Functional central limit theorems for epidemic models with varying infectivity

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Abstract. In this paper, we prove functional central limit theorems (FCLTs) for a stochastic epidemic model with varying infectivity and general infectious periods recently introduced in [9]. The infectivity process (total force of infection at each time) is composed of the independent infectivity random functions of each infectious individual, which starts at the time of infection. These infectivity random functions induce the infectious periods (as well as exposed, recovered or immune periods in full generality), whose probability distributions can be very general. The epidemic model includes the generalized non–Markovian SIR, SEIR, SIS, SIRS models with infection-age dependent infectivity. In the FCLT for the generalized SEIR model (including SIR as a special case), the limits for the infectivity and susceptible processes are a unique solution to a two-dimensional Gaussian-driven stochastic Volterra integral equations, and then given these solutions, the limits for the exposed/latent, infected and recovered processes are Gaussian processes expressed in terms of the solutions to those stochastic Volterra integral equations. We also present the FCLTs for the generalized SIS and SIRS models.

1. Introduction

It has been observed in recent studies of the Covid-19 pandemic (see e.g. [11]) that the infectivity of infectious individuals decreases from the epoch of symptom first appearing to full recovery. The varying infectivity characteristics also appears in many other epidemic diseases [14, 6]. We have presented a stochastic epidemic model with varying infectivity in [9], where the various individuals have i.i.d. infectivity random functions, and the total force of infection at each time is the aggregate infectivity of all the individuals that are currently infectious. We have proved a functional law of large numbers (FLLN) for the epidemic dynamics which results in a deterministic epidemic model, which is the model described as an “age-of-infection epidemic model” in [14, 5, 6]. For the early stage, we have studied the stochastic model directly starting with a small number of infectious individuals, and proved that the epidemic grows at an exponential rate on the event of non-extinction using an approximation by a non Markovian branching process. In addition, we have deduced the initial basic reproduction number $R_0$ from the limit process and computed its value for the case of the early phase of the Covid-19 epidemic in France. We have concluded a decreased value of $R_0$ induced by the decrease of the infectivity with age-infection.

In this paper we establish a functional central limit theorem (FCLT) for this stochastic epidemic model. As discussed in [9], the model can be regarded as a generalization of the SIR and SEIR models. In particular, the infectivity random function can take a very general form (see Assumption 2.1) and start with a value zero for a period of time, which then in turn determines the durations of the exposed and infectious periods. The joint distribution of these two periods is determined by the law of the random function, and can be very general. As in the FLLN, we study the mean infectivity process jointly with the proportions of susceptible, exposed, infected and recovered individuals. In case of the generalized SEIR model, in the FLLN, the infectivity and susceptible functions in the limit are uniquely determined by a two-dimensional Volterra integral equation, and given these

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two functions, the exposed, infectious and recovered functions in the limit are given by Volterra integral formulas. In the FCLT, we first show that the diffusion-scaled infectivity and susceptible processes converge jointly to a two-dimensional Gaussian-driven linear stochastic Volterra integral equation (Theorem 2.1). In particular, the diffusion-scaled instantaneous infectivity rate process has a limit that is a linear functional of the susceptible and infectivity limit processes. We then show the joint convergence of the exposed, infectious and recovered processes, whose limits are expressed in terms of the solution of the above stochastic Volterra integral equation (Theorem 2.2). These results extend the FCLT for the classical SEIR model with general exposed and infectious periods in [18]. They clearly include the generalized SIR model as a special case.

The main challenge in the proof of the FCLT lies in the convergence of the aggregate infectivity process. We allow the individual infectivity random functions to be piecewise continuous with a finite number of discontinuities as stated in Assumption 2.1, which covers many practical settings, see Examples 2.1–2.3. We use Poisson random measures (PRMs) induced by the laws of these random functions in the functional space $D$, and take advantage of some useful properties of stochastic integrals with respect to the corresponding compensated PRMs. We first give a useful decomposition of this process, and construct two auxiliary processes by replacing the random instantaneous infectivity rate process by its deterministic limit function in the FLLN. For these auxiliary processes, we employ the moment method to prove their tightness, using the criterion for tightness in [4, Theorem 13.5]. This, together with the convergence of finite dimensional distributions, proves their weak convergence. The martingale approach employed in [18] (see the proof of Lemma 4.2) could not be used for the aggregate infectivity process. It seems hard to construct appropriate martingales for the model considered in the present paper.

A major difficulty in the proof of tightness lies in the estimation of the expectation of the supremum of the square of the diffusion-scaled processes (see Lemma 3.5 and equation (3.35)), since we cannot use a semimartingale decomposition and exploit Doob’s maximal inequality. In order to establish this, we have developed a criterion to interchange sup and the expectation of the square of a stochastic process, provided that the process satisfies the well–known tightness criterion using the increments at three time points (see equation (3.2) as in Theorem 13.5 of [4], as well as the sufficient moment criterion in (3.4)). This result is stated in Theorem 3.1. It should be of independent interest and we expect that it will be useful for other purposes. For our model, we employ the moment formulas (specifically the mixed second moment formula in (3.6)) for the stochastic integrals with respect to the PRMs to establish the tightness moment bound for the increments of the infectivity process (see Propositions 3.1 and 3.2).

We also state the FCLTs without proofs for the generalized SIS and SIRS models with varying infectivity (which follow with a slight modification). For the SIS model, the epidemic dynamics is determined by the aggregate infectivity process and the infectious process, whose limits are given by a two-dimensional Gaussian-driven linear stochastic Volterra integral equation (Theorem 5.1). For the SIRS model, the epidemic dynamics is determined by the aggregate infectivity process and the infectious and recovered/immune processes, whose limits are given by a three-dimensional Gaussian-driven linear stochastic Volterra integral equation (Theorem 5.2).

This work contributes to the literature of stochastic epidemic models in the aspects of infection-age dependent infectivity, and general infectious periods. It establishes a useful result describing the fluctuations of the stochastic individual based model around its law of large numbers limit. The existing work on epidemic models with infection-age dependent (varying) infectivity has all been about the deterministic models, pioneered by Kermack and McKendrick in 1927 and 1932 [15, 16], and more recently proposed by Brauer [5] (see also [6, Chapter 4.5]), and the PDE models (see, e.g., [12, 22, 13, 17]). These were not considered as the FLLN limits of a well specified stochastic model either, except our work in [9]. For non-Markovian epidemic models with general infectious periods, although some deterministic models (including Volterra integral equations) appeared in the literature (see, e.g., [6, Chapter 4.5] and references therein), the works that rigorously establish
them as a FLLN from a stochastic model are limited. Both FLLNs and FCLTs are proved in [23, 24, 25] for population models, including an epidemic SIR model which has an infection rate dependent on the number of infectious individuals and general infectious periods for newly infected individuals, and which also has the remaining infectious periods of the initially infected individuals dependent on the elapsed infectious durations. A FLLN for the associated measure-valued processes was established for the SIR model with the infection rate dependent on time and the number of infected individuals using Stein’s method in [20]. For the general SIS, SIR, SEIR and SIRS models with a constant infection rate and the associated multi-patch models, both FLLNs and FCLTs were established in [18, 19]. The results in [18] are similar to those in [23, 24, 25] for the SIR model with general infectious periods, since the limits are given as Volterra integral equations, deterministic in the FLLNs and stochastic driven by Gaussian processes in the FCLTs. However, the proofs are rather different, and the FCLTs in [18] do not require any condition on the c.d.f. of the infectious periods, while those in [23, 24, 25] assume a $C^1$ condition. As mentioned at the beginning, the model with varying infectivity studied in this paper is much more general than that in [18]. It is also quite different from the models in [23, 24, 25, 20] since an infection rate dependent on the number of infectious individuals captures neither the randomness of infectivity as the i.i.d. random infectivity function of each individual, nor the age of infection dependent infectivity of each individual. For other aspects of epidemic models with general infectious periods, including the Sellke construction and the final size of an epidemic, we refer the readers to [21, 1, 2, 3] and the recent survey [7].

The paper is organized as follows. In Section 2.1, we provide a detailed description of the model and the assumptions, which is followed by the statement of the FCLT for the generalized SEIR model in Section 2.2. The main proofs are given in Section 3. We first state several technical preliminaries in Subsection 3.1, including the criterion for moment estimate of the supremum of stochastic processes in $D$ and the formulas of the Laplace functional and moments of stochastic integrals with respect to PRMs. We then proceed to proofs of the aggregate infectivity process in Subsections 3.2-3.6, with a proof map given in Subsection 3.2. The convergence of the exposed, infectious and recovered processes are given in Section 4. Finally, the FCLTs for the generalized SIR and SIRS models are stated in Sections 5.1 and 5.2, respectively.

2. Main Results

2.1. Generalized SEIR model with varying infectivity. In our epidemic model, each individual is associated with an infectivity random function at the epoch of infection, which exerts the infectivity to the susceptible individuals. Let the population size be $N$, and $S^N(t)$, $E^N(t)$, $I^N(t)$, $R^N(t)$ be the numbers of susceptible, exposed, infectious and recovered individuals at each time $t$, respectively. We have the balance equation $N = S^N(t) + E^N(t) + I^N(t) + R^N(t)$ for $t \geq 0$. Assume that $R^N(0) = 0$, $S^N(0) > 0$ and $I^N(0) > 0$.

Let $\{\lambda^0_i(\cdot)\}$, $\{\lambda^{0,j}_k(\cdot)\}$ and $\{\lambda_i(\cdot)\}$ be the infectivity processes associated with each initially exposed, infectious and newly exposed individual, respectively. Assume that the sequence $\{\lambda^0_i(\cdot)\}$ is i.i.d., and so are $\{\lambda^{0,j}_k(\cdot)\}$, and $\{\lambda_i(\cdot)\}$. These processes are only taking effect during the infectious periods, and generate the corresponding exposed and infectious periods. Assume that they all have càdlàg paths. In particular, the exposed and infectious periods $(\zeta_i, \eta_i)$ of a newly exposed individual are determined from $\lambda_i(\cdot)$ as follows:

$$
\zeta_i = \inf\{t > 0, \lambda_i(t) > 0\}, \quad \zeta_i + \eta_i = \inf\{t > 0, \lambda_i(r) = 0, \forall r \geq t\}. \tag{2.1}
$$

Similarly, the remaining exposed period and the infectious period $(\zeta^0_j, \eta^0_j)$ of an initially exposed individual are given as:

$$
\zeta^0_j = \inf\{t > 0, \lambda^{0,j}_j(t) > 0\} > 0, \quad \zeta^0_j + \eta^0_j = \inf\{t > 0, \lambda^{0,j}_j(r) = 0, \forall r \geq t\}. \tag{2.2}
$$
and the remaining infectious period $\eta^0_{k,t}$ of an initially infectious individual (note that $\inf\{t > 0, \lambda^0_{k,t} > 0\} = 0$) is:

$$\eta^0_{k,t} = \inf\{t > 0, \lambda^0_{k,t}(r) = 0, \forall r \geq t\}. \quad (2.3)$$

Under the i.i.d. assumptions of the corresponding infectivity processes, the random vectors $\{(\zeta, \eta^0_i) : i \in \mathbb{N}\}$ and $\{(\zeta^0_j, \eta^0_j) : j \in \mathbb{N}\}$ are i.i.d., and so is the sequence $\{\eta^0_k : k \in \mathbb{N}\}$. Let $H(du, dv)$ denote the law of $(\zeta, \eta^0_I(t))$ of $\zeta^0$ and $\eta^0_I(t)$ the c.d.f. of $\eta^0_{k,t}$. Define

$$\Phi(t) := \int_0^t \int_0^t H(du, dv) = \mathbb{P}(\zeta + \eta \leq t),$$

$$\Psi(t) := \int_0^t \int_\infty^{t-u} H(du, dv) = \mathbb{P}(\zeta \leq t < \zeta + \eta),$$

$$\Phi_0(t) := \int_0^t \int_0^{t-u} H_0(du, dv) = \mathbb{P}(\zeta^0 + \eta^0 \leq t),$$

$$\Psi_0(t) := \int_0^t \int_\infty^{t-u} H_0(du, dv) = \mathbb{P}(\zeta^0 \leq t < \zeta^0 + \eta^0),$$

and $F_{0,t}(t) := \mathbb{P}(\eta^0_{k,t} \leq t)$. We write $H(du, dv) = G(du)F(dv|u)$ and $H_0(du, dv) = G_0(du)F_0(dv|u)$, i.e., $G$ is the c.d.f. of $\zeta$ and $F(\cdot|u)$ is the conditional law of $\eta$, given that $\zeta = u$, $G_0$ is the c.d.f. of $\zeta^0$ and $F_0(\cdot|u)$ is the conditional law of $\eta^0$, given that $\zeta^0 = u$. In the case of independent exposed and infectious periods, it is reasonable that the infectious periods of the initially exposed individuals have the same distribution as the newly exposed ones, that is, $F_0 = F$. Note that in the independent case, $\Psi(t) = G(t) - \Phi(t)$ and $\Psi_0(t) = G_0(t) - \Phi_0(t)$. Also, let $G_0^c = 1 - G_0$, $G^c = 1 - G$, $F_0^c = 1 - F_{0,t}$, and $F^c = 1 - F$.

