mathbb{Z}_+^l \), and satisfying

\[
\langle e, \hat{q} \rangle \land \langle e, \hat{y} \rangle = 0, \quad \langle e, \hat{x} - \hat{q} \rangle = \langle e, -\hat{y} \rangle, \quad \text{and} \quad \hat{y}_j \leq N^m_j, \quad j \in J,
\]

it holds that \( \Phi(\hat{x} - \hat{q}, -\hat{y}) \in \hat{\mathbb{Z}}^n(\hat{x}) \). It follows from this lemma and Definition 2.3 that if \( \hat{x} \in \hat{\mathbb{X}}^n \), then the actions in \( \hat{\mathbb{Z}}^n(\hat{x}) \) are JWC.

2.4. The diffusion limit. The diffusion approximation or diffusion limit of the queueing model described above is an \( m \)-dimensional stochastic differential equation (SDE) of the form

\[
dX_t = b(X_t, U_t) dt + \sigma(X_t) dW_t, \quad X_0 = x \in \mathbb{R}^m. \tag{2.23}
\]

Here, \( \{W_t\}_{t \geq 0} \) is a standard \( m \)-dimensional Brownian motion, and the control \( U_t \) takes values in the set \( \Delta = \Delta_e \times \Delta_s \), The drift \( b \) can be derived as follows. Recall \( \mathbb{R}^\theta \) in (2.1). For \( u = (u^e, u^s) \in \Delta \), let \( \tilde{\Phi}[u] : \mathbb{R}^m \to \mathbb{R}^\theta \) be defined by

\[
\tilde{\Phi}[u](x) := \Phi(x - (e \cdot x)^+ u^e, -(e \cdot x)^- u^s),
\]

with \( \Phi \) as defined in (2.18). Then the drift \( b \) takes the form

\[
b_l(x, u) = \ell_i - \sum_{j \in J(i)} \mu_{ij} \tilde{\Phi}[u](x). \tag{2.25}
\]

By [13, Lemma 4.3], we also know that (2.25) can be expressed as

\[
b_l(x, u) = \ell - B_1(x - \langle e, x \rangle^+ u^e) + B_2 u^s \langle e, x \rangle^-, \tag{2.26}
\]

where \( B_1 \in \mathbb{R}^{m \times m} \) is a lower diagonal matrix with positive diagonal elements, and \( B_2 \in \mathbb{R}^{m \times J} \). Of course \( B_i \) in (2.26) and \( B_i^\theta \) in (2.21), \( i = 1, 2 \), have the same functional form with respect to \( \{\mu_{ij}\} \) and \( \{\mu_{ij}^\theta\} \), respectively.

The diffusion matrix \( \sigma \in \mathbb{R}^{m \times m} \) is constant, and

\[
a := a\sigma^T = \text{diag}(2\lambda_1, \ldots, 2\lambda_m).
\]

In addition, for \( f \in C^2(\mathbb{R}^m) \), we define

\[
\mathcal{L}_u f(x) := \frac{1}{2} \text{trace}(a\nabla^2 f(x)) + \langle b(x, u), \nabla f(x)\rangle, \tag{2.27}
\]

with \( \nabla^2 f \) denoting the Hessian of \( f \).

Remark 2.1. Let \( \hat{z} \in \hat{\mathbb{Z}}^n(\hat{x}) \). Then, \( \hat{\phi}^n(\hat{x}, \hat{z}) = \hat{\phi}^n(\hat{x}) = 0 \) for all \( \hat{x} \in \hat{\mathbb{X}}^n \), and in view of (2.21), for any \( \hat{x} \in \hat{\mathbb{X}}^n \), there exists \( u = u(\hat{x}, \hat{z}) \in \Delta \) such that

\[
b^n(\hat{x}, \hat{z}) = \ell^n - B_1^n(x - \langle e, x \rangle^+ u^e) + B_2 u^s \langle e, x \rangle^-.
\]

Since \( \ell^n \to \ell \) and \( B_i^n \to B_i \) as \( n \to \infty \), and by also comparing \( \hat{\mathcal{L}}_p^\theta \) in (2.13) to \( \mathcal{L}_u \) in (2.27), it is clear that Foster–Lyapunov equations for \( \mathcal{L}_u \) carry over to analogous equations for \( \hat{\mathcal{L}}^\theta \) on \( \hat{\mathbb{X}}^n \) uniformly over SWC policies. However, even though \( \hat{\mathbb{X}}^n \) fills the whole space as \( n \to \infty \), \( b \) and \( b^n \) differ in
functional form when $\hat{\theta}^n(\hat{x}, \hat{z}) \neq 0$, and this makes the stability analysis of multiclass multi-pool networks much harder than the ‘V’ network studied in [12].

3. Spare capacity, positive recurrence and transience

In this section we first introduce the concept of spare capacity for the $n$th system (prelimit) and the diffusion limit, and then show that if the space capacity is negative (zero), then both the prelimit and limiting diffusion are transient (not positive recurrent) under any stationary Markov control, in the absence of abandonment.

Recall the drifts characterizing the $n$th system (prelimit) and the diffusion limit given in (2.21) and (2.26), respectively, in Subsection 2.4. We define the spare capacity (or the safety staffing) for the $n$th system (prelimit) and the diffusion limit by

$$q_n := -\langle e, (B_1^n)^{-1} \ell \rangle, \quad \text{and} \quad q := -\langle e, B_1^{-1} \ell \rangle,$$

respectively. Note, of course, that $q_n \to q$ as $n \to \infty$ by (2.2). For the classes of models studied in Section 4 the converse statement to Theorem 3.1 below is established, namely, that if $q > 0$, then the limiting diffusion is uniformly exponentially ergodic. So $q$ indeed characterizes the spare capacity of the network for these models.

The results which follow in this section apply to any parallel server network under the hypotheses in Subsection 2.1. We start with the following important lemma.

**Lemma 3.1.** The drift in (2.26) satisfies $\inf_{u' \in \Delta} (1 + \langle e, B_1^{-1}B_2u^s \rangle) > 0$.

For the proof of this lemma, we need two properties which we state next.

**Fact 1.** The quantity $\langle e, B_1^{-1}B_2u^s \rangle$ is invariant under permutations of the states $x_i$. Indeed, with any permutation matrix $S$ the matrices get transformed as $B_1 = SB_1S^T$, and $B_2 = SB_2$ (since $S^T = S^{-1}$). So $\langle e, B_1^{-1}B_2u^s \rangle = \langle e, B_1^{-1}B_2u^s \rangle$. The quantity is clearly also invariant with respect to permutations of the pools. In this regard, with a permutation matrix $T$, we have $(B_2T)T^T(Tu^s) = B_2u^s$.

In order to motivate the second claim, we note that the matrices $B_1$ and $B_2$ depend on the basis used. For example, for the ‘N’ model, we obtain

$$\Phi = \begin{pmatrix} \beta_1 & \alpha_1 - \beta_1 \\ 0 & \alpha_2 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} \alpha_1 + \alpha_2 - \beta_2 & \beta_2 - \alpha_2 \\ 0 & \alpha_2 \end{pmatrix},$$

if we use as basis $(\alpha_1, \alpha_2, \beta_1)$, or $(\alpha_1, \alpha_2, \beta_2)$, respectively. For the second representation, the spare capacity gets scaled by a factor $\frac{\mu_2}{\mu_1}$ relative to the first, and $1 + \langle e, B_1^{-1}B_2u^s \rangle = u_1^s + \frac{\mu_2}{\mu_1}u_2^s$, whereas for the first representation we have $1 + \langle e, B_1^{-1}B_2u^s \rangle = \frac{\mu_1}{\mu_1}u_1^s + u_2^s$, so they are consistent with Proposition 3.1 which appears later in this section. With this in mind, we state and prove the following.

**Fact 2.** The positivity of $1 + \langle e, B_1^{-1}B_2u^s \rangle$ does not depend on the choice of the basis $D \in \{(\alpha, (\beta)_{-j}), j \in J\}$ of the representation.

**Proof of Fact 2.** Consider a representation whose basis contains $\beta_j$ for some arbitrary $j \in J$, and suppose that $1 + \langle e, B_1^{-1}B_2u^s \rangle > 0$. Eliminating $\beta_j$ leads of course to a unique representation of $\Phi$ with respect to $(\alpha, (\beta)_{-j})$, and therefore also to a uniquely defined $B_1$ and $B_2$ by (2.20). Let $v = v_j$ denote the $j$th column of $B_2$ and $e := (1, \ldots, 1)^T \in \mathbb{R}^J$ (as usual, $e^T$ is the corresponding vector in $\mathbb{R}^m$). It is rather easy to see that this operation results in matrices $\tilde{B}_1 = B_1 + ve^T$, and $\tilde{B}_2 = B_2 - ve^T$. Indeed, we start with the $\Phi$ matrix and to every element where $\beta_j$ appears we add $\alpha_1 + \ldots + \alpha_m - \beta_1 - \ldots - \beta_j \equiv 0$. This does not alter $\Phi$ as a map, but eliminates $\beta_j$. After multiplying the elements of $\Phi$ with $\{\mu_{ij}\}$, we recover the matrices $\tilde{B}_i$, $i = 1, 2$, as claimed.
Applying the Sherman–Morrison formula for the inverse of $\tilde{B}_1$ we have

\[
(B_1 + ve^T)^{-1} = B_1^{-1} - \frac{B_1^{-1}ve^TB_1^{-1}}{1 + e^TB_1^{-1}v}.
\]

Note that $1 + e^TB_1^{-1}v > 0$ by hypothesis. Multiply the expression above by $e^T$ on the left and by $\tilde{B}_2u^s$ on the right, to obtain

\[
\langle e, \tilde{B}_1^{-1}\tilde{B}_2u^s \rangle = e^T B_1^{-1} (B_2 - ve^T) u^s - \frac{e^T B_1^{-1}ve^TB_1^{-1}(B_2 - ve^T)u^s}{1 + e^TB_1^{-1}v}.
\]

\[
= \frac{1}{1 + e^TB_1^{-1}v} e^T B_1^{-1} (B_2 - ve^T) u^s
\]

\[
= \frac{1}{1 + e^TB_1^{-1}v} \left( e^T B_1^{-1}B_2u^s - e^TB_1^{-1}ve^T u^s \right)
\]

\[
= \frac{1 + e^TB_1^{-1}B_2u^s}{1 + e^TB_1^{-1}v} - 1 > -1,
\]

since $1 + e^TB_1^{-1}B_2u^s > 0$ by hypothesis (note we used $\tilde{e}^T u^s = 1$ in the last equality). The proof of Fact 2 is complete. \(\square\)

**Proof of Lemma 3.1.** We use induction. Suppose that the lemma holds for any network with $m$ classes and $J$ pools.

**Step 1 (adding a new class).** By Fact 1, we may assume that it is added with an edge to pool $J$. Let $\Phi \in \mathbb{R}^{m+1} \times \mathbb{R}^J$ be the representation of the linear map with respect to the basis $(\alpha, (\beta)_{-J})$, $B_i$, $i = 1, 2$, be the corresponding matrices, and $G$ denote the graph of the network. We have $\langle e, B_1^{-1}B_2u^s \rangle > -1$ for all $u^s \in \Delta_s$ by Fact 2.

Then the new $\tilde{\Phi} \in \mathbb{R}^{m+1} \times \mathbb{R}^J$ satisfies

\[
\tilde{\Phi}_{ij} = \begin{cases} 
\Phi_{ij}, & \text{if } (i,j) \in G, \\
0, & \text{if } i = m+1, j \neq J, \\
\alpha_{m+1}, & \text{if } i = m+1, j = J.
\end{cases}
\]

To see that this is the case, note that since $\beta_J$ is missing from the basis, all the rows of $\Phi$ have to add to $\alpha_i$’s and all the columns, except for the last one, to $\beta_j$’s. However the last column of $\Phi$ has to add to $\sum_i \alpha_i - \sum_{j=1, \ldots, J-1} \beta_j$, since this is the unique representation of $\beta_J$ in the basis chosen. With the addition of $\alpha_{m+1}$, this adds to the correct value for the map $\Phi$.