Let $A^N(t)$ be the number of individuals that are exposed in $(0, t]$, and $\tau^N_i$ denote the time of the $i$th individual that gets exposed. Let $\mathcal{I}^N(t)$ be the total force of infection which is exerted on the susceptibles at time $t$. By definition, we have

$$\mathcal{I}^N(t) = \sum_{j=1}^t \lambda^0_j(t) + \sum_{k=1}^t \lambda^0_k(t) + \sum_{i=1}^t \lambda_i(t - \tau^N_i), \quad t \geq 0. \quad (2.4)$$

Thus, the infection process $A^N(t)$ can be expressed by

$$A^N(t) = \int_0^t \int_\infty^t \mathbf{1}_{u \leq \mathcal{I}^N(s-)}Q(ds, du), \quad t \geq 0, \quad (2.5)$$

where

$$\mathcal{I}^N(t) := \frac{S^N(t)}{N} \mathcal{I}^N(t), \quad (2.6)$$

is the instantaneous infectivity rate function at time $t$, and $Q$ is a standard Poisson random measure (PRM) on $\mathbb{R}_+^2$.

The epidemic dynamics of the model can be described by

$$S^N(t) = S^N(0) - A^N(t),$$

$$E^N(t) = \sum_{j=1}^{E^N(0)} \mathbf{1}_{\zeta^0_j > t} + \sum_{i=1}^{A^N(t)} \mathbf{1}_{\tau^N_i + \zeta^0 > t},$$

$$I^N(t) = \sum_{j=1}^{E^N(0)} \mathbf{1}_{\zeta^0_j \leq t < \zeta^0_j + \eta^0_j} + \sum_{k=1}^{A^N(t)} \mathbf{1}_{\eta^0_k + \tau^N_i > t} + \sum_{i=1}^{A^N(t)} \mathbf{1}_{\tau^N_i + \zeta^0 < t < \tau^N_i + \zeta^0 + \eta_i}.$$
2.2. FCLTs for the generalized SEIR model. Let $D = D([0, +\infty), \mathbb{R})$ denote the space of \(\mathbb{R}\)-valued càdlàg functions defined on \([0, +\infty)\). Let $C$ denote its subspace of continuous functions. Throughout the paper, convergence in $D$ means convergence in the Skorohod $J_1$ topology, see Chapter 3 of [4]. Also, $D^k$ stands for the $k$-fold product equipped with the product topology.

We first make the following assumptions on the random infectivity functions and the distribution functions. We first state our assumptions on $\lambda^0$, $\lambda^{0, I}$ and $\lambda$. Let $\bar{\lambda}^0(t) = \mathbb{E}[\lambda^0(t)]$, $\bar{\lambda}^{0, I}(t) = \mathbb{E}[\lambda^{0, I}(t)]$ and $\bar{\lambda}(t) = \mathbb{E}[\lambda(t)]$ for $t \geq 0$. Also, let $v_0(t) = \text{Var}(\lambda^0(t))$, $v_{0, I}(t) = \text{Var}(\lambda^{0, I}(t))$ and $v(t) = \text{Var}(\lambda(t))$ for $t \geq 0$. Assume the mean and variance functions are all finite for each $t \geq 0$.

**Assumption 2.1.** The random functions $\lambda(t)$ (resp. $\lambda^0(t)$ and resp. $\lambda^{0, I}(t)$), of which $\lambda_1(t)$, $\lambda_2(t)$, \ldots (resp. $\lambda_1^0(t)$, $\lambda_2^0(t)$, \ldots and resp. $\lambda_1^{0, I}(t)$, $\lambda_2^{0, I}(t)$, \ldots) are i.i.d. copies, satisfying the following properties.

(i) There exists a constant $\lambda^* < \infty$ such that $\sup_{t \in [0, T]} \max\{\lambda^0(t), \lambda^{0, I}(t), \lambda(t)\} \leq \lambda^*$ almost surely.

(ii) There exist nondecreasing functions $\phi$ and $\psi$ in $C$ and $\alpha > 1/2$ and $\beta > 1$ such that for all $0 \leq r \leq s \leq t$, denoting $\bar{\lambda}^0(t) = \lambda^0(t) - \bar{\lambda}^0(t)$,

(a) $\mathbb{E}[ (\bar{\lambda}(0) - \bar{\lambda}(s))^2 ] \leq (\phi(s) - \phi(t))^\alpha$,

(b) $\mathbb{E}[ (\bar{\lambda}(t) - \bar{\lambda}(s))^2 (\bar{\lambda}(0) - \bar{\lambda}(r))^2 ] \leq (\psi(t) - \psi(s))^\beta$.

Similarly for the infectivity processes $\{\lambda_{k, I}^0\}_{k \geq 1}$.

(iii) Either $\lambda \in C$ and satisfies (2.8)–(2.9) below, or else there exist a given number $k \geq 1$, a random sequence $0 \leq \xi = \xi_0 < \xi_1 < \cdots < \xi^k = \zeta + \eta$ and random functions $\lambda^j \in C$, $1 \leq j \leq k$ such that

\[
\lambda(t) = \sum_{j=1}^{k} \lambda^j(t) \mathbb{1}_{[\xi^{j-1}, \xi^j)}(t).
\] (2.7)

Moreover, there exists a nondecreasing function $\varphi \in C$ satisfying

\[
\varphi(r) \leq Cr^\alpha, \quad \text{with } \alpha > 1/2 \text{ and } C > 0 \text{ arbitrary,}
\] (2.8)

such that

\[
|\lambda^j(t) - \lambda^j(s)| \leq \varphi(|t - s|), \quad \text{a.s.,}
\] (2.9)

for all $t, s \geq 0$, $1 \leq j \leq k$. Also, if $F_j$ denotes the c.d.f. of the r.v. $\xi_j$, then the exist $C'$ and $\rho > 0$ such that for any $0 \leq j \leq k$, $0 \leq s < t$,

\[
F_j(t) - F_j(s) \leq C'(t - s)^\rho,
\] (2.10)

and in addition, for any $1 \leq j \leq k$, $r > 0$,

\[
\mathbb{P}(\xi^j - \xi^{j-1} \leq r | \xi^{j-1}) \leq C^\prime r^\rho.
\] (2.11)

We remark that the conditions in Assumption 2.1 (ii) and the conditions in (2.9)–(2.11) are not required to establish the FLLN [9]. It is not surprising that the FCLT requires additional assumptions, compared with the FLLN.

These additional conditions are required to establish the moment criterion for tightness of the aggregate infectivity process (see Propositions 3.1 and 3.2 for the moment bounds for the increments of the associated processes). In fact, the condition (2.11) on $\{\xi^j\}$ is used only once in the end of the proof of Proposition 3.1, while (2.10) is used twice in the proofs of both Proposition 3.1 and 3.2. The conditions in (iii) allow fairly general random infectivity functions $\lambda(t)$, which can have càdlàg paths. They evidently impose a very mild condition on the distribution functions of the
exposed and infectious periods, in addition to being continuous. Recall that no condition is required on the distribution functions of the exposed and infectious periods in establishing the FLLNs and FCLTs of the standard SIR and SEIR models in [18]. The restriction on these distributions as a result of (2.10) and (2.11) is a compromise of allowing the random infectivity functions to have discontinuities. On the other hand, if we were to assume that \( \lambda(\cdot) \) is continuous satisfying (2.9), then it will allow the resulting exposed and infectious periods to take discrete values, and as a consequence, the same results of the FLLNs and FCLTs hold without requiring any condition on the distribution functions of the exposed and infectious periods.

We now describe examples of \( \lambda \)'s which satisfy the conditions in Assumption 2.1. We also refer the reader to [9, Section 2.5] for an example of \( \lambda \) adapted to the Covid–19 epidemic.

**Example 2.1.** Suppose we are given a random element \((\zeta, \eta)\) of \((0, +\infty)^2\), and let \(\lambda\) be a random function with trajectories in \(C\) which is such that
\[
\lambda(t) \begin{cases} 
> 0, & \text{for } \zeta < t < \zeta + \eta, \\
= 0, & \text{for } t \notin (\zeta, \zeta + \eta), 
\end{cases}
\]
and which satisfies (2.8)–(2.9). Then such a \(\lambda\) satisfies all the conditions in Assumption 2.1, and the law of the pair \((\zeta, \eta)\) is completely arbitrary.

**Example 2.2.** Suppose we are given again a random element \((\zeta, \eta)\) of \((0, +\infty)^2\), and \(\lambda\) a random function with now trajectories in \(D\) which is such that there exist \(0 \leq \zeta = \xi^0 < \xi^1 < \cdots < \xi^k = \zeta + \eta\) and random functions \(\lambda^j \in C, 1 \leq j \leq k\), such that \(\lambda(t)\) is as given in (2.7). We assume both that \(\lambda^1(\xi^0) = 0\) and \(\lambda^k(\xi^k) = 0\), that each \(\lambda^j\) satisfies (2.8)–(2.9), that \(F_1, \ldots, F_{k-1}\) satisfy (2.10), and that (2.11) is satisfied for all \(1 \leq j \leq k\). Our result applies to such a \(\lambda\), although neither \(F_0\) nor \(F_k\) is assumed to satisfy (2.10), and the reader can verify that our proofs extend to this situation. The reason is that we need not consider \(\xi^0\) or \(\xi^k\) as a time of discontinuity. \(\lambda^1\) (resp. \(\lambda^k\)) can be extended by 0 on \([0, \xi^0]\) (resp. on \([\xi^k, +\infty)\)). Again the law of the pair \((\zeta, \eta)\) is completely arbitrary. Both could be even deterministic. The only restrictions are that \(\xi^1, \ldots, \xi^{k-1}\), the times of discontinuity of \(\lambda\), satisfy both \(\zeta < \xi^1 < \cdots < \xi^{k-1} < \zeta + \eta\) and the fact that \(F_1, \ldots, F_{k-1}\) satisfy (2.10). Moreover, for \(2 \leq j \leq k - 1\), (2.11) must be satisfied.

**Example 2.3.** If we take the last example, but suppress the restriction \(\lambda^1(\xi^0) = 0\) and/or the restriction \(\lambda^k(\xi^k) = 0\), then we need to add the condition that the c.d.f. of \(\zeta\) and/or the c.d.f. of \(\zeta + \eta\) satisfies (2.10). That condition essentially rules out distributions with atoms. If we are mainly interested in realistic models, this is not a concern, since the uncertainty attached to those durations rather leads to distributions without atoms. On the other hand, sometimes deterministic durations are used, for their simplicity, as they might be reasonable approximations of reality for certain illnesses. The extension of the results of this paper to the case without any restriction on the \(F_j\)’s seems hard, and remains an open problem.

Let \(X^N = N^{-1}X^N\) for any process \(X^N\). Then under Assumption 2.1 (i) and (iii) (without (2.8) and (2.9)), assuming that there exist deterministic constants \(\bar{E}(0), \bar{I}(0) \in (0, 1)\) such that \(\bar{E}(0) + \bar{I}(0) < 1\), and \((\bar{E}^N(0), \bar{I}^N(0)) \to (\bar{E}(0), \bar{I}(0)) \in \mathbb{R}^2_+\) in probability as \(N \to \infty\), it is shown in [9, Theorem 2.1] that
\[
(\hat{S}^N, \hat{I}^N, \check{E}^N, \check{R}^N) \to (\hat{S}, \hat{I}, \check{E}, \check{R}) \text{ in } \mathbb{D}^5\text{ as } N \to \infty,
\]
in probability, locally uniformly in \(t\). The limit \((\hat{S}, \hat{I})\) is the unique solution of the following system of integral equations:
\[
\hat{S}(t) = 1 - \check{E}(0) - \hat{I}(0) - \int_0^t \hat{S}(s)\hat{I}(s)ds,
\]
\[ \tilde{S}(t) = E(0)\tilde{\lambda}(0) + \tilde{I}(0)\tilde{\lambda}^0(t) + \int_0^t \tilde{\lambda}(t-s)\tilde{S}(s)\tilde{S}(s)ds, \quad (2.14) \]

and the limits \((E, \tilde{I}, \tilde{R})\) are given by the following formulas:

\[ \begin{align*}
\tilde{E}(t) &= E(0)G^0(t) + \int_0^t G(t-s)\tilde{S}(s)\tilde{S}(s)ds, \\
\tilde{I}(t) &= I(0)F_0^c(t) + E(0)\Psi_0(t) + \int_0^t \Psi(t-s)\tilde{S}(s)\tilde{S}(s)ds, \\
\tilde{R}(t) &= \tilde{I}(0)F_{0,I}(t) + \tilde{E}(0)\Phi_0(t) + \int_0^t \Phi(t-s)\tilde{S}(s)\tilde{S}(s)ds.
\end{align*} \]

We also have \(\tilde{Y}^N \rightarrow \tilde{Y}\) in \(D\) in probability as \(N \rightarrow \infty\), where

\[ \tilde{Y}(t) := \tilde{S}(t)\tilde{S}(t), \quad t \geq 0. \quad (2.15) \]

Let \(\tilde{X}^N := \sqrt{N}(\tilde{X}^N - \tilde{X})\) for any process \(X^N\) with its fluid-scaled process \(\tilde{X}^N\) and limit \(X\). We make the following assumption on the initial quantities.

**Assumption 2.2.** There exist deterministic constants \(E(0), \tilde{I}(0) \in (0, 1)\) and random variables \(E(0), \tilde{I}(0)\) such that \((E(0), \tilde{I}(0)) := \sqrt{N}(E(0) - E(0), \tilde{I}(0) - \tilde{I}(0)) \Rightarrow (E(0), \tilde{I}(0))\) as \(N \rightarrow \infty\) and \(sup_N E[\tilde{E}(0)^2] < \infty\) and \(sup_N E[\tilde{I}(0)^2] < \infty\), and thus \(E[\tilde{E}(0)^2] < \infty\) and \(E[\tilde{I}(0)^2] < \infty\).

**Theorem 2.1.** Under Assumptions 2.1 and 2.2,

\[ (\tilde{S}^N, \tilde{Y}^N) \Rightarrow (\tilde{S}, \tilde{Y}) \text{ in } D^2 \text{ as } N \rightarrow \infty. \quad (2.16) \]

The limit process \((\tilde{S}, \tilde{Y})\) is the unique solution to the following system of stochastic integral equations:

\[ \begin{align*}
\tilde{S}(t) &= -\tilde{E}(0) - \tilde{I}(0) - \tilde{M}_A(t) + \int_0^t \tilde{Y}(s)ds, \\
\tilde{Y}(t) &= \tilde{I}(0)\tilde{\lambda}^0(t) + \tilde{E}(0)\tilde{\lambda}^0(t) + \tilde{\lambda}_0(t) + \tilde{\lambda}_0(t) + \tilde{\lambda}_1(t) + \tilde{\lambda}_2(t) + \int_0^t \tilde{\lambda}(t-s)\tilde{Y}(s)ds.
\end{align*} \quad (2.17, 2.18) \]

where

\[ \tilde{Y}(t) := \tilde{S}(t)\tilde{S}(t) + \tilde{S}(t)\tilde{Y}(t). \quad (2.19) \]

and \(\tilde{S}(t)\) and \(\tilde{Y}(t)\) are given by the unique solutions to the integral equations (2.13) and (2.14), \(\tilde{M}_A, \tilde{\lambda}_0, \tilde{\lambda}_1, \tilde{\lambda}_2\) are centered Gaussian processes which are globally independent of \((E(0), \tilde{I}(0))\). Moreover, the processes in \((\tilde{\lambda}_0, \tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{M}_A)\) are independent, and the covariances of each of those four processes (the last one being 2-dimensional) are given as follows:

\[ \begin{align*}
\text{Cov}(\tilde{\lambda}_0(t), \tilde{\lambda}_0(t')) &= \tilde{I}(0)\text{Cov}(\lambda^0(t), \lambda^0(t')), \\
\text{Cov}(\tilde{\lambda}_2(t), \tilde{\lambda}_2(t')) &= \tilde{E}(0)\text{Cov}(\lambda^0(t), \lambda^0(t')), \\
\text{Cov}(\tilde{\lambda}_1(t), \tilde{\lambda}_1(t')) &= \int_0^t \text{Cov}(\lambda(t-s), \lambda(t'-s))\tilde{S}(s)\tilde{S}(s)ds, \\
\text{Cov}(\tilde{\lambda}_2(t), \tilde{\lambda}_2(t')) &= \int_0^t \lambda(t-s)\lambda(t'-s)\tilde{S}(s)\tilde{S}(s)ds, \\
\text{Cov}(\tilde{M}_A(t), \tilde{M}_A(t')) &= \int_0^t \tilde{S}(s)\tilde{S}(s)ds, \\
\text{Cov}(\tilde{M}_A(t), \tilde{\lambda}_2(t')) &= \int_0^t \lambda(t'-s)\tilde{S}(s)\tilde{S}(s)ds.
\end{align*} \]
Concerning the pair $(\hat{M}_A, \hat{N}_2)$, $\hat{M}_A$ is a non–standard Brownian motion, and $\hat{N}_2(t) = \int_0^t \lambda(t - s)\hat{M}_A(ds)$. $\hat{S}$ has continuous paths, and if $\hat{\lambda}^0$ and $\hat{\lambda}^0.1$ are in $C$, then $\hat{\lambda}$ is also continuous.

We next state the FCLT for the processes $(\hat{E}^N, \hat{I}^N, \hat{R}^N)$, which extends Theorem 3.2 in [18].

**Theorem 2.2.** Given the convergence of $\hat{Y}^N \Rightarrow \hat{Y}$ in Theorem 2.1, under Assumptions 2.1(i) and 2.2,

$$
(\hat{E}^N, \hat{I}^N, \hat{R}^N) \Rightarrow (\hat{E}, \hat{I}, \hat{R}) \quad \text{in} \quad D^3 \quad \text{as} \quad N \to \infty,
$$

(2.20)

jointly with $(\hat{S}^N, \hat{N}^N)$ (i.e., $(\hat{S}^N, \hat{N}^N, \hat{E}^N, \hat{I}^N, \hat{R}^N) \Rightarrow (\hat{S}, \hat{N}, \hat{E}, \hat{I}, \hat{R})$ in $D^5$). The limit processes $\hat{E}$, $\hat{I}$ and $\hat{R}$ are given by the expressions:

$$
\hat{E}(t) = \hat{E}(0)G_0(t) + \hat{E}_0(t) + \hat{E}_1(t) + \int_0^t G^c(t - s)\hat{Y}(s)ds,
$$

$$
\hat{I}(t) = \hat{I}(0)F_0,1(t) + \hat{E}(0)\Psi_0(t) + \hat{I}_{0,1}(t) + \hat{I}_{0,2}(t) + \hat{I}_1(t) + \int_0^t \Psi(t - s)\hat{Y}(s)ds,
$$

$$
\hat{R}(t) = \hat{I}(0)F_0,1(t) + \hat{E}(0)\Phi_0(t) + \hat{R}_{0,1}(t) + \hat{R}_{0,2}(t) + \hat{R}_1(t) + \int_0^t \Phi(t - s)\hat{Y}(s)ds,
$$

where $\hat{Y}$ is given in (2.19). The limit $(\hat{I}_{0,1}, \hat{R}_{0,1})$ is a two-dimensional centered Gaussian processes which can be written as

$$
\hat{I}_{0,1}(t) = W_{0,1}([t, \infty)), \quad \hat{R}_{0,1}(t) = W_{0,1}([0, t])
$$

(2.21)

where $W_{0,1}$ is a Gaussian white noise on $\mathbb{R}_+$ satisfying

$$
\mathbb{E}[W_{0,1}(s, t)^2] = \hat{I}(0)(F_{0,1}(t) - F_{0,1}(s))(1 - (F_{0,1}(t) - F_{0,1}(s))
$$

for $0 \leq s \leq t$. The limits $(\hat{E}_0, \hat{I}_{0,2}, \hat{R}_{0,2})$ is a three-dimensional centered Gaussian processes which can be written as

$$
\hat{E}_0(t) = W_{0,2}([t, \infty] \times [0, \infty)), \quad \hat{I}_{0,2}(t) = W_{0,2}([0, t] \times [t, \infty)), \quad \hat{R}_{0,2}(t) = W_{0,2}([0, t] \times [0, t])
$$

(2.22)

where $W_{0,2}$ is a Gaussian white noise on $\mathbb{R}^2_+$ such that

$$
\mathbb{E}[W_{0,2}(s, t) \times [s', t')] = \hat{E}(0) \int_s^t (F_0(t' - y|y) - F_0(s' - y|y))G_0(dy)
$$

$$
\times \left(1 - \int_s^t (F_0(t' - y|y) - F_0(s' - y|y))G_0(dy)\right)
$$

for $0 \leq s \leq t$ and $0 \leq s' \leq t'$. The limit $(\hat{E}_1, \hat{I}_1, \hat{R}_1)$ is a three-dimensional centered Gaussian processes, independent of $\hat{E}_0$, $\hat{I}_{0,2}$, $\hat{R}_{0,2}$ and $\hat{I}(0)$, which can be written as

$$
\hat{E}_1(t) = W_1([0, t] \times [t, \infty] \times [0, \infty)),
$$

$$
\hat{I}_1(t) = W_1([0, t] \times [0, t] \times [t, \infty)),
$$

$$
\hat{R}_1(t) = W_1([0, t] \times [0, t] \times [0, t]),
$$

(2.23)

where $W_1$ is a Gaussian white noise on $\mathbb{R}^3_+$ such that

$$
\mathbb{E} \left[W_1([r, t] \times [a, b] \times [a', b'])^2\right]
$$

$$
= \int_r^t \left(\int_{a-s}^{b-s} (F(b' - y - s|y) - F(a' - y - s|y))G(dy)\right)\hat{S}(s)\hat{Y}(s)ds,
$$

for $0 \leq r \leq t$, $0 \leq a \leq b$ and $0 \leq a' \leq b'$. The white noise processes $W_{0,1}, W_{0,2}, W_1$ are mutually independent, and so are $(\hat{I}_{0,1}, \hat{R}_{0,1})$, $(\hat{E}_0, \hat{I}_{0,2}, \hat{R}_{0,2})$ and $(\hat{E}_1, \hat{I}_1, \hat{R}_1)$, which are globally independent of $\hat{E}(0), \hat{I}(0)$.
If $G_0$ and $F_{0,1}$ are continuous, then $\hat{E}_0$, $\hat{I}_{0,1}$, $\hat{I}_{0,2}$, $\hat{R}_{0,1}$ and $\hat{R}_{0,2}$ are continuous, and thus $\hat{E}$, $\hat{I}$ and $\hat{R}$ are continuous.

**Remark 2.1.** It is not hard to derive the following covariance functions from the white noise process representations for the Gaussian processes in Theorem 2.2. By (2.21), the limits $(\hat{I}_{0,1}, \hat{R}_{0,1})$ have covariance functions, for $t, t' \geq 0$,

\[
\text{Cov}(\hat{I}_{0,1}(t), \hat{I}_{0,1}(t')) = \hat{I}(0)(F_{0,1}^c(t \lor t') - F_{0,1}^c(t)F_{0,1}^c(t')),
\]

\[
\text{Cov}(\hat{R}_{0,1}(t), \hat{R}_{0,1}(t')) = \hat{I}(0)(F_{0,1}(t \land t') - F_{0,1}(t)F_{0,1}(t')).
\]

By (2.22), the limits $(\hat{E}_0, \hat{I}_{0,2}, \hat{R}_{0,2})$ have covariance functions, for $t, t' \geq 0$,

\[
\text{Cov}(\hat{E}_0(t), \hat{E}_0(t')) = \hat{E}(0)(G_0^c(t \lor t') - G_0^c(t)G_0^c(t')),
\]

\[
\text{Cov}(\hat{I}_{0,2}(t), \hat{I}_{0,2}(t')) = \hat{E}(0)\left(\int_0^{t \land t'} F_0^c(t \lor t' - s|s)dG_0(s) - \Psi_0(t)\Psi_0(t')\right),
\]

\[
\text{Cov}(\hat{R}_{0,2}(t), \hat{R}_{0,2}(t')) = \hat{E}(0)\left(\Phi_0(t \land t') - \Phi_0(t)\Phi_0(t')\right),
\]

\[
\text{Cov}(\hat{E}_0(t), \hat{I}_{0,2}(t')) = \hat{E}(0)1(t' \geq t)\left(\int_t^{t'} F_0^c(t' - s|s)dG_0(s) - G_0^c(t)\Psi_0(t')\right),
\]

\[
\text{Cov}(\hat{E}_0(t), \hat{R}_{0,2}(t')) = \hat{E}(0)1(t' \geq t)\left(\int_t^{t'} F_0(t' - s|s)dG_0(s) - G_0^c(t)\Phi_0(t')\right),
\]

\[
\text{Cov}(\hat{I}_{0,2}(t), \hat{R}_{0,2}(t')) = \hat{E}(0)1(t' \geq t)\left(\int_0^t (F_0(t' - s|s) - F_0(t - s|s))dG_0(s) - \Psi_0(t)\Phi_0(t')\right).
\]