Hence, the new matrices are given (in block form)

\[
\tilde{B}_1 = \begin{pmatrix} B_1 & 0 \\ 0 & \mu_{m+1,J} \end{pmatrix}, \quad \text{and} \quad \tilde{B}_2 = \begin{pmatrix} B_2 \\ 0 \ldots 0 \end{pmatrix}.
\]

Therefore, $\langle e, \tilde{B}_1^{-1}\tilde{B}_2u^s \rangle = \langle e, B_1^{-1}B_2u^s \rangle > -1$.

**Step 2 (adding a new pool).** Suppose a new pool $\beta_{J+1}$ is added with an edge to class $i$. By Property A, we may assume that $B_1$ is constructed canonically, so it is lower diagonal, with the last element of the ordered collection of $\alpha_i$ chosen as $\alpha_s$, that is, $i \equiv m$. By Fact 2, we have $\langle e, B_1^{-1}B_2u^s \rangle > -1$. Let $\beta_j$ be the variable missing from the basis.
Then the new $\Phi$ is given by (using $i \equiv m$)

\[
\tilde{\Phi}_{ij} = \begin{cases} 
\Phi_{ij}, & \text{if } i \neq m, \ j \leq J, \\
0, & \text{if } i \neq m, \ j = J + 1, \\
0, & \text{if } i = m, \ j \notin \{j, J + 1\}, \\
\Phi_{mj} - \beta_{J+1} & \text{if } i = m, \ j = j, \\
\beta_{J+1} & \text{if } i = m, \ j = J + 1.
\end{cases}
\]

Using the verification argument employed in Step 1, it is clear that the above is indeed the correct representation of $\Phi$.

So now we obtain that $B_1$ stays the same, and

\[
B_2 = \begin{cases} 
(B_2)_{ij}, & \text{if } j \leq J, \\
0, & \text{if } i \neq m, \ j = J + 1, \\
\mu_{mJ+1} - \mu_{mj} & \text{if } i = m, \ j = J + 1.
\end{cases}
\]

Then

\[
\langle e, \tilde{B}_1^{-1}\tilde{B}_2u^s \rangle = \langle e, B_1^{-1}B_2\tilde{u}^s \rangle + \left(\frac{\mu_{mJ+1}}{\mu_{mj}} - 1\right)u_{J+1}^s,
\]

where $\tilde{u}^s = (u_1^s, \ldots, u_J^s)^\top$. Now, normalize $\tilde{u}^s$ as $\bar{u}^s := \frac{\tilde{u}^s}{\|\tilde{u}^s\|_1}$, and write

\[
\langle e, \tilde{B}_1^{-1}\tilde{B}_2u^s \rangle = \langle e, B_1^{-1}B_2\bar{u}^s \rangle \|\bar{u}^s\|_1 + \left(\frac{\mu_{mJ+1}}{\mu_{mj}} - 1\right)u_{J+1}^s
\]

\[
> -\|\bar{u}^s\|_1 - u_{J+1}^s = -1,
\]

thus completing the induction argument. $\square$

We first show that $\varrho < 0$ implies transience for the diffusion limit.

**Theorem 3.1.** Suppose that $\varrho := -\langle e, B_1^{-1}\ell \rangle < 0$. Then the process $\{X_t\}_{t \geq 0}$ in (2.23) is transient under any stationary Markov control. In addition, if $\varrho = 0$, then $\{X(t)\}_{t \geq 0}$ cannot be positive recurrent.

**Proof.** Let $H(x) := \tanh(\beta\langle e, B_1^{-1}x \rangle)$, with $\beta > 0$. Then

\[
\text{trace}(a\nabla^2 H(x)) = \beta^2 \tanh''(\beta\langle e, B_1^{-1}x \rangle)\|\sigma^TB_1^{-1}e\|^2.
\]

We have

\[
\mathcal{L}_u H(x) = \frac{1}{2} \text{trace}(a\nabla^2 H(x)) + \langle b(x, u), \nabla H(x) \rangle
\]

\[
= -\beta^2 \tan^2(\beta\langle e, B_1^{-1}x \rangle)\left(\frac{\sigma^TB_1^{-1}e}{\cosh^2(\beta\langle e, B_1^{-1}x \rangle)} \right) \|\sigma^TB_1^{-1}e\|^2
\]

\[
+ \frac{\beta}{\cosh^2(\beta\langle e, B_1^{-1}x \rangle)} \left(\langle e, B_1^{-1}\ell \rangle + \langle e, x \rangle - \left(1 + \langle e, B_1^{-1}B_2u^s \rangle \right) \right). \tag{3.2}
\]

Thus, for $0 < \beta < \langle e, B_1^{-1}\ell \rangle\|\sigma^TB_1^{-1}e\|^2$, we obtain $\mathcal{L}_u H(x) > 0$ by Lemma 3.1. Therefore, $\{H(X_t)\}_{t \geq 0}$ is a bounded submartingale, so it converges almost surely. Since $X$ is irreducible, it can be either recurrent or transient. If it is recurrent, then $H$ should be constant a.e. in $\mathbb{R}^2$, which is not the case. Thus $X$ is transient.

We now turn to the case where $\varrho = 0$. Suppose that the process $\{X(t)\}_{t \geq 0}$ (under some stationary Markov control) has an invariant probability measure $\pi(dx)$. It is well known that $\pi$ must have
Since \( r \) is non-negative. Thus using dominated and monotone convergence, we can take limits in (3.3) to express the first and second order incremental quotients, together with (2.21) which implies that \( \beta \) vanishes as \( h \rightarrow 0 \). However, since \( h_1(x) \) is bounded and \( h_2(x) \) is non-negative. Thus using dominated and monotone convergence, we can take limits in (3.3) as \( r \rightarrow \infty \) for the terms on the right side to obtain

\[
\int_{\mathbb{R}^m} H(x)\pi(dx) - H(x) = t \sum_{i=1,2} \int_{\mathbb{R}^m} h_i(x)\pi(dx), \quad t \geq 0.
\]

Since \( H(x) \) is bounded, we can divide both sides by \( t \) and \( \beta \) and take the limit as \( t \rightarrow \infty \) to get

\[
\int_{\mathbb{R}^m} \beta^{-1}h_1(x)\pi(dx) + \int_{\mathbb{R}^m} \beta^{-1}h_2(x)\pi(dx) = 0. \tag{3.4}
\]

Since \( \beta^{-1}h_1(x) \) tends to 0 uniformly in \( x \) as \( \beta \downarrow 0 \), the first term on the left hand side of (3.4) vanishes as \( \beta \downarrow 0 \). However, since \( \beta^{-1}h_2(x) \) is bounded away from 0 on the open set \( \{x \in \mathbb{R}^m : \langle e, x \rangle^{-1} > 1\} \), this contradicts the fact that \( \pi(dx) \) has full support. \( \square \)

**Corollary 3.1.** Suppose that \( q_n < 0 \). Then the state process \( \{X^n(t)\}_{t \geq 0} \) of the \( n \)th system is transient under any stationary Markov scheduling policy.

**Proof.** The proof mimics that of Theorem 3.1. We apply the function \( H \) in that proof to the process \( \hat{X}^n \) in (2.13), and use the identity

\[
H\left(x \pm \frac{1}{\sqrt{n}}e_i\right) - H(x) = \frac{1}{n} \int_0^1 (1 - t) \partial_{x_i} H\left(x \pm \frac{t}{\sqrt{n}}e_i\right) dt, \tag{3.5}
\]

to express the first and second order incremental quotients, together with (2.21) which implies that

\[
\langle b^n(\hat{x}, \hat{z}), \nabla H(\hat{x}) \rangle = \frac{\beta}{\cosh^2(\beta\langle e, B_1^{-1}x \rangle)} \left( \langle e, (B_1^n)^{-1} \ell \rangle \right.
\]

\[
+ \left( \hat{\theta}^n(\hat{x}, \hat{z}) + \langle e, x \rangle^{-1} \right) \left( 1 + \langle e, (B_1^n)^{-1}B_2^n u^s \rangle \right) \right). 
\]

The rest follows exactly as in the proof of Theorem 3.1. \( \square \)

### 3.1. Spare capacity and average idleness

It is shown in [12, 20] that if the diffusion limit of the \( V \) network with no abandonment has an invariant distribution \( \pi \) under some stationary Markov control, then \( q \) represents the ‘average idleness’ of the system, that is, \( q = \int_{\mathbb{R}^m} \langle e, x \rangle^{-1} \pi(dx) \). In calculating this average for multi-pool networks, idle servers are not weighted equally across different pools and the term \( \langle e, B_1^{-1}B_2 u^s(x) \rangle \) appears in the expression. It is important to note that only the control on the idleness allocations among server pools \( u^s \) appears in the identity, and the control component \( u^c \) does not.

**Proposition 3.1.** Suppose that the limiting diffusion is positive recurrent under some stationary Markov control \( u \in U_{sm} \), and let \( \pi \) denote its invariant probability measure. Then

\[
q = \int_{\mathbb{R}^m} \left( 1 + \langle e, B_1^{-1}B_2 u^s(x) \rangle \right) \langle e, x \rangle^{-1} \pi(dx). \tag{3.6}
\]
Proof. The proof is similar to that of [20, Corollary 5.1], but more involved for the multiclass multi-pool networks. We first recall some definitions and notations. Let \( \chi_r(t) \), \( \tilde{\chi}_r(t) \), \( r > 1 \) be smooth, concave and convex functions, respectively, defined by

\[
\chi_r(t) = \begin{cases} 
  t, & t \leq r - 1, \\
  r - \frac{1}{2}, & t \geq r,
\end{cases}
\]

and

\[
\tilde{\chi}_r(t) = \begin{cases} 
  t, & t \geq 1 - r, \\
  \frac{1}{2} - r, & t \leq -r.
\end{cases}
\]

Let \( g_r(x) = \tilde{\chi}_r(\langle e, B_1^{-1}x \rangle) \), and \( f_r(x) = \chi_r(g_r(x)) \). A straightforward calculation shows that

\[
\langle b(x, u), \nabla f_r(x) \rangle = h_1(x) + h_2(x),
\]

\[
\frac{1}{2} \text{trace}(a(x) \nabla^2 f_r(x)) = h_3(x) + h_4(x),
\]

where

\[
\begin{align*}
  h_1(x) & := -g\chi'_r(f(x)) \tilde{x}_r'(\langle e, B_1^{-1}x \rangle), \\
  h_2(x) & := \left[ 1 + \langle e, B_1^{-1}B_2u^\star \rangle \right] \chi'_r(f(x)) \tilde{x}_r'(\langle e, B_1^{-1}x \rangle)\langle e, x \rangle^-, \\
  h_3(x) & := \frac{1}{2} \chi''_r(f(x)) \left( \tilde{x}_r'(\langle e, B_1^{-1}x \rangle) \right)^2 |\sigma^T B_1^{-1}e|^2, \\
  h_4(x) & := \frac{1}{2} \chi'_r(f(x)) \tilde{x}'_r(\langle e, B_1^{-1}x \rangle) |\sigma^T B_1^{-1}e|^2.
\end{align*}
\]

We note that \( g_r(x) \) is positive and bounded below away from 0, and \( f_r(x) \) is smooth, bounded, and has bounded derivatives. Also note that \( h_i, i = 1, 2, 3 \), are bounded, and \( h_2 \) is nonnegative. Therefore, if \( \{X(t)\}_{t \geq 0} \) is positive recurrent with an invariant probability measure \( \pi(dx) \), a straightforward application of Itô’s formula shows that \( \pi(\mathcal{L}_u f_r) = 0 \). Therefore, we obtain

\[
\pi(-h_1) = \pi(h_2) + \pi(h_3) + \pi(h_4). \tag{3.7}
\]

By the definition of \( \chi_r \) and \( \tilde{\chi}_r \), it is straightforward to verify that

\[
\lim_{r \to \infty} \pi(h_3) = \lim_{r \to \infty} \pi(h_4) = 0. \tag{3.8}
\]

In addition, using dominated convergence theorem

\[
\begin{align*}
  &\lim_{r \to \infty} \pi(h_1) = -\varrho, \\
  &\lim_{r \to \infty} \pi(h_2) = \int_{\mathbb{R}^m} \left( 1 + \langle e, B_1^{-1}B_2u^\star \rangle \right) \langle e, x \rangle^- \pi(dx).
\end{align*} \tag{3.9}
\]

Combining (3.7)–(3.9), we obtain (3.6). \( \square \)

4. Uniform exponential ergodicity of the diffusion limit

We start by reviewing the notion of uniform exponential ergodicity for a controlled diffusion. We do this under fairly general hypotheses. Consider a controlled diffusion process \( X = \{X_t, t \geq 0\} \) which takes values in the \( m \)-dimensional Euclidean space \( \mathbb{R}^m \), and is governed by the Itô equation

\[
dX_t = b(X_t, v(X_t)) \, dt + \sigma(X_t) \, dW_t. \tag{4.1}
\]

All random processes in (4.1) live in a complete probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \). The process \( W \) is a \( m \)-dimensional standard Wiener process independent of the initial condition \( X_0 \). The function \( v \) maps \( \mathbb{R}^m \) to a compact, metrizable set \( U \) and is Borel measurable. The collection of such functions comprising of the set of stationary Markov controls is denoted by \( \mathcal{M}_{sm} \).