By (2.23), the limits $(\hat{E}_1, \hat{I}_1, \hat{R}_1)$ have covariance functions, for $t, t' \geq 0$,

\[
\text{Cov}(\hat{E}_1(t), \hat{E}_1(t')) = \int_0^{t \lor t'} G^c(t \lor t' - s)\bar{S}(s)\bar{S}(s)ds,
\]

\[
\text{Cov}(\hat{I}_1(t), \hat{I}_1(t')) = \int_0^{t \lor t'} \int_0^{t \lor t' - s} F^c(t \lor t' - s - u|u)dG(u)\bar{S}(s)\bar{S}(s)ds,
\]

\[
\text{Cov}(\hat{R}_1(t), \hat{R}_1(t')) = \int_0^{t \lor t'} \Phi(t \land t' - s)\bar{S}(s)\bar{S}(s)ds,
\]

\[
\text{Cov}(\hat{E}_1(t), \hat{I}_1(t')) = 1(t' \geq t)\int_0^{t \lor t'} \int_{t - s}^{t'} F^c(t' - s - u|u)dG(u)\bar{S}(s)\bar{S}(s)ds,
\]

\[
\text{Cov}(\hat{E}_1(t), \hat{R}_1(t')) = 1(t' \geq t)\int_0^{t \lor t'} \int_{t - s}^{t'} F(t' - s - u|u)dG(u)\bar{S}(s)\bar{S}(s)ds,
\]

\[
\text{Cov}(\hat{I}_1(t), \hat{R}_1(t')) = 1(t' \geq t)\int_0^{t \lor t'} \int_{t - s}^{t'} (F(t' - s - u|u) - F(t - s - u|u))dG(u)\bar{S}(s)\bar{S}(s)ds.
\]

**Remark 2.2.** We discuss the covariance structures between the processes $\tilde{J}_{0,1}$, $\tilde{J}_{0,2}$, $\tilde{J}_1$ and $\tilde{J}_2$ in the integral expression of $\tilde{S}$ in (2.18), and the processes $(\hat{I}_{0,1}, \hat{R}_{0,1})$, $(\hat{E}_0, \hat{I}_{0,2}, \hat{R}_{0,2})$ and $(\hat{E}_1, \hat{I}_1, \hat{R}_1)$ in the integral expressions of $(\hat{E}, \hat{I}, \hat{R})$. They all follow easily from direct calculations by applying the Gaussian property, and also using PRM representations for the processes related to the newly infected individuals.

First, $\tilde{J}_{0,1}$ and $(\hat{I}_{0,1}, \hat{R}_{0,1})$ are driven by the common random source $\lambda^{0,1}(-)$, which gives the covariance functions, for $t, t' \geq 0$,

\[
\text{Cov}(\tilde{J}_{0,1}(t), \hat{I}_{0,1}(t')) = \tilde{I}(0)(\mathbb{E}[\lambda^{0,1}(t)1_{\lambda^{0,1} \geq t'}] - \tilde{\lambda}^{0,1}(t)F_{0,1}^c(t')),
\]

\[
\text{Cov}(\tilde{J}_{0,1}(t), \hat{R}_{0,1}(t')) = \tilde{I}(0)(\mathbb{E}[\lambda^{0,1}(t)1_{\lambda^{0,1} \leq t'}] - \tilde{\lambda}^{0,1}(t)F_{0,1}(t')).
\]
Second, $\hat{I}_{0,2}(t)$ and $(\hat{E}_0, \hat{I}_{0,2}, \hat{R}_{0,2})$ are driven by the common random source $\lambda^0(\cdot)$, which gives the covariance functions, for $t, t' \geq 0$,

$$\text{Cov}(\hat{I}_{0,2}(t), \hat{E}_0(t')) = E(0)(E[\lambda^0(t)1_{c^0 > t'}] - \bar{\lambda}^0(t) G^c_0(t')),$$

$$\text{Cov}(\hat{I}_{0,2}(t), \hat{I}_{0,2}(t')) = E(0)(E[\lambda^0(t)1_{c^0 \leq t' < \zeta^0 + \eta^0}] - \bar{\lambda}^0(t) \Psi_0(t')),$$

$$\text{Cov}(\hat{I}_{0,2}(t), \hat{R}_{0,2}(t')) = E(0)(E[\lambda^0(t)1_{c^0 + \eta^0 \leq t'}] - \bar{\lambda}^0(t) \Phi_0(t')).$$

Third, $\hat{I}_1$ and $(\hat{E}_1, \hat{I}_1, \hat{R}_1)$ are driven by the common random source $\lambda(\cdot)$, as well as the associated PRM, which gives the covariance functions, for $t, t' \geq 0$,

$$\text{Cov}(\hat{I}_1(t), \hat{E}_1(t')) = \int_0^{t' \wedge t} \left( E[\lambda(t-s)1_{\zeta > t'-s}] - \bar{\lambda}(t-s) G^c(t'-s) \right) \bar{S}(s) \bar{I}(s) ds,$$

$$\text{Cov}(\hat{I}_1(t), \hat{I}_1(t')) = \int_0^{t' \wedge t} \left( E[\lambda(t-s)1_{\zeta \leq t'-s < \zeta + \eta}] - \bar{\lambda}(t-s) \Psi(t'-s) \right) \bar{S}(s) \bar{I}(s) ds,$$

$$\text{Cov}(\hat{I}_1(t), \hat{R}_1(t')) = \int_0^{t' \wedge t} \left( E[\lambda(t-s)1_{\zeta + \eta \leq t'-s}] - \bar{\lambda}(t-s) \Phi(t'-s) \right) \bar{S}(s) \bar{I}(s) ds.$$

Fourth, $\hat{I}_2$ and $(\hat{E}_1, \hat{I}_1, \hat{R}_1)$ are driven by the associated PRM, only involving the deterministic $\bar{\lambda}(t)$, which gives the covariance functions, for $t, t' \geq 0$,

$$\text{Cov}(\hat{I}_2(t), \hat{E}_1(t')) = \int_0^{t' \wedge t} \bar{\lambda}(t-s) G^c(t'-s) \bar{S}(s) \bar{I}(s) ds,$$

$$\text{Cov}(\hat{I}_2(t), \hat{I}_1(t')) = \int_0^{t' \wedge t} \bar{\lambda}(t-s) \Psi(t'-s) \bar{S}(s) \bar{I}(s) ds,$$

$$\text{Cov}(\hat{I}_2(t), \hat{R}_1(t')) = \int_0^{t' \wedge t} \bar{\lambda}(t-s) \Phi(t'-s) \bar{S}(s) \bar{I}(s) ds.$$

In addition, $\bar{M}_A(t)$ can be written as $\bar{M}_A(t) = W_1([0, t] \times [0, \infty) \times [0, \infty))$. Thus, by (2.23), we also obtain for $t, t' \geq 0$,

$$\text{Cov}(\bar{M}_A(t), \hat{E}_1(t')) = \int_0^{t' \wedge t} G^c(t'-s) \bar{S}(s) \bar{I}(s) ds,$$

$$\text{Cov}(\bar{M}_A(t), \hat{I}_1(t')) = \int_0^{t' \wedge t} \Psi(t'-s) \bar{S}(s) \bar{I}(s) ds,$$

$$\text{Cov}(\bar{M}_A(t), \hat{R}_1(t')) = \int_0^{t' \wedge t} \Phi(t'-s) \bar{S}(s) \bar{I}(s) ds.$$

2.3. Generalized SIR model with varying infectivity. It is clear that the model includes the generalized SIR as a special case, where the random infectivity function $\lambda_i(t)$ does not equal to zero at time 0, that is, an infected individual is immediately infectious, so $\zeta = \zeta^0 = 0$ a.s., and there are no exposed individuals, $E^N(t) = 0$ for all $t \geq 0$. Let $F$ be the c.d.f. of the infectious duration $\eta$ of newly infected individuals and $F_{0, I}$ be the c.d.f. of the infectious duration $\eta^{0, I}$ of initially infectious individuals. The FLLN gives the limits $(\bar{S}, \bar{I})$ determined by the following two-dimensional integral equation:

$$\bar{S}(t) = 1 - \bar{I}(0) - \int_0^t \bar{S}(s) \bar{I}(s) ds,$$

$$\bar{I}(t) = \bar{I}(0) \bar{\lambda}^{0, I}(t) + \int_0^t \bar{\lambda}(t-s) \bar{S}(s) \bar{I}(s) ds,$$
and the limits $(\hat{I}, \hat{R})$ given by the following integral equations:

\[
\hat{I}(t) = \hat{I}(0)F_{0,I}(t) + \int_0^t F^c(t-s)\hat{S}(s)\hat{\lambda}(s)ds,
\]

\[
\hat{R}(t) = \hat{I}(0)F_{0,R}(t) + \int_0^t F(t-s)\hat{\lambda}(s)ds.
\]

The FCLT gives the limits $(\hat{S}, \hat{\lambda})$ determined by the solutions to the following two-dimensional stochastic integral equations:

\[
\hat{S}(t) = -\hat{I}(0) - \hat{M}_A(t) + \int_0^t \hat{\lambda}(s)ds,
\]

\[
\hat{\lambda}(t) = \hat{I}(0)\hat{\lambda}(t) + \hat{\lambda}_0(t) + \hat{\lambda}_1(t) + \hat{\lambda}_2(t) + \int_0^t \hat{\lambda}(t-s)\hat{\lambda}(s)ds,
\]

where the processes $\hat{M}_A, \hat{\lambda}_0, \hat{\lambda}_1, \hat{\lambda}_2$ and $\hat{\lambda}$ are as given in Theorem 2.1. And the limits $(\hat{I}, \hat{R})$ are given by the following stochastic integral equations:

\[
\hat{I}(t) = \hat{I}(0)F_{0,I}(t) + \hat{I}_{0,1}(t) + \hat{I}_1(t) + \int_0^t F^c(t-s)\hat{\lambda}(s)ds,
\]

\[
\hat{R}(t) = \hat{I}(0)F_{0,R}(t) + \hat{R}_{0,1}(t) + \hat{R}_1(t) + \int_0^t F(t-s)\hat{\lambda}(s)ds,
\]

where $\hat{I}_{0,1}$ and $\hat{R}_{0,1}$ are as given in Theorem 2.2, and the limits $(\hat{I}_1, \hat{R}_1)$ are two-dimensional continuous Gaussian processes, independent of $\hat{I}_{0,1}, \hat{R}_{0,1}$ and $\hat{I}(0)$, and can be written as

\[
\hat{I}_1(t) = W_1([0, t] \times [t, \infty)), \quad \hat{R}_1(t) = W_1([0, t] \times [0, t]),
\]

where $W_1$ is a Gaussian white noise generalized process on $\mathbb{R}_+^2$ with mean zero and

\[
\mathbb{E} \left[ W_1([s, t] \times [a, b])^2 \right] = \int_s^t (F(b - y - s) - F(a - y - s))\hat{S}(s)\hat{\lambda}(s)ds,
\]

for $0 \leq s \leq t$ and $0 \leq a \leq b$. They have the covariance functions: for $t, t' \geq 0$,

\[
\text{Cov}(\hat{I}_1(t), \hat{I}_1(t')) = \int_0^{t \wedge t'} F^c(t \wedge t' - s)\hat{S}(s)\hat{\lambda}(s)ds,
\]

\[
\text{Cov}(\hat{R}_1(t), \hat{R}_1(t')) = \int_0^{t \wedge t'} F(t \wedge t' - s)\hat{S}(s)\hat{\lambda}(s)ds,
\]

\[
\text{Cov}(\hat{I}_1(t), \hat{R}_1(t')) = 1(t' \geq t) \int_0^t (F(t' - s) - F(t - s))\hat{S}(s)\hat{\lambda}(s)ds.
\]

If $F_{0,I}$ is continuous, then $\hat{I}$ and $\hat{R}$ are continuous.

In addition, the process $\hat{\lambda}_{0,1}$ has covariance functions with the processes $\hat{I}_{0,1}$ and $\hat{R}_{0,1}$ as given in Theorem 2.2, and the process $\hat{\lambda}_1$ and $\hat{\lambda}_2$, independent of $\hat{I}_{0,1}$ and $\hat{R}_{0,1}$, have the following covariance functions with $\hat{I}_1$ and $\hat{R}_1$: for $t, t' \geq 0$,

\[
\text{Cov}(\hat{\lambda}_{0,1}(t), \hat{I}_1(t')) = \int_0^{t \wedge t'} \left( \mathbb{E}[\lambda(t-s)1_{u < t - s}] - \lambda(t-s)F^c(t' - s) \right)\hat{S}(s)\hat{\lambda}(s)ds,
\]

\[
\text{Cov}(\hat{\lambda}_1(t), \hat{R}_1(t')) = \int_0^{t \wedge t'} \left( \mathbb{E}[\lambda(t-s)1_{u > t - s}] - \lambda(t-s)F(t' - s) \right)\hat{S}(s)\hat{\lambda}(s)ds,
\]

\[
\text{Cov}(\hat{\lambda}_2(t), \hat{I}_1(t')) = \int_0^{t \wedge t'} \lambda(t-s)F^c(t' - s)\hat{S}(s)\hat{\lambda}(s)ds,
\]
\[ \text{Cov}(\hat{\mathcal{S}}_2(t), \hat{R}_1(t')) = \int_0^{t \wedge t'} \lambda(t - s)F(t' - s)\tilde{S}(s)\tilde{J}(s)ds . \]

\( \hat{M}_A \) is independent of \( \hat{I}_{0,1} \) and \( \hat{R}_{0,1} \), and has covariance functions with \( \hat{I}_1 \) and \( \hat{R}_1 \): for \( t, t' \geq 0 \),
\[ \text{Cov}(\hat{M}_A(t), \hat{I}_1(t')) = \int_0^{t \wedge t'} Fc(t' - s)\tilde{S}(s)\tilde{J}(s)ds , \quad \text{Cov}(\hat{M}_A(t), \hat{R}_1(t')) = \int_0^{t \wedge t'} F(t' - s)\tilde{S}(s)\tilde{J}(s)ds . \]

Now consider the SEIR model, but suppose that we do not care to follow the numbers or proportions of exposed and infectious individuals, but only the number or proportion of infected individual (where infected means either exposed or infectious). Formally, the SEIR model then reduces to a SIR model, with the function \( \lambda \) being allowed to be zero on the interval \([0, \zeta]\), with \( \zeta > 0 \). For such a model, the FLLN and the FCLT are exactly those described in this subsection.

3. Proofs for the Convergence of \( (\hat{S}^N, \hat{J}^N) \)

In this section we prove Theorem 2.1 on the convergence of \( (\hat{S}^N, \hat{J}^N) \). We first provide the following technical preliminaries used in the proofs.

3.1. Technical Preliminaries. In this subsection, we gather a few technical results which will be used later in our proofs. Recall the moment criterion in Theorem 13.5 of [4]. We will first deduce from that condition a uniform moment estimate. Let us now introduce some notations which will be used in the following proof. For \( x \) a map from \([0, T]\) into \( \mathbb{R} \), we define \( w(x, \delta) = \sup_{0 \leq s, t \leq T} |x(t) - x(s)| \), \( w(x, [a, b]) = \sup_{0 \leq s < t < b} |x(t) - x(s)| \), \( w'(x, \delta) = \inf_{t \in T} \max_{1 \leq i \leq v} w(x, [t_{i-1}, t_i]) \), where the infimum extends over all sets \( 0 = t_0 < t_1 < \cdots < t_v = T \) such that \( \min_{1 \leq i \leq v} (t_i - t_{i-1}) > \delta \), and finally
\[ w''(x, \delta) = \sup_{0 \leq t_1 \leq t_2 \leq t(t_1 + \delta) \wedge T} \{ |x(t) - x(t_1)| \wedge |x(t_2) - x(t)| \} . \]

Suppose now that a sequence \( \{X^N\} \) satisfies the original condition in Theorem 13.5 of [4]. Under an additional mild condition, it implies that \( \{X^N\} \) is tight in \( D \), hence in particular that for any \( T > 0 \), \( \sup_{0 \leq t \leq T} |X^N(t)| \) is a tight sequence of \( \mathbb{R}_+ \)-valued r.v.’s. We now show how we can deduce a bound on the second moment of that supremum.

**Theorem 3.1.** Let \( X^N \) be a sequence in \( D([0, T]) \). Suppose that the two following conditions are satisfied
\[ \sup_{N \geq 1} \sup_{0 \leq t \leq T} E \left[ (X^N(t))^2 \right] < \infty , \quad (3.1) \]
and that for some \( \alpha > 1/2, \beta > 1/2 \), and for all \( 0 \leq r \leq s \leq t \leq T, N \geq 1, \lambda > 0 \),
\[ P \left( |X^N(s) - X^N(r)| \wedge |X^N(t) - X^N(s)| \geq \lambda \right) \leq \frac{\|G(t) - G(r)\|^2\alpha}{\lambda^{4\beta}} , \quad (3.2) \]
where \( G \) is a non decreasing continuous function satisfying \( G(0) = 0 \). Then
\[ \sup_{N \geq 1} \left[ \sup_{0 \leq t \leq T} E \left( |X^N(t)|^2 \right) \right] < \infty . \quad (3.3) \]

As noted in [4], a sufficient condition for (3.2) is that
\[ E \left[ |X^N(s) - X^N(r)|^{2\beta} |X^N(t) - X^N(s)|^{2\beta} \right] \leq \|G(t) - G(r)\|^{2\alpha} . \quad (3.4) \]
We shall use this result with \( \beta = 1 \). Condition (3.2) with any \( \beta \geq 0 \) plus a minor condition, much weaker than (3.1), implies tightness of the sequence \( X^N \). Note that going from (3.1) to (3.3) is usually easy to achieve if \( X^N \) is a semimartingale, using Doob’s inequality for the martingale part. In our non Markovian setup, we do not have enough martingales at our disposal, and the above theorem will be useful to us.
Proof. We shall exploit several arguments from [4], replacing [0, 1] by [0, T]. As shown in the proof of Theorem 13.5 of [4], (3.2) implies that
\[ \mathbb{P}[w''(X^N, \delta) \geq \epsilon] \leq \frac{K_\delta}{\epsilon^{1/3}}. \]
Then
\[ \mathbb{E}[w''(X^N, \delta)^2] = \int_0^\infty \mathbb{P}(w''(X^N, \delta) > \sqrt{x})dx \]
\[ \leq \int_0^\infty 1 \wedge \frac{K_\delta}{x^{2/3}}dx \]
\[ = C_{\delta, \beta} < \infty, \tag{3.5} \]
where \( C_{\delta, \beta} \) is independent of \( N \).

Now let us connect \( w''(X^N, \delta) \) with \( \|X^N\|_\infty \). We first deduce from (12.33) in [4] that
\[ w'(X^N, \delta/2) \leq 24(w''(X^N, \delta) \lor |X^N(\delta) - X^N(0)| \lor |X^N(T^-) - X^N(T - \delta)|). \]
Finally, from line -14 page 140 of [4], if we choose a sequence \( s_j \) with \( 0 = s_0 < s_1 < \cdots < s_k = T \) and \( s_j - s_{j-1} < \delta/2 \), then
\[ \|X^N\|_\infty \leq \max_{0 \leq j \leq k} |X^N(s_j)| + w'(X^N, \delta/2) + 1. \]
Combining the two last inequalities, we see that there exists a finite subset \( t_1, \ldots, t_\ell \subset [0, T] \) and a constant \( c \) such that
\[ \|X^N\|_\infty \leq c \left( \max_{0 \leq j \leq \ell} |X^N(t_j)| + w''(X^N, \delta) \right). \]
Hence, for another constant \( c' \),
\[ \mathbb{E}\left[ \sup_{0 \leq t \leq T} |X^N(t)|^2 \right] \leq c' \left( \sup_{0 \leq t \leq T} \mathbb{E}[|X^N(t)|^2] + \mathbb{E}[w''(X^N, \delta)^2] \right). \]
The result now follows from (3.1) and (3.5). \( \square \)

We shall also use the following technical lemma.

**Lemma 3.1.** Let \( \{X^N\}_{N \geq 1} \) be a sequence of random elements in \( \mathcal{D} \) such that \( X^N(0) = 0 \). If for all \( \epsilon > 0 \), as \( \delta \to 0 \),
\[ \limsup_N \sup_{0 \leq t \leq T} \frac{1}{\delta} \mathbb{P}\left( \sup_{0 \leq u \leq \delta} |X^N(t + u) - X^N(t)| > \epsilon \right) \to 0, \]
then the sequence \( X^N \) is tight in \( \mathcal{D} \).

**Proof.** The result follows from a combination of the Corollary of Theorem 7.4, combined with (12.7), the fact that the condition implies that \( \sup_{0 \leq t \leq T} |X^N(t) - X^N(t^-)| \to 0 \), as \( N \to 0 \), Theorem 13.2 and its Corollary from [4]. \( \square \)

Let us recall well–known formulas for the exponential moment and moments of an integral with respect to a compensated PRM. These follow, e.g., rather easily from Theorem VI.2.9 and Exercise VI.2.21 in [8].

**Lemma 3.2.** Let \( Q \) be a PRM on some measurable space \( (E, \mathcal{E}) \), with mean measure \( \nu \), and \( \tilde{Q} \) the associated compensated measure. Let \( f : E \to \mathbb{C} \) be measurable and such that \( e^f - 1 - f \) is \( \nu \) integrable. Then
\[ \mathbb{E}\left[ \exp\left( \int_E f(x) \tilde{Q}(dx) \right) \right] = \exp\left( \int_E [e^{f(x)} - 1 - f(x)] \nu(dx) \right). \]
If $\nu(f^2) < \infty$,
\[
\mathbb{E} \left[ \left( \int_E f(x) \bar{Q}(dx) \right)^2 \right] = \int_E f(x)^2 \nu(dx).
\]
If $\nu(f^2 + g^2 + f^2g^2) < \infty$,
\[
\mathbb{E} \left[ \left( \int_E f(x) \bar{Q}(dx) \right)^2 \left( \int_E g(x) \bar{Q}(dx) \right)^2 \right] = \int_E f^2(x)g^2(x)\nu(dx) + 2 \left( \int_E f(x)g(x)\nu(dx) \right)^2
\]
\[
+ \int_E f^2(x)\nu(dx) \times \int_E g^2(x)\nu(dx).
\]

3.2. Representation of $(\check{S}^N, \check{J}^N)$ and proof roadmap. By the expressions of $\check{S}^N$ in (2.4) and $\check{J}$ in (2.14), we obtain
\[
\check{J}^N(t) = \hat{I}^N(0)\check{\lambda}^0(t) + \hat{E}^N(0)\check{\lambda}^0(t) + \check{\lambda}^N_{0,1}(t) + \check{\lambda}^N_{0,2}(t) + \check{\lambda}^N_1(t) + \check{\lambda}^N_2(t) + \int_0^t \check{\lambda}(t-s)\check{\Upsilon}^N(s)ds,
\]
where
\[
\check{\lambda}^N_{0,1}(t) := \frac{1}{\sqrt{N}} \sum_{k=1}^{N(0)} (\lambda^0_{k,1}(t) - \check{\lambda}^0_{k,1}(t)),
\]
\[
\check{\lambda}^N_{0,2}(t) := \frac{1}{\sqrt{N}} \sum_{j=1}^{E^N(0)} (\lambda^0_j(t) - \check{\lambda}^0_j(t)),
\]
\[
\check{\lambda}^N_1(t) := \frac{1}{\sqrt{N}} \sum_{i=1}^{A^N(t)} (\lambda_i(t - \tau^N_i) - \check{\lambda}(t - \tau^N_i)),
\]
and
\[
\check{\lambda}^N_2(t) := \frac{1}{\sqrt{N}} \left( \sum_{i=1}^{A^N(t)} \check{\lambda}(t - \tau^N_i) - \int_0^t \check{\lambda}(t-s)\check{\Upsilon}^N(s)ds \right).
\]
For the process $A^N(t)$, we have the decomposition
\[
A^N(t) = M^N_A(t) + \int_0^t \check{\Upsilon}^N(s)ds,
\]
where
\[
M^N_A(t) = \int_0^t \int_0^\infty 1_{u \leq \check{\Upsilon}^N(s-)} Q(ds, du),
\]
with $Q(ds, du) := Q(ds, du) - dsdu$ being the compensated PRM. The process $\{M^N_A(t) : t \geq 0\}$ is a square-integrable martingale with respect to the filtration $\{\mathcal{F}^N_t : t \geq 0\}$ defined by
\[
\mathcal{F}^N_t := \sigma \left\{ I^N(0), E^N(0), \lambda^0_j(\cdot)_{j \geq 1}, \lambda^0_{k,1}(\cdot)_{k \geq 1}, \lambda_i(\cdot)_{i \geq 1}, Q_{[0,t] \times \mathbb{R}_+} \right\}.
\]
It has the quadratic variation (see e.g. [8, Chapter VI])
\[
\langle M^N_A \rangle(t) = N^{-1} \int_0^t \check{\Upsilon}^N(s)ds, \quad t \geq 0.
\]
Under Assumption 2.1(i), we have
\[ 0 \leq N^{-1} \int_s^t \Upsilon^{N}(u)du \leq \lambda^{*}(t-s), \text{ w.p. 1 for } 0 \leq s \leq t. \tag{3.11} \]
It is shown in Section 4.1 of [9] that
\[ \int_0^t \bar{\Upsilon}^{N}(s)ds = \int_0^t \bar{S}(s)\bar{\Upsilon}^N(s)ds \Rightarrow \int_0^t \bar{S}(s)\bar{\Upsilon}(s)ds \text{ in } D \text{ as } N \to \infty, \tag{3.12} \]
and
\[ \bar{A}^N \Rightarrow \bar{A} = \int_0^s \bar{S}(s)\bar{\Upsilon}(s)ds \text{ in } D \text{ as } N \to \infty. \]
By (3.10), we have
\[ \bar{A}^N(t) = \sqrt{N}(A^N(t) - \bar{A}(t)) = \dot{M}^N_A(t) + \int_0^t \dot{\Upsilon}^N(s)ds, \]
where
\[ \dot{M}^N_A(t) = \frac{1}{\sqrt{N}} \int_0^t \int_0^\infty 1_{u \leq \Upsilon(t)-v} \bar{Q}(ds,du), \]
and
\[ \dot{\Upsilon}^N(t) = \sqrt{N}(\dot{S}^N(t)\bar{\Upsilon}^N(t) - \bar{S}(t)\bar{\Upsilon}(t)) = \dot{S}^N(t)\bar{\Upsilon}^N(t) + \bar{S}(t)\bar{\Upsilon}^N(t). \tag{3.13} \]
It then follows that
\[ \dot{S}^N(t) = \dot{S}^N(0) - \bar{A}^N(t) = -\bar{E}^N(0) - \bar{I}^N(0) - \dot{M}^N_A(t) - \int_0^t \dot{\Upsilon}^N(s)ds. \tag{3.14} \]