The parameters of the equation (4.1) satisfy the following:
(1) **Local Lipschitz continuity:** The functions \( b: \mathbb{R}^m \times U \rightarrow \mathbb{R}^m \) and \( \sigma: \mathbb{R}^m \rightarrow \mathbb{R}^{m \times m} \) are continuous, and satisfy
\[
|b(x, u) - b(y, u)| + \|\sigma(x) - \sigma(y)\| \leq C_R |x - y| \quad \forall x, y \in B_R, \forall u \in U.
\]
for some constant \( C_R > 0 \) depending on \( R > 0 \).

(2) **Affine growth condition:** For some \( C_0 > 0 \), we have
\[
\sup_{u \in U} \langle b(x, u), x \rangle^+ + \|\sigma(x)\|^2 \leq C_0(1 + |x|^2) \quad \forall x \in \mathbb{R}^m.
\]

(3) **Nondegeneracy:** For each \( R > 0 \), it holds that
\[
\sum_{i,j=1}^m a^{ij}(x)\xi_i \xi_j \geq C_R^{-1} |\xi|^2 \quad \forall x \in B_R,
\]
and for all \( \xi = (\xi_1, \ldots, \xi_m)^T \in \mathbb{R}^m \), where \( a = \frac{1}{2} \sigma \sigma^T \).

It is well known that, under hypotheses (1)–(2), (4.1) has a unique strong solution which is also a strong Markov process for any \( v \in \mathcal{U}_{sm} \) [34]. We let \( E^v_x \) denote the expectation operator on the canonical space of the process controlled by \( v \), with initial condition \( X_0 = x \). Let \( \tau(A) \) denote the first exit time from the set \( A \in \mathbb{R}^m \).

We say that the process \( \{X_t\}_{t \geq 0} \) is uniformly exponentially ergodic if for some ball \( B_0 \) there exist \( \delta_0 > 0 \) and \( x_0 \in \overline{B_0} \) such that \( \sup_{v \in \mathcal{U}_{sm}} E^{v}_{x_0} [e^{\delta_0 \tau(B^c_0)}] < \infty \).

We let \( \widehat{A} \) denote the operator
\[
\widehat{A} \phi(x) := \frac{1}{2} \text{trace} \left( a(x) \nabla^2 \phi(x) \right) + \max_{u \in U} \langle b(x, u), \nabla \phi(x) \rangle, \quad x \in \mathbb{R}^m,
\]
for \( \phi \in C^2(\mathbb{R}^m) \). For a locally bounded, Borel measurable function \( f: \mathbb{R}^m \rightarrow \mathbb{R} \), which is bounded from below in \( \mathbb{R}^m \), i.e., \( \inf_{\mathbb{R}^m} f > -\infty \), we define the generalized principal eigenvalue of \( \widehat{A} + f \) by
\[
\Lambda(f) := \inf \left\{ \lambda \in \mathbb{R} : \exists \varphi \in W^{2,m}_\text{loc}(\mathbb{R}^m), \varphi > 0, \widehat{A} \varphi + (f - \lambda) \varphi \leq 0 \ \text{a.e. in} \ \mathbb{R}^m \right\}.
\]

We have the following equivalent characterizations of uniform exponential ergodicity. This is a straightforward extension of [20, Theorem 3.1] for controlled diffusions, and is stated without proof. Recall that a map \( f: \mathbb{R}^m \rightarrow \mathbb{R} \) is called coercive, or inf-compact, if \( \inf_{B^c} f \rightarrow \infty \) as \( r \rightarrow \infty \).

**Theorem 4.1.** The following are equivalent.

(a) For some ball \( B_0 \) there exists \( \delta_0 > 0 \) and \( x_0 \in \overline{B_0} \) such that \( \sup_{v \in \mathcal{U}_{sm}} E^{v}_{x_0} [e^{\delta_0 \tau(B^c_0)}] < \infty \).

(b) For every ball \( B \) there exists \( \delta > 0 \) such that \( \sup_{v \in \mathcal{U}_{sm}} E^{v}_{x} [e^{\delta \tau(B^c)}] < \infty \) for all \( x \in B \).

(c) For every ball \( B \), there exists a coercive function \( \mathcal{V} \in W^{2,p}_\text{loc}(\mathbb{R}^m) \), \( p > d \), with \( \inf_{\mathbb{R}^m} \mathcal{V} \geq 1 \), and positive constants \( \kappa_0 \) and \( \delta \) such that
\[
\widehat{A} \mathcal{V}(x) \leq \kappa_0 (1_{B}(x) - \delta \mathcal{V}(x)) \quad \forall x \in \mathbb{R}^m.
\]

(d) **Equation (4.1)** is recurrent, and \( \Lambda(1_{B^c}) < 1 \) for every ball \( B \).

**Remark 4.1.** Recall (1.2). Let \( P^n_t(x, dy) \) denote the transition probability of \( \{X_t\}_{t \geq 0} \) in (4.1) under a control \( v \in \mathcal{U}_{sm} \). It is well known that (4.2) implies that there exist constants \( \gamma \) and \( C_\gamma \) which do not depend on the control \( v \) chosen, such that
\[
\left\| P^n_t(x, \cdot) - \pi_v(\cdot) \right\|_\mathcal{P} \leq C_\gamma \mathcal{V}(x) e^{-\gamma t}, \quad \forall x \in \mathbb{R}^m, \forall t \geq 0,
\]
where \( \pi_v \) denotes the invariant probability measure of \( \{X_t\}_{t \geq 0} \) under the control \( v \).
A class of intrinsic Lyapunov functions for the queueing network model. As seen in Theorem 4.1, uniform exponential ergodicity is equivalent to the Foster–Lyapunov inequality in (4.2). In establishing this property for the diffusion limit of stochastic networks, a proper choice of a Lyapunov function is of tantamount importance. We first describe an intrinsic class of such functions.

We fix a convex function \( \psi \in C^2(\mathbb{R}) \) with the property that \( \psi(t) \) is constant for \( t \leq -1 \), and \( \psi(t) = t \) for \( t \geq 0 \). This is defined by

\[
\psi(t) := \begin{cases} 
-\frac{1}{2}, & t \leq -1, \\
(t + 1)^3 - \frac{1}{2}(t + 1)^4 - \frac{1}{2} & t \in [-1, 0], \\
t & t \geq 0.
\end{cases}
\]

For \( \varepsilon > 0 \) we define

\[
\psi_\varepsilon(t) := \psi(\varepsilon t),
\]

Thus \( \psi_\varepsilon(t) = \varepsilon t \) for \( t > 0 \). A simple calculation also shows that \( \psi''_\varepsilon(t) \leq \frac{3}{2}\varepsilon^2 \).

**Definition 4.1.** Suppose that \( B_1 = \text{diag}(\tilde{\mu}_1, \ldots, \tilde{\mu}_m) \). Using the function \( \psi_\varepsilon \) introduced above, we let

\[
\Psi(x) := \sum_{i \in \mathcal{I}} \frac{\psi_\varepsilon(x_i)}{\tilde{\mu}_i},
\]

with

\[
\varepsilon := \frac{\vartheta}{3m} \left( \sum_{i \in \mathcal{I}} \frac{\lambda_i(3\tilde{\mu}_i + 2)}{\tilde{\mu}_i^2} \right)^{-1}.
\]

We also define

\[
V_1(x) := \exp(\theta \Psi(-x)), \quad V_2(x) := \exp(\Psi(x)), \quad \text{and} \quad V(x) := V_1(x) + V_2(x),
\]

with \( \theta \) a positive constant.

As a result of fixing the value of \( \varepsilon \) in (4.5), \( \Psi \) depends only on the parameter \( \theta \). This simplifies the statements of the results in the rest of the paper.

We review some useful properties of the function \( \psi_\varepsilon \). First, for the choice of \( \psi \) above, we have

\[
\psi''(-1/2) = 1/2, \quad \text{from which we obtain}
\]

\[
\sum_{i \in \mathcal{I}} \psi'_\varepsilon(x_i)x_i \geq \varepsilon\|x^+\|_1 - \frac{m}{2}, \quad \text{and} \quad -\sum_{i \in \mathcal{I}} \psi'_\varepsilon(-x_i)x_i \geq \varepsilon\|x^-\|_1 - \frac{m}{2}.
\]

where \( \mathcal{I} := \{1, 2, \ldots, m\} \). Note also that

\[
-\sum_{i \in \mathcal{I}} \psi'_\varepsilon(-x_i)x_i \leq \varepsilon\langle e, x \rangle \leq \sum_{i \in \mathcal{I}} \psi'_\varepsilon(x_i)x_i.
\]

The function \( V \) in **Definition 4.1**, scaled by the parameter \( \theta \) which are suppressed in the notation, is our choice of a Lyapunov function when \( B_1 \) is a diagonal matrix. The reader will notice the similarities in **Definition 4.1** and [12, Definition 4.2]. However the function \( V \) used in this paper is the sum of the two exponential functions \( V_1 \) and \( V_2 \), whereas their product is used in [12, Lemma 2.1]. As will be seen later, in the case of multiclass multi-pool networks, the analysis is considerably more complex.

The class of functions is particularly useful for the multiclass multi-pool networks resulting in a diagonal matrix \( B_1 \) in the drift. This includes the following families of networks:
(i) **Networks with a dominant server pool.** In such networks, only one server pool can serve multiple classes, and all classes can be served by this server pool. The diameter of their graphs is equal to three or four. This family of networks includes the standard ‘N’, ‘M’ networks and generalized ‘N’ and ‘M’ networks, as shown in Figure 1. In fact, following the leaf-elimination algorithm in [13], it is easy to show that in order for the matrix $B_1$ in the drift of the diffusion limit to be diagonal for any set of values of the parameters, it is necessary and sufficient that the network has at most one nonleaf server node.

(ii) **Networks with class-dependent service rates.**

In the following subsections, we establish the uniform exponential ergodicity of these networks. To help with the exposition, we study the ‘N’ network in detail and then proceed to the general networks with a dominant server pool, and networks with class-dependent service rates.

In establishing the desired drift inequalities, we often partition the space appropriately, and focus on the subsets of the partition. The following cones appear quite often in the analysis.

**Definition 4.2.** For $\delta \in [0, 1]$, we define the cones

$$K_0^+ := \{ x \in \mathbb{R}^n : \langle e, x \rangle \geq \delta \| x \|_1 \},$$

$$K_0^- := \{ x \in \mathbb{R}^n : \langle e, x \rangle \leq -\delta \| x \|_1 \}.$$  

It is clear that $K_0^+$ ($K_0^-$) corresponds to the nonnegative (nonpositive) canonical half-space, and $K_0^+$ ($K_0^-$) is the nonnegative (nonpositive) closed orthant.