Roadmap of the proofs ahead. In what follows, we will first prove the joint convergence of \((\hat{\Upsilon}^{N}_{0,1}, \hat{\Upsilon}^{N}_{0,2})\) in Lemma 3.3, which is analogous to the proof of CLT for i.i.d. random processes, but with a variation of a random summation term \(I^N(0)\) or \(E^N(0)\). We then prove the joint convergences of \((\bar{\Upsilon}^{N}_{1}, \bar{\Upsilon}^{N}_{2})\). We observe that both processes \(\hat{\Upsilon}^{N}_{1}\) and \(\hat{\Upsilon}^{N}_{2}\) can be written as stochastic integrals with respect to the associated PRMs: \(\hat{\Upsilon}^{N}_{1}\) uses the PRM on \(\mathbb{R}^2\) \times \text{Law of } \lambda \text{ involving the law of the random infectivity function } \lambda(\cdot)\), while \(\hat{\Upsilon}^{N}_{2}\) uses the standard PRM on \(\mathbb{R}^2\). We employ the moment criterion in Theorem 13.5 of [4] to establish the tightness of these two processes which requires calculations of moments of the process increments. In order to apply the formulas in Lemma 3.2, we introduce two associated processes \(\bar{\Upsilon}^{N}_{1}\) and \(\bar{\Upsilon}^{N}_{2}\) by replacing the stochastic intensity by a deterministic function, and provide the required moment estimates in Section 3.4. The next crucial step is to prove the finiteness of the second moment of the supremum of the processes \(\hat{\Upsilon}^{N}_{1}\) and \(\hat{\Upsilon}^{N}_{2}\), which results in the same property of \(\hat{\Upsilon}^{N}\) (Lemma 3.5). For that, we employ the new criterion in Theorem 3.1 to first establish the bounds for the supremum of the processes \(\bar{\Upsilon}^{N}_{1}\) and \(\bar{\Upsilon}^{N}_{2}\) using the moment estimates of their increments in Propositions 3.1 and 3.2, and then establish the bounds for the supremum of the differences \(\hat{\Upsilon}^{N}_{1} - \hat{\Upsilon}^{N}_{1}\) and \(\bar{\Upsilon}^{N}_{2} - \bar{\Upsilon}^{N}_{2}\). This estimate of the supremum of processes in Lemma 3.5 will be used in the proof of the asymptotic equivalence of \((\hat{\Upsilon}^{N}_{1}, \hat{\Upsilon}^{N}_{2})\) and \((\bar{\Upsilon}^{N}_{1}, \bar{\Upsilon}^{N}_{2})\) in Lemmas 3.7 and 3.8, as well as the convergence of \((\bar{E}^{N}_{1}, \bar{I}^{N}_{1}, \bar{R}^{N}_{1})\) in Lemma 4.2.

3.3. Convergence of \((\hat{\Upsilon}^{N}_{0,1}, \hat{\Upsilon}^{N}_{0,2})\).

Lemma 3.3. Under Assumptions 2.1(ii) and 2.2,
\[ (\hat{\Upsilon}^{N}_{0,1}, \hat{\Upsilon}^{N}_{0,2}) \Rightarrow (\hat{\Upsilon}_{0,1}, \hat{\Upsilon}_{0,2}) \text{ in } D^2 \text{ as } N \to \infty, \tag{3.15} \]
where \((\hat{\Upsilon}_{0,1}, \hat{\Upsilon}_{0,2})\) is as given in Theorem 2.1.
Proof. Define
\[ \tilde{\mathcal{N}}_{0,1}(t) := \frac{1}{\sqrt{N}} \sum_{k=1}^{N \mathcal{I}(0)} (\lambda_k^{0,I}(t) - \bar{\lambda}(t)) \quad (3.16) \]
and
\[ \tilde{\mathcal{N}}_{0,2}(t) := \frac{1}{\sqrt{N}} \sum_{j=1}^{N \mathcal{E}(0)} (\lambda_j^0(t) - \bar{\lambda}(t)) \quad (3.17) \]
By the CLT for the random elements in \( \mathbf{D} \) (see Theorem 2 in [10], whose conditions (i) and (ii) are satisfied thanks to Assumption 2.1 (i) (a) and (b), respectively) and by the independence of the sequences \( \{\lambda_j^0\}_{j \geq 1} \) and \( \{\lambda_{k,I}^0\}_{k \geq 1} \), we obtain
\[ (\tilde{\mathcal{N}}_{0,1}, \tilde{\mathcal{N}}_{0,2}) \Rightarrow (\mathcal{N}_{0,1}, \mathcal{N}_{0,2}) \quad (3.18) \]
It then suffices to show that
\[ (\tilde{\mathcal{N}}_{0,1}, \tilde{\mathcal{N}}_{0,2}) \Rightarrow (\mathcal{N}_{0,1}, \mathcal{N}_{0,2}) \quad (3.19) \]
in probability. We focus on \( \tilde{\mathcal{N}}_{0,1} - \tilde{\mathcal{N}}_{0,2} \Rightarrow 0 \). It is clear from the definition in (3.17) and the i.i.d. property of \( \lambda_j^0(\cdot) \) that for each \( t \geq 0 \), \( \mathbb{E}[\tilde{\mathcal{N}}_{0,2}(0) - \tilde{\mathcal{N}}_{0,2}(t)] = 0 \), and
\[ \mathbb{E}[(\tilde{\mathcal{N}}_{0,2}(t) - \tilde{\mathcal{N}}_{0,2}(t)^2)] = v_0(t)\mathbb{E}[\|\tilde{E}(0) - \tilde{E}^N(0)\|] \rightarrow 0 \quad (3.20) \]
where \( v_0(t) < \infty \), and the convergence follows from Assumption 2.2 and the dominated convergence theorem. It then remains to show that \( \{\mathcal{N}_{0,2} - \mathcal{N}_{0,2} : N \in \mathbb{N}\} \) is tight in \( \mathbf{D} \). We have
\[ \tilde{\mathcal{N}}_{0,2}(t) - \tilde{\mathcal{N}}_{0,2}(t) = \text{sign}(\tilde{E}(0) - \tilde{E}^N(0)) \frac{1}{\sqrt{N}} \sum_{j=N(\tilde{E}(0) \land \tilde{E}^N(0)) + 1}^{N(\tilde{E}(0) \lor \tilde{E}^N(0))} (\check{\lambda}^0_j(t) - \bar{\lambda}(t)) \quad (3.21) \]
We use the moment criterion in Theorem 13.5 of [4], and consider the moment: for \( t' \leq t \leq t'' \),
\[ \mathbb{E}[|\tilde{\mathcal{N}}_{0,2}(t) - \tilde{\mathcal{N}}_{0,2}(t')|^2 | (\tilde{\mathcal{N}}_{0,2}(t) - \tilde{\mathcal{N}}_{0,2}(t') - \tilde{\mathcal{N}}_{0,2}(t'' - \tilde{\mathcal{N}}_{0,2}(t''))|^2] \quad (3.22) \]
Recall \( \check{\lambda}^0_j(t) = \lambda_j(t) - \bar{\lambda}(t) \), and we drop the subscript \( j \) for the generic variable \( \check{\lambda}^0(t) \). Then by the i.i.d. and mean zero properties of \( \check{\lambda}^0_j(t) \), and by the independence between \( \tilde{E}^N(0) \) and \( \check{\lambda}^0_j(t) \), we obtain that the moment above is equal to
\[ \frac{1}{N^2} \mathbb{E} \left[ \left( \sum_{j=N(\tilde{E}(0) \lor \tilde{E}^N(0)) + 1}^{N(\tilde{E}(0) \land \tilde{E}^N(0))} (\check{\lambda}^0_j(t) - \check{\lambda}^0(t')) \right)^2 \left( \sum_{j=N(\tilde{E}(0) \land \tilde{E}^N(0)) + 1}^{N(\tilde{E}(0) \lor \tilde{E}^N(0))} (\check{\lambda}^0_j(t) - \check{\lambda}^0(t'')) \right)^2 \right] \]
\[ = \frac{1}{N^2} \mathbb{E} \left[ \mathbb{E}[N\tilde{E}^N(0) - \tilde{E}(0)] \mathbb{E}[(\check{\lambda}^0(t) - \check{\lambda}^0(t'))^2(\check{\lambda}^0(t) - \check{\lambda}^0(t''))^2] \right] + \mathbb{E}[N\tilde{E}^N(0) - \tilde{E}(0)] \mathbb{E}[(\check{\lambda}^0(t) - \check{\lambda}^0(t'))^2(\check{\lambda}^0(t) - \check{\lambda}^0(t''))^2] \]
\[ + 2\mathbb{E}[N\tilde{E}^N(0) - \tilde{E}(0)] \mathbb{E}[(\check{\lambda}^0(t) - \check{\lambda}^0(t'))(\check{\lambda}^0(t) - \check{\lambda}^0(t''))^2] \]
\[ \leq \frac{1}{N^2} \mathbb{E} \left[ \mathbb{E}[N\tilde{E}^N(0) - \tilde{E}(0)] \mathbb{E}[(\check{\lambda}^0(t) - \check{\lambda}^0(t'))^2(\check{\lambda}^0(t) - \check{\lambda}^0(t''))^2] \right] + 3\mathbb{E}[N\tilde{E}^N(0) - \tilde{E}(0)] \mathbb{E}[(\check{\lambda}^0(t) - \check{\lambda}^0(t'))^2(\check{\lambda}^0(t) - \check{\lambda}^0(t''))^2] \]
\[ \leq \frac{1}{N} \mathbb{E}[\tilde{E}^N(0) - \tilde{E}(0)] (\psi(t'') - \psi(t'))^\beta + 3\mathbb{E}[\tilde{E}^N(0) - \tilde{E}(0)]^2 (\phi(t'') - \phi(t'))^{2\alpha}, \]

where we have used Cauchy-Schwarz inequality in the first inequality, and conditions (a) and (b) in Assumption 2.1 (ii) in the second inequality. By Assumption 2.2, \( \mathbb{E}[|E^N(0) - E(0)|] \to 0 \) and \( \mathbb{E}[|\tilde{E}^N(0) - \tilde{E}(0)|^2] \to 0 \) as \( N \to \infty \). Thus we conclude that the tightness of \( \{\tilde{\gamma}_{0,1}^N - \tilde{\gamma}_{0,1}^N : N \in \mathbb{N}\} \) follows from [4, Theorem 13.5]. This completes the proof. \( \square \)

3.4. **Moment estimates associated with the processes** \( \hat{\gamma}_1^N, \hat{\gamma}_2^N \) **and** \( \hat{\gamma}^N \). We first consider \( \hat{\gamma}_1^N \). We introduce a PRM \( \tilde{Q} \) on \( \mathbb{R}_+ \times D \times \mathbb{R}_+ \), which to the point \( \tau_i^N \) associates the copy \( \lambda_i \) of the random function \( \lambda \), so that the mean measure of the PRM is

\[
ds \times \text{Law of } \lambda \times du.
\]

With the notation \( \tilde{\lambda} := \lambda - \bar{\lambda} \), \( \hat{\gamma}_1^N \) can be written

\[
\hat{\gamma}_1^N(t) = N^{-1/2} \int_0^t \int_D \int_0^\infty \tilde{\lambda}(t-s)1_{u \leq \hat{\gamma}^N(s-)} \tilde{Q}(ds,d\lambda,du).
\]

(3.23)

We note that if we replace in the above \( \tilde{Q} \) by its mean measure, then the resulting integral vanishes. Consequently we also have

\[
\hat{\gamma}_1^N(t) = N^{-1/2} \int_0^t \int_D \int_0^\infty \tilde{\lambda}(t-s)1_{u \leq \hat{\gamma}^N(s-)} \tilde{Q}(ds,d\lambda,du),
\]

(3.24)

where \( \tilde{Q} \) is the compensated PRM of \( \tilde{Q} \). Hence \( \mathbb{E}[\hat{\gamma}_1^N(t)] = 0 \) and

\[
\mathbb{E}[(\hat{\gamma}_1^N(t))^2] = \mathbb{E} \left[ \int_0^t \tilde{\lambda}^2(t-s)\hat{\gamma}^N(s)ds \right].
\]

(3.25)

We will need the following bounds on the increments of the infectivity function \( \lambda(\cdot) \).