The following identities are very useful.

$$\langle e, x^+ \rangle = \frac{1 + \delta}{2} \| x \|_1, \quad \langle e, x^- \rangle = \frac{1 - \delta}{2} \| x \|_1 \quad \text{for } x \in \partial K_0^\pm, \, \delta \in [0, 1]. \quad (4.9)$$

In addition, it is straightforward to show that

$$\sum_{i \in I} \psi_e(x_i) \leq -\sum_{i \in I} \psi_e(-x_i) \quad \text{if } x \in K_0^-.$$  \quad (4.10)

Also, the following inequality is true in general for any $I' \subset I$.

$$\sum_{i \in I'} \psi_e(x_i) x_i - \varepsilon \sum_{i \in I'} x_i = \sum_{x_i < 0, i \in I'} (\psi_e'(x_i) - \varepsilon) x_i \geq 0. \quad (4.11)$$

**Remark 4.2.** There is an important scaling of the drift which we employ. Note that if we let $\zeta = \frac{\varrho}{m} e + B_1^{-1} \ell$, with $\varrho$ as in (3.1), then a mere translation of the origin of the form $\tilde{X}_t = X_t + \zeta$ results in a diffusion of with the same drift as (2.25), except that the vector $\ell$ gets replaced by $\ell = -\frac{\varrho}{m} B_1 e$. Therefore, we may assume without any loss of generality that the drift in (2.26) takes the form

$$b(x, u) = -\frac{\varrho}{m} B_1 e - B_1 (x - \langle e, x \rangle^+ u^c) + B_2 u^s \langle e, x \rangle^-. \quad (4.12)$$

4.2. **The case of the ‘N’ network.** Here, $m = 2$, and the matrices $B_i$, $i = 1, 2$, in (2.26) are given by $B_1 = \text{diag}(\mu_{12}, \mu_{22})$, and $B_2 = \text{diag}(\mu_{11} - \mu_{12}, 0)$. Thus, using (4.12), the drift $b: \mathbb{R}^2 \to \mathbb{R}^2$ for the ‘N’ network is given by

$$b(x, u) = -\frac{\varrho}{m} \begin{pmatrix} \mu_{12} & 0 \\ 0 & \mu_{22} \end{pmatrix} x - \langle e, x \rangle^+ u^c + \begin{pmatrix} (\mu_{11} - \mu_{12}) u_1^s \\ 0 \end{pmatrix} \langle e, x \rangle^-.$$  \quad (4.13)

Note that for the ‘N’ network, we have $\Psi(x) = \frac{\psi_e(x_1)}{\mu_{12}} + \frac{\psi_e(x_2)}{\mu_{22}}$ by (4.4). Recall the definition of the cube $K_\varepsilon$ in (1.1). We have the following lemma.

**Lemma 4.1.** Consider an ‘N’ network satisfying $\varrho > 0$. Let $\delta \in (0, 1)$, $\theta \geq 2 \eta \vee \eta^{-1}$, with $\eta := \frac{\mu_{11}}{\mu_{12}}$, and $V(x)$ be as in Definition 4.1. Then, there exist positive constants $c_0$ and $c_1$ such that

$$\langle b(x, u), \nabla V(x) \rangle \leq c_0 - \varepsilon c_1 \| x \|_1 V(x) \quad \forall (x, u) \in (K_\delta^+)^c \times \Delta \quad (4.14a)$$
\[ \langle b(x, u), \nabla V_2(x) \rangle \leq -\frac{\rho \varepsilon}{2} V_2(x) \quad \forall (x, u) \in \mathcal{K}_0^+ \times \Delta. \] (4.14b)

**Proof.** To simplify the notation we define
\[ F_i(x, u) := \frac{1}{V_i(x)} \langle b(x, u), \nabla V_i(x) \rangle, \quad i = 1, 2. \] (4.15)
We use (4.13), and apply (4.7) and the inequalities \( \frac{\theta}{2} \sum_{i \in I} \psi'_\varepsilon \leq \rho \varepsilon, \) and
\[ \psi'_\varepsilon(-x_1)(\eta - 1) u_i^\eta \langle e, x \rangle \leq -\varepsilon(1 - \eta)^+ \langle e, x \rangle \leq \varepsilon(1 - \eta)^+ \| x^- \|_1 \quad \forall (x, u) \in \mathcal{K}_0^- \times \Delta. \]

\[ \frac{1}{\theta} F_1(x, u) = \frac{\theta}{2} \sum_{i \in I} \psi'_\varepsilon(-x_i) + \sum_{i \in I} \psi'_\varepsilon(-x_i)x_i - \sum_{i \in I} \psi'_\varepsilon(-x_i)(\eta - 1) u_i^\eta \langle e, x \rangle \]
\[ \leq 1 + \rho \varepsilon - \varepsilon\| x^- \|_1 + \varepsilon(1 - \eta)^+ \| x^- \|_1 \leq (1 + \rho \varepsilon) - \varepsilon(\eta \wedge 1) | x^- |_1 \leq (1 + \rho \varepsilon) - \varepsilon(\eta \wedge 1)| x |_1 \quad \forall (x, u) \in \mathcal{K}_0^- \times \Delta. \] (4.16)

Similarly, we have
\[ F_2(x, u) = -\frac{\theta}{2} \sum_{i \in I} \psi'_\varepsilon(x_i) - \sum_{i \in I} \psi'_\varepsilon(x_i)x_i + \sum_{i \in I} \psi'_\varepsilon(x_i)(\eta - 1) u_i^\eta \langle e, x \rangle \]
\[ \leq \varepsilon(1 + (\eta - 1)^+) \| x \|_1 \leq \varepsilon(\eta \vee 1) \| x \|_1 \quad \forall (x, u) \in \mathcal{K}_0^- \times \Delta. \] (4.17)

Note that, due to (4.10) and the choice of \( \theta, \) we have \( V_1 \geq V_2^2 \) on \( \mathcal{K}_0^- \). Thus, since \( V_1 \) has exponential growth in \( \| x \|_1 \) on \( \mathcal{K}_0^- \), combining (4.16) and (4.17) and choosing an appropriate cube \( \mathcal{K}_r \), we obtain
\[ \langle b(x, u), \nabla V(x) \rangle \leq \left( \theta(1 + \rho \varepsilon) - \frac{\varepsilon}{4}(\eta \wedge 1) | x |_1 \right) V(x) \quad \forall (x, u) \in (\mathcal{K}_0^- \setminus \mathcal{K}_r) \times \Delta. \] (4.18)

We continue with estimates on \( \mathcal{K}_0^+ \). A straightforward calculation shows that
\[ \frac{1}{\theta} F_1(x, u) = \frac{\theta}{2} \sum_{i \in I} \psi'_\varepsilon(-x_i) + \sum_{i \in I} \psi'_\varepsilon(-x_i)x_i - \sum_{i \in I} \psi'_\varepsilon(-x_i)(\eta - 1) u_i^\eta \langle e, x \rangle \]
\[ \leq 1 + \rho \varepsilon - \varepsilon\| x^- \|_1 + \varepsilon(1 - \eta)^+ \| x^- \|_1 \quad \forall (x, u) \in \mathcal{K}_0^+ \times \Delta. \] (4.19)

Again using (4.7), we have
\[ \frac{1}{\theta} F_2(x, u) \leq 1 + \rho \varepsilon - \varepsilon\| x^- \|_1 \quad \forall (x, u) \in \mathcal{K}_0^+ \times \Delta. \] (4.20)

We break the estimate of \( F_2 \) in two parts. First, for any \( \delta \in (0, 1), \) using (4.7), we obtain
\[ F_2(x, u) \leq -\frac{\rho \varepsilon}{2} + 1 - \varepsilon\| x^+ \|_1 + \varepsilon \langle e, x \rangle \]
\[ \leq -\frac{\rho \varepsilon}{2} + 1 - \varepsilon\| x^- \|_1 \quad \forall (x, u) \in (\mathcal{K}_0^+ \setminus \mathcal{K}_0^+) \times \Delta. \] (4.21)
Combining (4.19) and (4.20), we get
\[ \langle b(x, u), \nabla V(x) \rangle \leq \left( \theta(1 + \rho \varepsilon) - \frac{\varepsilon(1 - \delta)}{2} \| x \|_1 \right) V(x) \quad \forall (x, u) \in (\mathcal{K}_0^+ \setminus \mathcal{K}_0^+) \times \Delta, \] (4.21)
Next, using (4.7), we have
\[ F_2(x, u) \leq -\frac{\varrho \varepsilon}{2} \quad \forall (x, u) \in K^+_\delta \times \Delta, \]
and this completes the proof. \(\square\)

Recall the definition of the operator \(L_u\) in (2.27).

**Theorem 4.2.** Consider an ‘N’ network satisfying \(\varrho > 0\). Then for any \(\theta \geq 2\eta \sqrt{\eta}^{-1}\), with \(\eta := \frac{n_1}{n_2}\), the function \(V\) in (4.6) satisfies the Foster–Lyapunov equation
\[ L_u V(x) \leq C_0 - \frac{\varrho \varepsilon}{4} V(x) \quad \forall (x, u) \in \mathbb{R}^m \times \Delta, \quad (4.22) \]
for some positive constant \(C_0\).

**Proof.** A straightforward calculation, using the fact that \(\psi^\varepsilon(t) \leq \frac{3}{2} \varepsilon^2\), shows that
\[ \frac{1}{2} \text{trace}(a \nabla^2 V_2(x)) \leq \varepsilon^2 \sum_{i \in I} \lambda_i (3\mu_i + 2) \frac{2\mu_i^2}{V_2(x)} \quad \forall x \in \mathbb{R}^m. \]
Therefore, the choice of \(\varepsilon\) in (4.5) implies that \(\frac{1}{2} \text{trace}(a \nabla^2 V_2(x)) \leq \frac{\varrho \varepsilon}{8} V_2\) for all \(x \in \mathbb{R}^2\), and thus
\[ L_u V_2(x) \leq -\frac{3\varrho \varepsilon}{8} V_2(x) \quad \forall (x, u) \in K^+_\delta \times \Delta \quad (4.23) \]
by (4.14b). Since \(|x^+| \geq \frac{1+\delta}{\delta} |x^-|\) for all \(x \in K^+_{\delta}\), we may select \(\delta\) sufficiently close to 1 such that \(V_2 \geq V_1^2\) on \(K^+_{\delta} \cap K^-_{\delta}\) for some \(r > 0\). Since \(V_2\) has exponential growth in \(|x|_1\) on \(K^+_{\delta}\) and \(L_u V_1(x) \leq C(1 + |x|_1) V_1(x)\) on \(K^+_{\delta} \times \Delta\), it then follows that (4.22) holds on \(K^+_{\delta} \times \Delta\) by (4.23). It is also clear that (4.22) also holds on \((K^+_{\delta})^c \times \Delta\) by (4.14a). This completes the proof. \(\square\)

**Remark 4.3.** Using Proposition 3.1, it is easy to verify that (3.6) in the case of the ‘N’ network reduces to
\[ \varrho = \int_{\mathbb{R}^m} \frac{\mu_{11} u^{11}(x) + \mu_{12} u^{12}(x)}{\mu_{12}} (e, x)^{-1} \pi(dx). \]
Note that this identify holds under all stationary Markov controls. Naturally, \(\pi\) depends on the control.

4.3. Networks with a dominant server pool. This network has one nonleaf server node, which, without loss of generality, we label as \(j = 1\). As in Section 2, the customer nodes are denoted by \(I = \{1, 2, \ldots, m\}\), and the server nodes by \(J = \{1, 2, \ldots, J\}\). Recall that \(J(i)\) is the collection of server nodes connected to customer \(i\). Owing to the tree structure of the network, server 1 \(\in J(i)\) for all \(i \in \{1, 2, \ldots, m\}\). Let \(J_1(i) := J(i) \setminus \{1\}\) for all \(i \in I\). Recall the form of the drift in (2.25).

Using (2.18), it is simple to show that the matrix \(\hat{\Phi}_{ij}[u]\) for this network is given by
\[ \hat{\Phi}_{ij}[u](x) = \begin{cases} x_i - \langle e, x \rangle u_i^c + \sum_{j \in J_1(i)} \langle e, x \rangle u_j^c & \text{for } j = 1, \\ -\langle e, x \rangle u_j^c & \text{for } j \in J_1(i), \\ 0 & \text{otherwise.} \end{cases} \quad (4.24) \]
Using (4.24), the drift takes the following simple form:
\[ b_i(x, u) = \ell_i - \mu_{11} \langle x - u_i^c (e, x)^+ \rangle + \sum_{j \in J_1(i)} \mu_{11} (\eta_{ij} - 1) u_j^c \langle e, x \rangle^-, \quad i \in I, \quad (4.25) \]
with \(\eta_{ij} := \frac{\mu_{ij}}{\mu_{i1}}\) for \(j \in J_1(i)\) and \(i \in I\). Note that \(B_1 = \text{diag}(\mu_{11}, \ldots, \mu_{m1})\), and so \(\ell = -\frac{\varrho}{\mu} B_1 e\), where \(\varrho\) is given by (3.1). We define
\[ \bar{\eta} := \max_{i \in I} \max_{j \in J_1(i)} \eta_{ij}, \quad \text{and} \quad \underline{\eta} := \min_{i \in I} \min_{j \in J_1(i)} \eta_{ij}. \]
for any $\delta$ and $\varrho$ that it is easy to verify using Definition 4.2 and (4.9) that

$\eta$ as in Definition 4.2. Consider a network with a dominant server pool, such that (4.29) is equivalent to (4.26).

□

$\eta$ as in Definition 4.1. Recall the definitions in (4.15). A straightforward calculation using (4.25) shows that

$$\frac{1}{\theta} F_1(x,u) = \frac{\varrho}{m} \sum_{i \in I} \psi'_i(-x_i) + \sum_{i \in I} \psi'_i(-x_i)(x_i - u_i(e,x)^+) - \langle e,x \rangle \sum_{i \in I} \sum_{j \in J_1(i)} \psi'_i(-x_i)(\eta_{ij} - 1)u^s_j,$$

$$F_2(x,u) = -\frac{\varrho}{m} \sum_{i \in I} \psi'_i(x_i) - \sum_{i \in I} \sum_{j \in J_1(i)} \psi'_i(x_i)(x_i - u_i(e,x)^+) + \langle e,x \rangle \sum_{i \in I} \sum_{j \in J_1(i)} \psi'_i(x_i)(\eta_{ij} - 1)u^s_j.$$

Let $\eta := \min_{ij} \eta_{ij}$, and $\bar{\eta} := \max_{ij} \eta_{ij}$. Noting that

$$-\langle e,x \rangle \sum_{i \in I} \sum_{j \in J_1(i)} \psi'_i(x_i)(\eta_{ij} - 1)u^s_j \leq (1 - \eta^+)\langle e,x \rangle^-,$$

it is easy to verify using Definition 4.2 and (4.9) that

$$\frac{1}{\theta} F_1(x,u) \leq \frac{\varrho e + m}{2} - \frac{\varepsilon}{m} \|x^-\|_1 + \varepsilon(1 - \eta^+)\|x^-\|_1 \leq \frac{\varrho e + m}{2} - \frac{\varepsilon(1 - \delta)}{2}(\eta \land 1)\|x\|_1 \quad \forall (x,u) \in (K^+_\delta \setminus (K^+_0 \cup K^+_\delta')) \times \Delta.$$

(4.27)

Note that the drift equations on $K^+_0$ are similar to those of the ‘N’ model, with the only exception that $\frac{\varrho}{m}$ is replaced by $\frac{\varrho e}{m}$, and the sum ranges from $i = 1, \ldots, m$ instead of $i = 1, 2$. Hence, we obtain

$$F_2(x,u) \leq -\frac{\varrho e}{m} + \frac{m}{2} - \varepsilon\|x^+\|_1 + \varepsilon\langle e,x \rangle \leq -\frac{\varrho e}{m} + \frac{m}{2} - \varepsilon\|x^+\|_1 \quad \forall (x,u) \in (K^+_0 \cup K^+_\delta') \times \Delta,$$

(4.28)

and

$$F_2(x,u) \leq \frac{\varrho e}{m} - \varepsilon\langle e,x \rangle + \varepsilon\langle e,x \rangle \leq -\frac{\varrho e}{m} \quad \forall (x,u) \in K^+_0 \times \Delta,$$

(4.29)

for any $\delta \in (0,1)$.

The choice of $\theta$ implies that $V_1 \geq V_2^2$ on $K_0^-$. Thus (4.14a) holds by (4.27) and (4.28), while (4.29) is equivalent to (4.26).

□

Figure 1. Examples of multiclass multi-pool networks with a dominant pool (square–customer classes, circle–server pools, solid circle–the dominant server pool).
4.4. **Networks with class-dependent service rates.** We consider in this part arbitrary networks where the service rates are dictated by the customer type; namely \( \mu_{ij} = \mu_i \) for all \((i, j) \in \mathcal{E} \). Recall the definition in (2.24). Using (2.18) and (2.25), the drift of this network takes the form

\[
b_i(x, u) = \ell_i - \sum_{j \in J(i)} \mu_{ij} \hat{\Phi}_{ij}[u](x) = \ell_i - \mu_i (x_i - u_i^c(e, x)^+) , \quad \forall i \in \mathcal{I}.
\]

(4.30)

Note then that \( B_2 = 0 \).

**Remark 4.4.** Networks with only class-dependent service rates have a limiting diffusion with the same drift structure studied in [12], and that paper shows that when \( \rho > 0 \), then the diffusion (and the prelimit) is uniformly exponentially ergodic in the presence or absence of abandonment. However, the proof of uniform exponential ergodicity of the prelimit for models with class-dependent service rates does not seem to carry through with the Lyapunov function used in [12]. Thus for the sake of proving the result for the \( n \)th system in Section 5, we adopt here the Lyapunov function in (4.6).

**Lemma 4.3.** Consider a network satisfying \( \mu_{ij} = \mu_i \) for all \( i \in \mathcal{I} \), and \( \rho > 0 \). Let \( \delta \in (0, 1) \), and \( \theta \geq \theta_0 := 2 \frac{\mu_{\max}}{\mu_{\min}} \). Then the conclusions of Lemma 4.2 follow.

**Proof.** A simple calculation using (4.30) shows that

\[
\frac{1}{\theta} F_1(x, u) = \frac{\rho}{m} \sum_{i \in \mathcal{I}} \psi_i'(-x_i) + \sum_{i \in \mathcal{I}} \psi_i'(-x_i)(x_i - u_i^c(e, x)^+) ,
\]

\[
F_2(x, u) = -\frac{\rho}{m} \sum_{i \in \mathcal{I}} \psi_i'(x_i) - \sum_{i \in \mathcal{I}} \psi_i'(x_i)(x_i - u_i^c(e, x)^+). \tag{4.31}
\]

Using (4.31), we obtain

\[
\frac{1}{\theta} F_1(x, u) \leq \rho \epsilon + \frac{m}{2} - \frac{\epsilon}{2} \|x\|_1 \quad \forall (x, u) \in \mathcal{K}_0^- \times \Delta,
\]

Therefore, (4.14a) holds on \( \mathcal{K}_0^- \times \Delta \) by this inequality and the choice of \( \theta \).

On \( \mathcal{K}_0^+ \times \Delta \), the equations in (4.31) are identical to the corresponding ones for a network with a dominant server pool, for which the result has already been established in Lemma 4.2. This completes the proof.

4.5. **Main result.** We state the counterpart of Theorem 4.2.

**Theorem 4.3.** Consider a network with a dominant server pool, or with class-dependent service rates, and assume \( \rho > 0 \). Then, for any \( \theta \geq \theta_0 := 2 \frac{\mu_{\max}}{\mu_{\min}} \), the function \( V \) in (4.6) satisfies the Foster–Lyapunov equation

\[
\mathcal{L}_u V(x) \leq C_0 - \frac{\rho \epsilon}{3m} V(x) \quad \forall (x, u) \in \mathbb{R}^m \times \Delta,
\]

for some positive constant \( C_0 \).

**Proof.** This follows exactly as in the proof of Theorem 4.2 using Lemmas 4.2 and 4.3. \( \square \)

**Remark 4.5.** Using Proposition 3.1, it is easy to verify that (3.6) in the case of a network with a dominant server pool

\[
\rho = \int_{\mathbb{R}^m} \left[ 1 + \sum_{i \in \mathcal{I}} \sum_{j \in J(i)} \left( \frac{\mu_j}{\mu_i} - 1 \right) u_j^c \right] \langle e, x \rangle^- \pi(dx),
\]

while in the case of a network with class-dependent service rates, it takes the form

\[
\rho = \int_{\mathbb{R}^m} \langle e, x \rangle^- \pi(dx).
\]
5. Uniform exponential ergodicity of the $n$th system

In this section we show that if $\varrho_n > 0$ then the prelimit of a network with a dominant server pool, or with class-dependent service rates, is uniformly exponentially ergodic and the invariant distributions have exponential tails.

Recall that $\{\tilde{\mu}_i, i \in \mathcal{I}\}$ are the elements of the diagonal matrix $B^n_1$ in (2.21). Throughout this section $V$ denotes the function in (4.6), with $\varepsilon$ given by

$$
\varepsilon = \varepsilon_n := \frac{\varrho_n}{3m} \left( \sum_{i \in \mathcal{I}} \frac{1}{n} \frac{\lambda^n_i (3\tilde{\mu}^n_i + 2)}{n} \right)^{-1} \exp \left( -\frac{1}{\sqrt{n}} \sum_{i \in \mathcal{I}} \frac{1}{n} \tilde{\mu}^n_i \right). \tag{5.1}
$$

Recall the definition of the operator $\hat{L}_0^n$ in (2.13), and the definitions of $S^n$ and $\bar{Z}^n(\hat{x})$ in Definitions 2.1 and 2.3. We start with the following simple assertion.

**Lemma 5.1.** Let $V$ be the function in (4.6) with $\varepsilon$ as in (5.1), and $\theta$ fixed at some value. Suppose that for any $\delta \in (0,1)$ there exist positive constants $c_0$ and $c_1$ such that the drift $b^n$ in (2.14) satisfies

$$
\langle b^n(\hat{x}, \hat{\varepsilon}), \nabla V(\hat{x}) \rangle \leq c_0 - c_1 \varepsilon_1 \|x\|_1 V(\hat{x}) \quad \forall \hat{x} \in S^n \setminus K^+_\delta, \quad \forall \hat{\varepsilon} \in \bar{Z}^n(\hat{x}),
$$

$$
\langle b^n(\hat{x}, \hat{\varepsilon}), \nabla V_2(\hat{x}) \rangle \leq -\frac{\varrho_n \varepsilon_n}{2m} V_2(\hat{x}) \quad \forall \hat{x} \in S^n \cap K^+_\delta, \quad \forall \hat{\varepsilon} \in \bar{Z}^n(\hat{x}). \tag{5.2}
$$

Then, there exists a constant $\tilde{C}_0$ such that

$$
\hat{L}_0^n V(\hat{x}) \leq \tilde{C}_0 - \frac{\varrho_n \varepsilon_n}{4m} V(\hat{x}) \quad \forall \hat{x} \in S^n, \quad \forall \hat{\varepsilon} \in \bar{Z}^n(\hat{x}). \tag{5.3}
$$

**Proof.** A simple calculation shows that

$$
\int_0^1 (1-t) \partial_{x,x} V_2(\hat{x} \pm \frac{t}{\sqrt{n}} e_i) \, dt \leq \frac{\varepsilon_n^2}{2} \left( \sum_{i \in \mathcal{I}} \frac{3\tilde{\mu}^n_i + 1}{(2\tilde{\mu}^n_i)^2} \right) \exp \left( \frac{1}{\sqrt{n}} \sum_{i \in \mathcal{I}} \frac{1}{n} \tilde{\mu}^n_i \right) V_2(\hat{x}).
$$