**Lemma 3.4.** For \( t \geq s \geq 0 \),

\[
|\lambda(t) - \lambda(s)| \leq \varphi(t-s) + \lambda^* \sum_{j=1}^{k} 1_{s \leq \xi_j \leq t},
\]

and

\[
|\tilde{\lambda}(t) - \tilde{\lambda}(s)| \leq \varphi(t-s) + \lambda^* \sum_{j=1}^{k} (F_j(t) - F_j(s)).
\]

Moreover,

\[
\mathbb{E}[|\lambda(t) - \lambda(s)|] \leq 2\varphi(t-s) + 2\lambda^* \sum_{j=1}^{k} (F_j(t) - F_j(s)),
\]

\[
\mathbb{E}[|\tilde{\lambda}(t) - \tilde{\lambda}(s)|^2] \leq 8\varphi(t-s)^2 + 4(\lambda^*)^2 \sum_{j=1}^{k} (F_j(t) - F_j(s))^2 + 4(\lambda^*)^2 \left( \sum_{j=1}^{k} (F_j(t) - F_j(s)) \right)^2.
\]

**Proof.** We have

\[
\lambda(t) - \lambda(s) = \sum_{j=1}^{k} (\lambda^j(t) - \lambda^j(s))1_{\xi_j \leq t} + \sum_{j=1}^{k} 1_{s \leq \xi_j \leq t}.
\]

Thus the first statement follows from Assumption 2.1 (iii). The other statements follow readily from the first one. \( \square \)
We shall also use several times the following obvious inequality: for any $1 \leq j \leq k$, $0 \leq r \leq t$,
\[
\int_0^r (F_j((t-s) - F_j(r-s))ds = \int_0^t F_j(s)ds - \int_0^r F_j(s)ds \\
\leq \int_t^r F_j(s)ds \leq t - r .
\] (3.26)

Let
\[
\tilde{F}_N^N(t) := N^{-1/2} \int_0^t \int_D \int_0^\infty \hat{\lambda}(t-s)1_{u \leq \lambda N^\delta(s)}Q(ds, d\lambda, du).
\] (3.27)

In the next proposition, we prove the moment bound for the increments of $\tilde{F}_N^N$, which will be used in the proof of its convergence. Note that by (3.11) and (3.12), we have $\hat{Y}(t) \leq \lambda^\ast$ for any $t \geq 0$.

**Proposition 3.1.** There exist $C > 0$ and $\beta > 0$ such that for any $r < t < v$,
\[
\mathbb{E} \left[ \left| \tilde{F}_N^N(t) - \tilde{F}_N^N(r) \right|^2 \tilde{F}_N^N(t) - \tilde{F}_N^N(v) \right|^2 \right] \leq C(v-r)^{1+\beta}.
\] (3.28)

**Proof.** We deduce from (3.6) the following identity
\[
\mathbb{E} \left[ \left| \tilde{F}_N^N(t) - \tilde{F}_N^N(r) \right|^2 \tilde{F}_N^N(t) - \tilde{F}_N^N(v) \right|^2 \right] = \frac{1}{N} \mathbb{E} \int_r^t \tilde{\lambda}^2(t-s) \left[ \tilde{\lambda}(v-s) - \tilde{\lambda}(t-s) \right]^2 \tilde{Y}(s)ds \\\n+ \frac{1}{N} \mathbb{E} \int_0^r \left[ \tilde{\lambda}(t-s) - \tilde{\lambda}(r-s) \right]^2 \left[ \tilde{\lambda}(v-s) - \tilde{\lambda}(t-s) \right]^2 \tilde{Y}(s)ds \\\n+ 2 \left( \frac{1}{N} \mathbb{E} \int_r^t \tilde{\lambda}(t-s) \left[ \tilde{\lambda}(v-s) - \tilde{\lambda}(t-s) \right] \tilde{Y}(s)ds \right) \\\n+ \frac{1}{N} \mathbb{E} \int_0^r \left[ \tilde{\lambda}(t-s) - \tilde{\lambda}(r-s) \right] \left[ \tilde{\lambda}(v-s) - \tilde{\lambda}(t-s) \right] \tilde{Y}(s)ds \\\n+ \left( \frac{1}{N} \mathbb{E} \int_r^t \tilde{\lambda}^2(t-s) \tilde{Y}(s)ds + \frac{1}{N} \mathbb{E} \int_0^r \left[ \tilde{\lambda}(t-s) - \tilde{\lambda}(r-s) \right]^2 \tilde{Y}(s)ds \right) \\\n\times \left( \frac{1}{N} \mathbb{E} \int_t^v \tilde{\lambda}^2(v-s) \tilde{Y}(s)ds + \frac{1}{N} \mathbb{E} \int_t^r \left[ \tilde{\lambda}(v-s) - \tilde{\lambda}(t-s) \right] \tilde{Y}(s)ds \right).
\] (3.29)

We now bound each of the four terms on the right of (3.29). It follows from Lemma 3.4 and (2.8), (2.9) and (2.10) in Assumption 2.1 that for some constant $C$, the first term on the right of (3.29) is bounded by
\[
C \frac{(\lambda^\ast)^3}{N} (t-r)((v-t)^{2\alpha} + (v-t)^{\beta}) \leq C(v-r)^{1+\rho}(2\alpha).
\]

We next consider the third term. The absolute value of the first summand in the square is clearly bounded by $(\lambda^\ast)^3(t-r)$. The second summand is bounded by $\lambda^\ast r \varphi(t-r)\varphi(v-t)$, plus
\[
C \sum_{j=1}^k \int_0^t (F_j(t-s) - F_j(r-s) + F_j(v-s) - F_j(t-s))ds \leq 2Ck[(t-r) + (v-t)],
\]
where we have used twice (3.26). Finally the third term is bounded by a constant times
\[
\varphi^{4\alpha}(v-r) + (v-r)^2 = 2(v-r)^{1+(4\alpha-1)/1}.
\]

By similar arguments, the first factor in the last term is bounded by a constant times
\[
t - r + \varphi^{2\alpha}(t-r) + \sum_{j=1}^k \int_0^t [F_j(t-s) - F_j(r-s)]ds \leq t-r + \varphi^{2\alpha}(t-r) + k(t-r).
\]
Since the second factor can be estimated in the same way, with however \( t - r \) replaced by \( v - t \), we conclude that the last term is again bounded by a constant times \( (v - r)^2 + \varphi^{4\alpha}(v - r) \).

It remains to consider the second term, which is a bit more delicate. We disregard the factor \( 1/N \). Define \( A_i(s) = \{ r - s < \xi^i \leq t - s \} \), \( B_j(s) = \{ t - s < \xi^j \leq v - s \} \), and note that \( \mathbb{P}(A_i(s)) = F_i(t - s) - F_i(r - s) \), \( \mathbb{P}(B_j(s)) = F_j(v - s) - F_j(t - s) \). We disregard the factor \( 1/N \).

The integrand in the integral from \( s = 0 \) to \( s = r \) is bounded from above by a constant times

\[
\left[ \varphi^{2\alpha}(t - r) + \sum_{i=1}^{k} (1_{A_i(s)} - \mathbb{P}(A_i(s)))^2 \right] \times \left[ \varphi^{2\alpha}(v - t) + \sum_{i=1}^{k} (1_{B_j(s)} - \mathbb{P}(B_j(s)))^2 \right]
\]

\[
\leq \varphi^{4\alpha}(v - r) + \varphi^{2\alpha}(v - r) \sum_{i=1}^{k} (1_{A_i(s)} - \mathbb{P}(A_i(s)))^2
\]

\[
+ \varphi^{2\alpha}(t - r) \sum_{i=1}^{k} (1_{B_j(s)} - \mathbb{P}(B_j(s)))^2
\]

\[
+ \sum_{i,j=1}^{k} (1_{A_i(s)} - \mathbb{P}(A_i(s)))^2 (1_{B_j(s)} - \mathbb{P}(B_j(s)))^2
\]

The first term on the right hand side of the last inequality is bounded by \( (v - r)^{4\alpha} \). The expectation of the second term is bounded by \( \varphi^{2\alpha}(t - r)\mathbb{P}(A_i(s)) \), whose integral from \( s = 0 \) to \( s = r \) is bounded by \( (t - r)\varphi^{2\alpha}(v - r) \), and the next term is treated exactly in the same way. We finally treat the last term. We note that

\[
(1_{A_i(s)} - \mathbb{P}(A_i(s)))^2 (1_{B_j(s)} - \mathbb{P}(B_j(s)))^2
\]

is the square of a real number which is between \(-1\) and \(+1\), hence it is bounded by its absolute value. Consequently

\[
\mathbb{E} \left\{ (1_{A_i(s)} - \mathbb{P}(A_i(s)))^2 (1_{B_j(s)} - \mathbb{P}(B_j(s)))^2 \right\}
\]

\[
\leq \mathbb{E} \left( 1_{A_i(s) \cap B_j(s)} + 1_{A_i(s)}\mathbb{P}(B_j(s)) + 1_{B_j(s)}\mathbb{P}(A_i(s)) + \mathbb{P}(A_i(s))\mathbb{P}(B_j(s)) \right)
\]

\[
= \mathbb{P}(A_i(s) \cap B_j(s)) + 3\mathbb{P}(A_i(s))\mathbb{P}(B_j(s))
\]

It remains to bound each of those two last terms. Note that \( \mathbb{P}(A_i(s) \cap B_j(s)) \neq 0 \) only if \( i < j \), and in that case

\[
\mathbb{P}(A_i(s) \cap B_j(s)) \leq \mathbb{P}(A_i(s)) \times \mathbb{P}(B_j(s)|A_i(s))
\]

\[
\leq \mathbb{P}(A_i(s)) \times \sup_u \mathbb{P}(\xi_{i+1} - \xi_i \leq v - r|\xi_i = u)
\]

\[
\leq C'(v - r)^\alpha \mathbb{P}(A_i(s)),
\]

thanks to (2.11) in Assumption 2.1. Since by (3.26), \( \int_0^t \mathbb{P}(A_i(s))ds \leq t - r \), the integral from \( s = 0 \) to \( s = r \) of the first term has the right bound. The second term is bounded by the same expression, using this time (2.10) instead of (2.11).

Concerning the process \( \tilde{\mathcal{J}}_2^N(t) \), we can represent it as

\[
\tilde{\mathcal{J}}_2^N(t) := \frac{1}{\sqrt{N}} \int_0^t \int_0^\infty \tilde{\lambda}(t - s) 1_{u \leq \mathbb{I}^{N(s-v),l}_{\mathbb{I}^N(\mathbb{I}^{N(s-v)}\mathbb{Q}(ds, du))}. \tag{3.30}
\]

Define

\[
\tilde{\mathcal{J}}_2^N(t) := \frac{1}{\sqrt{N}} \int_0^t \int_0^\infty \tilde{\lambda}(t - s) 1_{u \leq \mathbb{I}^{N(s-v),l}_{\mathbb{I}^N(\mathbb{I}^{N(s-v)}\mathbb{Q}(ds, du))}. \tag{3.31}
\]

In the next proposition, we prove the moment bound for the increments of \( \tilde{\mathcal{J}}_2^N \).
Proposition 3.2. There exist $C > 0$ and $\beta > 0$ such that for any $r < t < v$,
\[ \mathbb{E}\left[ |\tilde{J}_2^N(t) - \tilde{J}_2^N(r)|^2 |\tilde{J}_2^N(t) - \tilde{J}_2^N(v)|^2 \right] \leq C(v-r)^{1+\beta}. \quad (3.32) \]

Proof. As in the proof of Proposition 3.2, we exploit (3.6). We obtain
\[
\mathbb{E}\left[ |\tilde{J}_2^N(t) - \tilde{J}_2^N(r)|^2 |\tilde{J}_2^N(t) - \tilde{J}_2^N(v)|^2 \right]
= \frac{1}{N} \int_r^t \lambda^2(t-s) \left[ \bar{\lambda}(v-s) - \bar{\lambda}(t-s) \right]^2 \Upsilon(s) ds
+ \frac{1}{N} \int_r^T \left[ \bar{\lambda}(t-s) - \bar{\lambda}(r-s) \right]^2 \left[ \bar{\lambda}(v-s) - \bar{\lambda}(t-s) \right]^2 \Upsilon(s) ds
+ 2 \left( \int_r^t \bar{\lambda}(t-s) \left[ \bar{\lambda}(v-s) - \bar{\lambda}(t-s) \right] \Upsilon(s) ds \right)
+ \left( \int_r^T \bar{\lambda}(t-s) \Upsilon(s) ds + \int_r^r \left[ \bar{\lambda}(t-s) - \bar{\lambda}(r-s) \right]^2 \Upsilon(s) ds \right)
\times \left( \int_r^T \bar{\lambda}(v-s) \Upsilon(s) ds + \int_r^t \left[ \bar{\lambda}(v-s) - \bar{\lambda}(t-s) \right]^2 \Upsilon(s) ds \right). \quad (3.33) \]