Thus, using (3.5) to express the first and second order incremental quotients in (2.13), we obtain

$$
\hat{L}_0^n V_2(\hat{x}) \leq \frac{\varrho_n \varepsilon_n}{4m} V_2(\hat{x}) + \langle b^n(\hat{x}, \hat{\varepsilon}), \nabla V_2(\hat{x}) \rangle.
$$

The rest follows as in the proof of Theorem 4.2 by selecting $\delta$ sufficiently close to 1. \hfill \Box

**Remark 5.1.** Recall (2.21). In direct analogy to Remark 4.2, if we let $\zeta^n = \frac{\varrho_n}{m} e + (B^n_1)^{-1} \ell^n$, with $\varrho_n$ as in (3.1), then a mere translation of the origin of the form $\tilde{X}^n = \bar{X}^n + \zeta^n$ results in a diffusion of with the same drift as (2.21), except that the vector $\ell^n$ gets replaced by $\ell_0^n = -\frac{\varrho_n}{m} (B^n_1)^{\nu} e$. Therefore, we may assume without any loss of generality that the drift in (2.21) takes the form

$$
b^n(\hat{x}, \hat{\varepsilon}) = -\frac{\varrho_n}{m} B^n_1 e - B^n_1 (\hat{x} - \langle \varepsilon, \hat{x} \rangle^+ u^c) + B^n_2 u^s(\varepsilon, \hat{x})^+ + \dot{\theta}^n(\hat{x}, \hat{\varepsilon})(B^n_1 u^c + B^n_2 u^s). \tag{5.4}
$$

Note that this centering has the effect of translating the ‘equilibrium’ allocations $\tilde{z}^n_{ij}$ given in (2.7). Since this translation is of $O(n^{-1/2})$, it has no affect on the results for large $n$. However, in the interest of providing precise estimates we calculate then new values of $\tilde{z}^n_{ij}$. Note that $\langle \varepsilon, \zeta^n \rangle = 0$, and recall the map $\Phi$ in (2.18). Let $\tilde{z}^n_{ij} = \Phi(\zeta^n_i, 0)$. Then, the centering of $\hat{x}$ that results in (5.4) is given by (compare with (2.7))

$$
\tilde{z}^n_{ij} = \frac{1}{n} \xi^s_{ij} N^n_j + \frac{\tilde{z}^n_{ij}}{\sqrt{n}}, \quad \tilde{x}^n_i := \sum_{j \in \mathcal{J}} \tilde{z}^n_{ij}. \tag{5.5}
$$

Throughout this section the family $\{\tilde{z}^n_{ij} (i,j) \in \mathcal{E}\}$ is as given in (5.5).
**Definition 5.1.** Let $\varepsilon_n$ and $\bar{z}_{ij}^n$ as in (5.1) and (5.5), respectively. For a network with a dominant server pool as in Subsection 4.3 define

$$n_0 := \max \left\{ n \in \mathbb{N} : \frac{1}{\sqrt{n}} \geq \varepsilon_n \min_{i \in I} \bar{z}_{i1}^n \right\},$$

while for network with class-dependent service rates, we let

$$n_0 := \max \left\{ n \in \mathbb{N} : \frac{1}{\sqrt{n}} \geq \frac{\varepsilon_n}{2m} \min_{i \sim j} \bar{z}_{ij}^n \right\}.$$

Since $\{\varepsilon_n\}$ and $\{\bar{z}_{i1}^n\}$ are bounded away from 0 by the convergence of the parameters in (2.2), the number $n_0$ is finite.

The next theorem is the main result for the uniform exponential ergodicity of the prelimit processes.

**Theorem 5.1.** Assume that $\varrho_n > 0$, and let $n_0$ be as in Definition 5.1. Then any network with a dominant server pool, or with class-dependent service rates, satisfies (5.3) for all $n > n_0$. In particular, due to the convergence of the parameters, $\hat{C}_0$ can be selected independent of $n$, and this implies that the prelimit dynamics are uniformly exponentially ergodic and the invariant distributions have exponential tails.

**Proof.** In Lemmas 5.4 and 5.5 in the section which follows, we establish (5.2) for these networks. Thus the proof of the theorem follows directly from Lemma 5.1. □

**Remark 5.2.** Since the process $\hat{X}^n$ is irreducible and aperiodic under any stationary Markov scheduling $\hat{z} \in \hat{Z}^n$ (see Definition 2.3), a convergence property completely analogous to (4.3) follows from (5.3). Namely, there exist positive constants $\gamma$ and $C_\gamma$ not depending on $n \geq 0$ or $\hat{z}$, such that if $P_t^{n,\hat{z}}$ and $\pi^n_{\hat{z}}$ denote the transition probability and the stationary distribution, respectively, of $\hat{X}^n(t)$ under a policy $\hat{z} \in \hat{Z}^n$, then we have

$$\left\| P_t^{n,\hat{z}}(\hat{x}, \cdot) - \pi^n_{\hat{z}}(\cdot) \right\|_V \leq C_\gamma V(\hat{x}) e^{-\gamma t}, \quad \forall \hat{x} \in \hat{X}^n, \forall t \geq 0.$$

**5.1. Four technical lemmas.** In this section, we establish the technical results used in the proof of Theorem 5.1.

Let

$$\tilde{X}^n := \{ \hat{x} \in S^n : \hat{\vartheta}_n^\ast(\hat{x}) \neq 0 \}, \quad (5.6)$$

with $\hat{\vartheta}_n^\ast$ as in Definition 2.3. As seen in Subsection 2.3, the set $\tilde{X}^n$ in (2.22) is contained in $S^n \setminus \tilde{X}^n$. In establishing (5.2) on $S^n \setminus \tilde{X}^n$, the results in Section 4 pave the way, since the drift of of the controlled generator $\bar{L}^n_{\hat{z}}$ over the class of SWC stationary Markov policies $\hat{Z}^n$ (see (2.28)) has the same functional form as the drift of the diffusion in (2.27). So it remains to establish (5.2) in $\tilde{X}^n$. We start by establishing a bound for $\hat{\vartheta}_n^\ast$ in Definition 2.2 over all SWC policies.

As done earlier in the interest of notational economy, we suppress the dependence on $n$ in the diffusion scaled variables $\hat{z}^n$ and $\tilde{z}^n$ in (2.10).

**Lemma 5.2.** There exists a number $\chi_n^\ast < 1$ depending only on the parameters of the network such that

$$\hat{\vartheta}_n^\ast(\hat{x}, \tilde{z}) = \hat{\vartheta}_n^\ast(\hat{x}) \leq \chi_n^\ast(|\hat{x}^+| \wedge |\hat{x}^-|) \quad \forall \tilde{z} \in \tilde{Z}^n(\hat{x}).$$

In addition, due to the convergence of the parameters in (2.2), such a constant $\chi_n < 1$ may be selected which does not depend on $n$. 

Proof. Let $\hat{x} \in \tilde{X}^n$, $\hat{z} \in \tilde{Z}^n(\hat{x})$, and define

$$\tilde{J} := \left\{ j \in J : \sum_{i \in J(i)} \hat{z}_{ij} < 0 \right\}, \quad \text{and} \quad \tilde{I} := \left\{ i \in I : (i, j) \in \mathcal{E} \text{ for some } j \in \tilde{J} \right\},$$

and $\tilde{E} := \left\{ (i, j) \in \mathcal{E} : (i, j) \in \tilde{I} \times \tilde{J} \right\}$. Work conservation implies that $x_i^n = \sum_{j \in \tilde{J}(i)} z_{ij}^n$ for all $i \in \tilde{I}$. Let $\hat{i} \in I$ be such that $\hat{q}_i > 0$, and consider the unique path (since the graph of the network is a tree) connecting $\hat{i}$ and $\tilde{I}$, that is, a path $(\hat{i}, j_1), (\hat{i}, j_1, j_2), \ldots, (\hat{i}, j_k)$, with $j_\ell \in J \setminus \tilde{J}$ for $\ell = 1, \ldots, k$, $i_\ell \in I \setminus \tilde{I}$ for $\ell = 1, \ldots, k - 1$, and $i_k \in I$. We claim that $z_{i_kj_k}^n = 0$, or equivalently, that $z_{ij_k}^n = -\hat{z}_{ij}/\sqrt{n}$, with $z_{ij}^n$ as defined in (5.5). If not, then we can move a job of class $\hat{i}$ from pool $j_k$ to some pool in $\tilde{I}$, and proceeding along the path to place one additional job from class $\hat{i}$ into service, thus contradicting the hypothesis that $\hat{z} \in \tilde{Z}^n(\hat{x})$. Removing all such paths, we are left with a strict subnetwork (possibly disconnected) $\tilde{G}_o = (I_0 \cup J_0, \tilde{E}_o)$, with $I_0 \supset \tilde{I}, J_0 \supset \tilde{J}$, and $\mathcal{E}_o := \left\{ (i, j) \in \mathcal{E} : (i, j) \in I_0 \times J_0 \right\}$, such that

$$x_i^n = \sum_{j \in \tilde{J}(i) \cap \tilde{J}_o} z_{ij}^n, \quad \forall i \in I_0. \quad (5.7)$$

Let $\mathcal{E}_o' := (I_0 \times (J \setminus J_0)) \cap \mathcal{E}$. By (5.7) we have

$$\sum_{(i, j) \in \mathcal{E}_o'} z_{ij}^n = 0.$$

Thus we have

$$|\hat{x}^-| \geq -\sum_{i \in I_0} \hat{x}_i = \sqrt{n} \sum_{(i, j) \in \mathcal{E}_o} \hat{z}_{ij} - \sum_{(i, j) \in \tilde{E}_o} \hat{z}_{ij} \geq \sqrt{n} \sum_{(i, j) \in \mathcal{E}_o} \hat{z}_{ij}^n + \hat{\vartheta}_*(\hat{x}) \quad (5.8)$$

by the construction above. By (5.8), we obtain

$$\hat{\vartheta}_*(\hat{x}) \leq \frac{\sum_{(i, j) \in \tilde{E}_o} \hat{z}_{ij}^n - \sum_{(i, j) \in \mathcal{E}_o} \hat{z}_{ij}}{\sum_{i \in I_0} \hat{x}_i} \quad (5.9)$$

Similarly

$$|\hat{x}^+| \geq \sum_{i \in I \setminus I_0} \hat{x}_i = \sqrt{n} \sum_{(i, j) \in \mathcal{E}_o} \hat{z}_{ij}^n + \hat{\vartheta}_*(\hat{x}). \quad (5.10)$$

Using the bound $\hat{\vartheta}_*(\hat{x}) \leq \sqrt{n} \sum_{(i, j) \in \mathcal{E}_o} \hat{z}_{ij}^n$ we obtain from (5.10) that

$$\hat{\vartheta}_*(\hat{x}) \leq \frac{\sum_{i \in I \setminus I_0} \hat{x}_i - \sqrt{n} \sum_{(i, j) \in \mathcal{E}_o} \hat{z}_{ij}^n \land \sqrt{n} \sum_{(i, j) \in \tilde{E}_o} \hat{z}_{ij}}{\sum_{i \in I \setminus I_0} \hat{x}_i} \quad (5.11)$$

It should be now clear how to construct $\mathfrak{x}_o^n$. For any given subset $J' \subseteq J$, let

$$I_{\mathcal{J}'} := \bigcup_{j \in J'} I(j), \quad \mathcal{E}_{\mathcal{J}'} := \left\{ (i, j) \in \mathcal{E} : (i, j) \in I_{\mathcal{J}'} \times J' \right\}, \quad \text{and} \quad \mathfrak{X}_o^n(\mathcal{J}').
and \( \mathcal{E}_{\mathcal{J}'} := (\mathcal{I}_{\mathcal{J}'} \times (\mathcal{J} \setminus \mathcal{J}')) \cap \mathcal{E} \), and define
\[
\xi_0^n := \max_{\mathcal{J}' \subseteq \mathcal{J}} \sum_{(i,j) \in \mathcal{E}_{\mathcal{J}'}} z_{ij}^n + \sum_{(i,j) \in \mathcal{E}_{\mathcal{J}'}^c} z_{ij}^n.
\]
Then the result clearly follows from (5.9) and (5.11) since
\[
|\hat{x}^-| \geq -\sum_{i \in I_0} \hat{x}_i, \quad \text{and} \quad |\hat{x}^+| \geq \sum_{i \in I \setminus I_0} \hat{x}_i.
\]
This completes the proof. \(\square\)

**Remark 5.3.** It is easy to see that the estimates of the bounds on \( \hat{\varrho}^n \) can be improved. It is clear from (5.8) and (5.9), that
\[
\hat{\varrho}^n(\hat{x}) \leq -\xi_0^n \sum_{i \in I_0} \hat{x}_i \wedge \left( \sum_{i \in I \setminus I_0} \hat{x}_i \right) \quad \forall \hat{x} \in \widetilde{X}^n.
\]

Also, since there can be at most \( \sum_{j \in \mathcal{J}} N_j^n \) idle servers, it follows that \( \widetilde{\xi}_0^n \in (0, 1) \), such that
\[
-\sum_{i \in I_0} \hat{x}_i \geq \widetilde{\xi}_0^n \|\hat{x}^-\|_1 \quad \forall \hat{x} \in \widetilde{X}^n,
\]
where the constant \( \widetilde{\xi}_0^n \in (0, 1) \) can be selected as
\[
\widetilde{\xi}_0^n := \left( \sum_{j \in \mathcal{J}} N_j^n \right)^{-1} n \min_{i,j} z_{ij}^n.
\]

Due to the convergence of the parameters in (2.2), \( \widetilde{\xi}_0^n \) is bounded away from 0 uniformly in \( n \in \mathbb{N} \).

Even though the ‘N’ network is a special case of networks with a dominant server pool we first establish the result for this network in Lemma 5.3 in order to exhibit with simpler calculations how Lemma 5.2 is applied.

Throughout the proofs of Lemmas 5.3 to 5.5 we use the functions (compare with (4.15))
\[
F_i^n(\hat{x}, \hat{z}) := \frac{1}{V_i(x)} \langle b^n(\hat{x}, \hat{z}), \nabla V_1(\hat{x}) \rangle, \quad i = 1, 2,
\]
and let \( n_0 \) be as in Definition 5.1. Moreover, we suppress the dependence on \( n \) in the variables \( \hat{\varrho}^n \), \( \hat{\psi}^n \), and \( \hat{\varrho}^n \) in Definition 2.2, and from \( \varepsilon_n \) in (5.1).

**5.1.1. The diffusion-scale of the ‘N’ network.** We recall here [16]. In this work, Stolyar considers the ‘N’ network with \( O(\sqrt{n}) \) safety staffing in pool 2, under the priority discipline that class 2 has priority in pool 2 and class 1 prefers pool 1, and shows tightness of the invariant distributions. First note that for any stationary Markov scheduling policy \( z \), such that class 2 has priority in pool 2 we have \( z_{22}^n(x) = x_2^n \wedge N_2^n \), and it is clear that such a policy is SWC. The same applies to Markov policies under which class 1 prefers pool 1 (here \( z_{11}^n(x) = x_1^n \wedge N_1^n \)). As a result, SWC policies are more general than the particular policy considered in [16].

Recall that the matrices \( B_1^n \) and \( B_2^n \) in the drift (5.4) are given by
\[
B_1^n = \begin{pmatrix} \mu_{11}^n & 0 \\ 0 & \mu_{22}^n \end{pmatrix}, \quad \text{and} \quad B_2^n = \begin{pmatrix} \mu_{11}^n - \mu_{12}^n & 0 \\ 0 & 0 \end{pmatrix} \quad (5.12)
\]

It is also worth noting here, that the spare capacity \( \varrho_n \) of the \( n^{th} \) system is given by
\[
\varrho_n = -\frac{1}{\sqrt{n}} \left( \frac{\lambda_1^n}{\mu_{12}^n} + \frac{\lambda_2^n}{\mu_{22}^n} - \frac{\mu_{11}^n N_1^n + \mu_{12}^n \xi_{12}^n N_2^n - \mu_{22}^n \xi_{22}^n N_2^n}{\mu_{12}^n} \right),
\]
Lemma 5.3. Consider the ‘N’ network, and assume that \( g_n > 0 \). Then for any \( \theta \geq \theta_0 := \frac{\mu_{12} \land \mu_{22}^{\nu_n}}{\mu_{12} \lor \mu_{22}^{\nu_n}} \), and \( \delta \in (0, 1) \), there exist positive constants \( c_0 \) and \( c_1 \) such that (5.2) holds for all \( n \geq n_0 \).

Proof. As discussed earlier, it suffices to establish (5.2) in \( \tilde{X}^n \). It is clear that \( \hat{x}_1^- = \hat{y}_1 + \sqrt{n}z_{12} \), and \( \hat{x}_2 = \hat{q}_2 + \sqrt{n}z_{12} \) for all \( \hat{z} \in \tilde{Z}^n(\hat{x}) \) and \( \hat{x} \in \tilde{X}^n \), with \( \tilde{X}^n \) as defined in (5.6). Hence \( u_1^i = 0 \), \( u_2^i = 1 \), \( u_1^i = 1 \), and \( u_2^i = 0 \). Also, by the definitions of \( \psi_x \) and \( n_0 \) we have

\[
\psi_x(\hat{x}) = \psi_x(-\hat{x}) = 0 \quad \forall \hat{x} \in \tilde{X}^n, \quad \forall n \geq n_0. \tag{5.13}
\]

By (5.4) and (5.12), we have

\[
\frac{1}{\theta} F^n(\hat{x}, \hat{z}) = \frac{\theta n}{2} \sum_{i \in I} \psi_x(\hat{x}_i) - \sum_{i \in I} \psi_x(\hat{x}_i)(\hat{x}_i - u_1^i(\hat{x}_i)^+) - \psi_x(-\hat{x}_1)(\eta^n - 1)(\hat{x}_2)^- \\
= \hat{\theta} \left( \psi_x(\hat{x}_2) + \psi_x(-\hat{x}_1)(\eta^n - 1) \right), \tag{5.14}
\]

\[
F_2^n(\hat{x}, \hat{z}) = -\frac{\theta n}{2} \sum_{i \in I} \psi_x(\hat{x}_i) - \sum_{i \in I} \psi_x(\hat{x}_i)(\hat{x}_i - u_1^i(\hat{x}_i)^+) + \psi_x(\hat{x}_1)(\eta^n - 1)(\hat{x}_2)^- \\
= \hat{\theta} \left( \psi_x(\hat{x}_2) + \psi_x(-\hat{x}_1)(\eta^n - 1) \right). \tag{5.15}
\]

Using the fact that \( \hat{x}_1^- - \hat{x}_2 = \hat{y}_1 - \hat{q}_2 \), and \( \hat{\theta} = \hat{q}_2 \) when \( \langle e, \hat{x}_1 \rangle \leq 0 \), we obtain from (5.13) and (5.14) that

\[
\frac{1}{\theta} F^n(\hat{x}, \hat{z}) = \frac{\theta n}{2} \psi_x(\hat{x}_1) - \psi_x(\hat{x}_1)(\eta^n - 1)(\hat{x}_2)^- - \psi_x(-\hat{x}_1)(\eta^n - 1)(\hat{x}_2)^-
= \varepsilon \left( \frac{\theta n}{2} - \hat{x}_1^- - (\eta^n - 1)(\hat{x}_1^- - \hat{x}_2^+) - (\eta^n - 1) \hat{\theta}^n \right) \tag{5.16}
\]

\[
\leq \varepsilon \left( \frac{\theta n}{2} - (\eta^n - 1) \hat{x}_1^- \right) \tag{5.16}
\]

Similarly, from (5.15), we obtain

\[
F_2^n(\hat{x}, \hat{z}) = -\frac{\theta n}{2} \sum_{i \in I} \psi_x(\hat{x}_i) - \sum_{i \in I} \psi_x(\hat{x}_i)(\eta^n - 1)(\hat{x}_2)^- \\
= 1 - \varepsilon \|\hat{x}_1^-\|_1 + \varepsilon \varphi_0 \left( \|\hat{x}_1^-\|_1 \wedge \|\hat{x}_1^+\|_1 \right) \tag{5.17}
\]

where we also use (4.7) and Lemma 5.2.
We continue with the estimate on $K_0^n$. We have

\[
\frac{1}{\theta} F^n_1 (\hat{x}, \hat{z}) \leq \frac{\theta_n}{2} \psi_\varepsilon'(-\hat{x}_1) - \psi_\varepsilon'(-\hat{x}_1) \hat{x}_1 - \psi_\varepsilon'(-\hat{x}_1)(\eta^n - 1) \hat{\theta} \\
= \varepsilon \left( \frac{\theta_n}{2} - \frac{1}{1} \hat{x}_1 (\eta^n - 1) \hat{\theta} \right) \\
\leq \varepsilon \left( \frac{\theta_n}{2} - (\eta^n - 1) \hat{x}_1 \right) \\
\forall (\hat{x}, \hat{z}) \in (\bar{X}^n \cap K_0^n) \times \hat{Z}^n (\hat{x}), \ \forall n \geq n_0,
\]

where in the last inequality we also use Lemma 5.2.

We break the estimate of $F^n_2$ in two parts. First, using (4.7), (4.9), (5.13) and Lemma 5.2, we obtain

\[
F^n_2 (\hat{x}, \hat{z}) \leq \frac{-\theta_n}{2} - \varepsilon \hat{x}_2 + \varepsilon (\hat{e}, \hat{x}) + \varepsilon \zeta_0 \left( \hat{x}_1 \wedge \hat{x}_2 \right) \\
\leq \frac{-\theta_n}{2} - \varepsilon (1 - \zeta_0) \hat{x}_1 \\
\leq \begin{cases} 
\frac{-\theta_n}{2} - \varepsilon \left( \frac{1-\delta}{2} \right) (1 - \zeta_0) \|\hat{x}\|_1 & \text{for } (\hat{x}, \hat{z}) \in (\bar{X}^n \cap (K_0^n \setminus K_0^n)) \times \hat{Z}^n (\hat{x}) \\
\frac{-\theta_n}{2} & \text{for } (\hat{x}, \hat{z}) \in (\bar{X}^n \cap K_0^n) \times \hat{Z}^n (\hat{x}).
\end{cases}
\]

Thus, (5.2) follows by (4.9) and (5.16)–(5.19). This completes the proof. \Box

5.1.2. The diffusion scale of networks with a dominant pool. We describe these networks exactly as in Subsection 4.3 where the dominant server pool is $j = 1$. We first note that the spare capacity $\varrho_n$ of the $n^{th}$ system is given by

\[
\varrho_n = -\frac{1}{\sqrt{n}} \left( \sum_{i \in \mathcal{I}} \lambda_i^n - \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}(i)} \frac{\mu_{ij}^n}{\mu_{i1}^n} \xi_{ij}^n \mu_j \right),
\]

where $\xi_{ij}^n$ satisfies

\[
\sum_{j \in \mathcal{J}(i)} \mu_{ij} \xi_{ij}^n \mu_j = \lambda_i.
\]

This is again due to (2.4), (2.12), (3.1), and (4.30).

Recall that the drift takes the following form:

\[
b^n_i (\hat{x}, \hat{z}) = -\frac{\varrho_n}{m} \mu_{i1}^n \mu_{i1}^n (\hat{x}_i - u_i^c (\hat{e}, \hat{x}) + \sum_{j \in \mathcal{J}(i)} \mu_{ij}^n (\eta_{ij}^n - 1) u_j^c (\hat{e}, \hat{x}) \\
+ \hat{\theta} \left( \mu_{i1}^n u_i^e + \sum_{j \in \mathcal{J}(i)} \mu_{ij}^n (\eta_{ij}^n - 1) u_j^e \right), \quad i \in \mathcal{I},
\]

with $\eta_{ij}^n := \frac{\mu_{ij}^n}{\mu_{i1}^n}$ for $j \in \mathcal{J}(i) := \mathcal{J}(i) \setminus \{1\}$ and $i \in \mathcal{I}$. In analogy to Subsection 4.3, we define We define

\[
\bar{\eta}_n := \max_{i \in \mathcal{I}} \max_{j \in \mathcal{J}(i)} \eta_{ij}^n, \quad \text{and} \quad \underline{\eta}_n := \min_{i \in \mathcal{I}} \min_{j \in \mathcal{J}(i)} \eta_{ij}^n.
\]

Lemma 5.4. Consider a network with a dominant server pool, and assume $\varrho_n > 0$. Then for any $\theta \geq \theta_0^n := 2_{\max_{i \in \mathcal{I}}} \frac{\mu_{i1}^n}{\min_{i \in \mathcal{I}}} \lambda_i^n$ and $\delta \in (0, 1)$, there exist positive constants $c_0$ and $c_1$ such that (5.2) holds for all $n \geq n_0$. 
Proof. Suppose \( \hat{x} \in \hat{\mathbb{X}}^n \). A simple calculation using (5.20) shows that
\[
\frac{1}{\theta} F_1^n(\hat{x}, \hat{z}) = \frac{\rho_n}{m} \sum_{i \in I} \psi'_e(-\hat{x}_i) + \sum_{i \in I} \psi'_e(-\hat{x}_i)(\hat{x}_i - u_i^e(\hat{e}, \hat{x})^+) \\
- \sum_{i \in I} \sum_{j \in J_1(i)} \psi'_e(-\hat{x}_i)(\eta^n_{ij} - 1) u_j^e(\hat{e}, \hat{x})^- \\
- \hat{\theta} \left( \sum_{i \in I} \psi'_e(-\hat{x}_i) u_i^e + \sum_{i \in I} \sum_{j \in J_1(i)} \psi'_e(-\hat{x}_i)(\eta^n_{ij} - 1) u_j^e \right),
\]
and
\[
F_2^n(\hat{x}, \hat{z}) = -\frac{\rho_n}{m} \sum_{i \in I} \psi'_e(\hat{x}_i) - \sum_{i \in I} \psi'_e(\hat{x}_i)(\hat{x}_i - u_i^e(\hat{e}, \hat{x})^+) \\
+ \sum_{i \in I} \sum_{j \in J_1(i)} \psi'_e(\hat{x}_i)(\eta^n_{ij} - 1) u_j^e(\hat{e}, \hat{x})^- \\
+ \hat{\theta} \left( \sum_{i \in I} \psi'_e(\hat{x}_i) u_i^e + \sum_{i \in I} \sum_{j \in J_1(i)} \psi'_e(\hat{x}_i)(\eta^n_{ij} - 1) u_j^e \right).
\]

By (5.21) we obtain
\[
\frac{1}{\theta} F_1^n(\hat{x}, \hat{z}) \leq \rho_n e + \frac{m}{2} - \varepsilon \sum_{i \in I} \hat{x}_i^- + \varepsilon (1 - \frac{n}{2})^+ \langle e, \hat{x} \rangle^- + \varepsilon (1 - \frac{n}{2})^+ \hat{\theta}^n \\
\leq \rho_n e + \frac{m}{2} - \varepsilon \sum_{i \in I} \hat{x}_i^- + \varepsilon (1 - \frac{n}{2})^+ \left( \sum_{i \in I} \|\hat{x}_i^- - \hat{x}_i^+\|_1 \land \|\hat{x}_i^+\|_1 \right) \\
\leq \rho_n e + \frac{m}{2} - \varepsilon (\frac{n}{2} \land 1) \|\hat{x}_i^-\|_1 \\
\leq \rho_n e + \frac{m}{2} - \varepsilon (1 - \delta) \left( \frac{n}{2} \land 1 \right) \|\hat{x}_i^-\|_1 \quad \forall (\hat{x}, \hat{z}) \in (\hat{\mathbb{X}}^n \setminus \mathbb{K}_0^+) \times \hat{\mathbb{Z}}^n(\hat{x}), \forall n \in \mathbb{N},
\]
where we used (4.7) in the first inequality, Lemma 5.2 in the second, and (4.9) in the fourth.

Next, we estimate a bound for \( F_2^n(\hat{x}, \hat{z}) \). Recall the definitions of \( I_0, J_0, E_0, \) and \( E_0' \) in the proof of Lemma 5.2. Since \( x \in \hat{\mathbb{X}}^n \), we have \( u_i^e = 0 \) for all \( i \in I_0 \), and \( u_j^e = 0 \) for all \( j \in J_0' \). Additionally, 
\( \hat{x}_i \leq -\sum_{(i,j) \in E_0} \hat{z}_{ij} \) for \( i \in I_0 \), which implies that \( \psi'_e(x_i) = 0 \) for all \( i \in I_0 \) and \( n > n_0 \), by Definition 5.1. Hence, since \( \sum_{i \in I_0} \hat{x}_i > 0 \), where \( I_0' \equiv I \setminus I_0 \), we have
\[
\sum_{i \in I_0} \psi'_e(\hat{x}_i)(\hat{x}_i - u_i^e(\hat{e}, \hat{x})^+) \geq \sum_{i \in I_0} \psi'_e(\hat{x}_i) \hat{x}_i - \varepsilon \sum_{i \in I_0} \hat{x}_i - \varepsilon \sum_{i \in I_0} \hat{x}_i \geq -\varepsilon \sum_{i \in I_0} \hat{x}_i
\]
by (4.11). Using (5.22) together with Remark 5.3 and (5.24), we obtain
\[
F_2^n(\hat{x}, \hat{z}) \leq -\frac{\rho_n e}{m} + \varepsilon \hat{\theta}^n + \varepsilon \sum_{i \in I_0} \hat{x}_i + \hat{\theta} \left( \sum_{i \in I_0} \sum_{j \in J_1(i)} \psi'_e(\hat{x}_i)(\eta^n_{ij} - 1) u_j^e \right) \\
\leq -\frac{\rho_n e}{m} + \varepsilon (1 - \frac{n}{2}) \sum_{i \in I_0} \hat{x}_i \\
\leq \begin{cases} 
-\frac{\rho_n e}{m} - \varepsilon (1 - \delta) \|\hat{x}_i^-\|_1 & \text{for } (\hat{x}, \hat{z}) \in (\hat{\mathbb{X}}^n \cap (\mathbb{K}_0^+ \setminus \mathbb{K}_0^+)) \times \hat{\mathbb{Z}}^n(\hat{x}) \\
-\frac{\rho_n e}{m} & \text{for } (\hat{x}, \hat{z}) \in (\hat{\mathbb{X}}^n \cap \mathbb{K}_0^+) \times \hat{\mathbb{Z}}^n(\hat{x}),
\end{cases}
\]
for all \( n \geq n_0 \). Thus, the result follows by (5.23) and (5.25), noting also that the choice of \( \theta \) implies that \( V_1 \geq V_2^2 \) on \( \mathbb{K}_0^+ \).
\[\square\]
5.1.3. The diffusion-scale of networks with class-dependent service rates. Recall from Subsection 4.3 that the drift in (5.4) reduces to
\[ b^n(\hat{x}, \hat{z}) = -\frac{\eta_n}{m} B^n_1 e - B^n_1 \langle \hat{x} - \langle e, \hat{x} \rangle^+, u^e \rangle + \hat{\theta}^n(\hat{x}, \hat{z}) B^n_1 u^e. \] (5.26)
where \( B^n_1 = \text{diag}(\mu^n_1, \ldots, \mu^n_m) \). Thus, the spare capacity \( g_n \) is given by
\[ g_n = -\frac{1}{\sqrt{n}} \left( \sum_{i \in I} \frac{\lambda^n_i}{\mu^n_i} - \sum_{i \in I} \sum_{j \in J(i)} \xi^n_{ij} N^n_j \right). \]

Lemma 5.5. Suppose that \( \mu^n_{ij} = \mu^n_i \), for all \( i \in I \), and \( g_n > 0 \). Then, for any \( \theta \geq \theta^n_0 := 2 \frac{\mu^n_{\max}}{\mu^n_{\min}}, \) and \( \delta \in (0,1) \), the conclusions of Lemma 5.4 follow.

Proof. Suppose \( \hat{x} \in \tilde{X}^n \). A simple calculation using (5.26) shows that
\[ \frac{1}{\theta} F^n_1(\hat{x}, \hat{z}) = \frac{\eta_n}{m} \sum_{i \in I} \psi'_c(\hat{x}_i) + \sum_{i \in I} \psi'_c(\hat{x}_i) \langle \hat{x}_i - u^c_i (\hat{x}, \hat{z}) \rangle + \hat{\theta} \sum_{i \in I} \psi'_c(\hat{x}_i) u^c_i, \] (5.27)
\[ F^n_2(\hat{x}, \hat{z}) = \frac{\eta_n}{m} \sum_{i \in I} \psi'_e(\hat{x}_i) - \sum_{i \in I} \psi'_e(\hat{x}_i) (\hat{x}_i - u^e_i (\hat{x}, \hat{z}) \rangle + \hat{\theta} \sum_{i \in I} \psi'_e(\hat{x}_i) u^e_i. \] (5.28)

By (5.27), we obtain
\[ \frac{1}{\theta} F^n_1(\hat{x}, \hat{z}) \leq g_n \varepsilon + \frac{m}{2} - \varepsilon \| \hat{x}^- \|_1 \]
\[ \leq g_n \varepsilon + \frac{m}{2} - \varepsilon (1 - \delta) \| \hat{x}^- \|_1 \quad \forall (\hat{x}, \hat{z}) \in (\tilde{X}^n \setminus \mathcal{K}^-_\delta) \times \hat{Z}^n(\hat{x}), \forall n \geq n_0. \]

In computing the analogous bound to (5.25), there is a difference here. It is not the case here that \( \psi'_e(x_i) = 0 \) for all \( i \in I_0 \) and \( n > n_0 \).

So instead, recalling that \( u^e_i = 0 \) for all \( i \in I_0 \), and since \( \hat{x} \in \mathcal{K}^+_0 \), we write
\[ -\sum_{i \in I} \psi'_e(\hat{x}_i) (\hat{x}_i - u^e_i (\hat{x}, \hat{z}) \rangle + \hat{\theta} \sum_{i \in I} \psi'_e(\hat{x}_i) u^e_i \leq -\sum_{i \in I} \psi'_e(\hat{x}_i) \hat{x}_i + \varepsilon (\hat{x}, \hat{z}) + \varepsilon \hat{\theta} \]
\[ \leq \varepsilon \left( \hat{\theta} - \sum_{i \in I_0} \hat{x}^-_i \right) - \sum_{i \in I_0} \psi'_e(\hat{x}_i) \hat{x}_i \]
\[ \leq \left( \sum_{i \in I_0} \psi'_e(\hat{x}_i) \hat{x}_i - \varepsilon \sum_{i \in I_0} \hat{x}_i \right). \] (5.29)

The third term on the right-hand side is nonpositive by (4.11). We also have
\[ \hat{\theta} - \sum_{\ell \in I_0} \hat{x}^-_\ell = -\sqrt{n} \sum_{(i,j) \in E^+_0} z^n_{ij}, \] (5.30)
and
\[ -\sum_{i \in I_0} \psi'_e(\hat{x}_i) \hat{x}_i \leq \sum_{i \in I_0} \hat{x}^-_i \leq \frac{1}{\sqrt{n}} \min_{i \sim j} \hat{z}^n_{ij}. \] (5.31)

Therefore, by (5.28), (5.29)–(5.31) and Remark 5.3, we obtain
\[ F^n_2(\hat{x}, \hat{z}) \leq \frac{\varepsilon}{2} \left( \hat{\theta} - \sum_{\ell \in I_0} \hat{x}^-_\ell \right) \]
\[ \leq \frac{\varepsilon}{2} (1 - \varepsilon^n_0) \| \hat{x}^- \|_1 \quad \forall (\hat{x}, \hat{z}) \in (\tilde{X}^n \cap \mathcal{K}^+_0) \times \hat{Z}^n(\hat{x}). \]
The rest follows as in Lemma 5.4. \qed
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References