For the first term on the right hand side, by Lemma 3.4 as well as the assumptions in (2.8), (2.9) and (2.10), we obtain the upper bound: for some constant $C > 0$,
\[ C \frac{(\lambda^*)^3}{N} (t-r) \left[ \varphi^2(v-t) + (v-t)^{2\rho} \right] \lesssim C(v-r)^{1+2(\alpha \wedge \rho)}. \]

For the second term, we obtain the upper bound
\[ CT \frac{\lambda^*}{N} \varphi^4(t-r) + (\lambda^*)^4(t-r) \rho \sum_{j=1}^k \int_0^r (F_j(t-s) - F_j(r-s)) ds \]
\[ \lesssim C(v-r)^{4\alpha \wedge (1+\rho)}, \]
where we have used (2.10) and (3.26). For the third term, we obtain
\[ C \left( t-r + \varphi^2(t-r) + t-r \right)^2 \lesssim C(v-r)^{4\alpha \wedge 2} \]
Finally, we get the following bound for the last term,
\[ \left( (\lambda^*)^3(t-r) + \lambda^* TC \left[ \varphi^2(t-r) + t-r \right] \right) \times \left( (\lambda^*)^3(v-t) + \lambda^* TC \left[ \varphi^2(v-t) + v-t \right] \right) \]
\[ \leq \left( (\lambda^*)^3(v-r) + \lambda^* TC \left[ \varphi^2(v-r) + v-r \right] \right)^2 \]
\[ \lesssim C(v-r)^{4\alpha \wedge 2}. \]
Therefore we have obtained the desired upper bound, with $\beta = (4\alpha - 1) \wedge \rho \wedge 1 \wedge 2\alpha$. \(\square\)

Lemma 3.5. For any $T > 0$, the following hold
\[ \sup_N \mathbb{E} \left[ \sup_{0 \leq t \leq T} \tilde{S}(t)^2 \right] < \infty, \quad \text{and} \quad \sup_N \mathbb{E} \left[ \sup_{0 \leq t \leq T} \tilde{\gamma}(t)^2 \right] < \infty, \quad (3.34) \]
and consequently,
\[ \sup_N \mathbb{E} \left[ \sup_{t \in [0,T]} \tilde{\gamma}^N(t)^2 \right] < \infty. \quad (3.35) \]
Proof. We first show that for any $T > 0$,
\[
\sup_N \sup_{0 \leq t \leq T} \mathbb{E}[\hat{S}(t)^2] < \infty, \quad \text{and} \quad \sup_N \sup_{0 \leq t \leq T} \mathbb{E}[\hat{J}(t)^2] < \infty, \quad (3.36)
\]
which combined with (3.13) implies that
\[
\sup_N \sup_{t \in [0, T]} \mathbb{E}[\hat{T}^N(t)^2] < \infty. \quad (3.37)
\]
We shall use (3.13) and the two integral representations in (3.7) and (3.14). We first obtain the
moment property for \( \hat{\bar{A}} \). For the processes \( \hat{\bar{N}}_1 \) and \( \hat{\bar{N}}_2 \), we deduce from a computation similar to the one leading to (3.22) and Theorem 3.1 that
\[
\sup_N \sup_{t \in [0, T]} \mathbb{E}[(\hat{\bar{N}}(0)\bar{X}^0(t))^2] \leq (\lambda^*)^2 C, \quad \text{and} \quad \sup_N \sup_{t \in [0, T]} \mathbb{E}[(\hat{\bar{N}}(0)\bar{X}(t))^2] \leq (\lambda^*)^2 C.
\]
It is also clear that
\[
\sup_N \sup_{t \in [0, T]} \mathbb{E}[(\hat{\bar{N}}_1(t))^2] = \sup_N \sup_{t \in [0, T]} \mathbb{E}[\hat{I}(0)\bar{X}(t)] \leq \sup_{t \in [0, T]} v(t) \leq \infty,
\]
\[
\sup_N \sup_{t \in [0, T]} \mathbb{E}[(\hat{\bar{N}}_2(t))^2] = \sup_N \sup_{t \in [0, T]} \mathbb{E}[\bar{X}(0)] \leq \sup_{t \in [0, T]} v(t) \leq \infty.
\]
By (3.25) and (3.11), we have
\[
\sup_{t \in [0, T]} \mathbb{E}[\hat{\bar{N}}_1(t)^2] = \sup_{t \in [0, T]} \mathbb{E}\left[\int_0^t \bar{X}^2(t-s)\bar{Y}(s)ds\right] \leq \lambda^* \int_0^T v(s)ds < \infty,
\]
and again by (3.11), we obtain
\[
\sup_{t \in [0, T]} \mathbb{E}[\hat{\bar{N}}_2(t)^2] = \sup_{t \in [0, T]} \mathbb{E}\left[\int_0^t \bar{X}(t-s)^2\bar{Y}(s)ds\right] \leq (\lambda^*)^2 T.
\]
Combining (3.7), (3.13) and (3.14) with the simple bounds \( \hat{S}(t) \leq 1 \) and \( \hat{\bar{N}}(t) \leq \lambda^*(\hat{\bar{N}}(0) + \hat{\bar{A}}(t)) \leq 2\lambda^* \), and Gronwall’s inequality, we obtain the claims in (3.36).

We next prove (3.34). By Doob’s maximal inequality, we have
\[
\sup_N \mathbb{E}\left[\sup_{t \in [0, T]} |\hat{\bar{A}}_N(t)|^2 \right] \leq 4\sup_N \mathbb{E}\left[|\hat{\bar{A}}_N(T)|^2 \right] \leq 4\lambda^* T,
\]
and then by (3.14) and (3.37), and by applying Gronwall’s inequality, we obtain the moment property for \( \hat{S}^N(t) \) in (3.34) holds.

For \( \hat{\bar{N}}(t) \), we first consider the two processes \( \hat{\bar{N}}_0(t) \) and \( \hat{\bar{N}}_2(t) \), which can be treated in the same way, so we focus on \( \hat{\bar{N}}_2(t) \) as in the proof of Lemma 3.3. Recall \( \hat{\bar{N}}_2(t) \) in (3.17). Under Assumption 2.1 (ii), we deduce from a computation similar to the one leading to (3.22) and Theorem 3.1 that
\[
\sup_N \mathbb{E}\left[\sup_{t \in [0, T]} \hat{\bar{N}}_2(t)^2 \right] < \infty. \quad (3.38)
\]
We next consider \( \hat{\bar{N}}_0(t) - \hat{\bar{N}}_2(t) \) which is given in (3.20). By (3.21) and (3.22), applying again Theorem 3.1, we obtain that the moment property in (3.38) holds for \( \hat{\bar{N}}_0(t) - \hat{\bar{N}}_2(t) \). Combining these two moment estimates, we deduce that the moment property in (3.38) holds for \( \hat{\bar{N}}_0(t) \).

For the processes \( \hat{\bar{N}}(t) \), we write it as \( \hat{\bar{N}}(t) = \hat{\bar{N}}_1(t) + (\hat{\bar{N}}(t) - \hat{\bar{N}}_1(t)) \) and treat the decomposed terms separately. By Proposition 3.1, applying Theorem 3.1, we obtain that the moment property in (3.38) holds for \( \hat{\bar{N}}_1(t) \). Now for the difference \( \hat{\bar{N}}_1(t) - \hat{\bar{N}}_1(t) \), we have
\[
\hat{\bar{N}}_1(t) - \hat{\bar{N}}_1(t) = \frac{1}{\sqrt{N}} \int_0^t \int_D \int_{\{\bar{Y}(s) \wedge \bar{Y}(t)\}} \lambda(t-s)\text{sign}(\bar{Y}(s) - \bar{Y}(t))\hat{Q}(ds, d\lambda, du). \quad (3.39)
\]
Observe that
\[
\left| \hat{N}^N(t) - \tilde{N}^N(t) \right| \leq \frac{1}{\sqrt{N}} \lambda^* \int^t_0 \int_D \int_{N(T^N(s) \lor T(s))} Q(ds, d\lambda, du),
\]
is increasing in \( t \). Thus,
\[
\mathbb{E} \left[ \sup_{t \in [0, T]} \left| \hat{N}^N(t) - \tilde{N}^N(t) \right| \right] \leq \mathbb{E} \left[ \left( \frac{1}{\sqrt{N}} \lambda^* \int^T_0 \int_D \int_{N(T^N(s) \lor T(s))} Q(ds, d\lambda, du) \right)^2 \right] = (\lambda^*)^2 \int^T_0 \left| \bar{T}^N(s) - \bar{T}(s) \right| ds \to 0 \quad \text{as} \quad N \to \infty.
\]
Combining the above arguments, we obtain that the moment property in (3.38) holds for \( \hat{N}^N(t) \).

Similarly, we write \( \hat{N}^N(t) = \hat{N}^N_2(t) + (\hat{N}^N_1(t) - \hat{N}^N_2(t)) \) and treat the decomposed terms separately. We obtain the moment property in (3.38) for \( \hat{N}^N_2(t) \) by Proposition 3.2 and Theorem 3.1. For the difference \( \hat{N}^N_2(t) - \tilde{N}^N_2(t) \), we have
\[
\hat{N}^N_2(t) - \tilde{N}^N_2(t) = \frac{1}{\sqrt{N}} \int^t_0 \int_{N(T^N(s) \lor T(s))} \lambda(t-s) \text{sign}(\bar{T}^N(s) - \bar{T}(s))Q(ds, du)
- \frac{1}{\sqrt{N}} \int^t_0 \lambda(t-s)(\bar{T}^N(s) - \bar{T}(s))ds.
\]
Observe that
\[
\left| \hat{N}^N_2(t) - \tilde{N}^N_2(t) \right| \leq \frac{1}{\sqrt{N}} \lambda^* \int^t_0 \int_{N(T^N(s) \lor T(s))} Q(ds, du) + \sqrt{N} \lambda^* \int^t_0 \left| \bar{T}^N(s) - \bar{T}(s) \right| ds.
\]
Both terms on the right hand side are increasing in \( t \). Thus,
\[
\mathbb{E} \left[ \sup_{t \in [0, T]} \left| \hat{N}^N_2(t) - \tilde{N}^N_2(t) \right| \right] \leq 2 \mathbb{E} \left[ \left( \frac{1}{\sqrt{N}} \lambda^* \int^T_0 \int_{N(T^N(s) \lor T(s))} Q(ds, du) \right)^2 \right] + 2 \mathbb{E} \left[ \left( \sqrt{N} \lambda^* \int^T_0 \left| \bar{T}^N(s) - \bar{T}(s) \right| ds \right)^2 \right] = 2(\lambda^*)^2 \left( \int^T_0 \left| \bar{T}^N(s) - \bar{T}(s) \right| ds + \mathbb{E} \left[ \left( \int^T_0 \left| \bar{T}^N(s) \right|^2 ds \right) \right] \right).
\]

Here the first integral converges to zero as \( N \to \infty \), and the second term is bounded by (3.37). Thus combining these, we have shown that the moment property in (3.38) holds for \( \hat{N}^N_2(t) \). Finally, combining (3.7) with the above estimates yields that \( \hat{N}^N \) satisfies (3.34).

3.5. Joint convergence of \( \hat{N}^N \) and \( \hat{N}^N_2 \). In this subsection, we will show that \( (\hat{N}^N, \hat{N}^N_2) \Rightarrow (\hat{1}, \hat{2}) \) in \( D^2 \), as \( N \to \infty \). We shall first show that \( (\hat{N}^N_1, \hat{N}^N_2) \Rightarrow (\hat{1}, \hat{2}) \) in \( D^2 \), as \( N \to \infty \). Given the results in Proposition 3.1 and 3.2, Theorem 13.5 from [4] tells us that it remains to prove that the finite dimensional distributions of \( (\hat{N}^N_1, \hat{N}^N_2) \) converge to those \( (\hat{1}, \hat{2}) \), which we establish in the first Lemma which follows. It will then remain to prove that \( \hat{N}^N_1 - \tilde{N}^N_1 \to 0 \) and \( \hat{N}^N_2 - \tilde{N}^N_2 \to 0 \) in \( D \) in probability, as \( N \to \infty \) which will be done in the next two Lemmas.

**Lemma 3.6.** For any \( k \geq 1 \), \( 0 < t_1 < t_2 < \cdots < t_k \), as \( N \to \infty \),
\[
\left( (\hat{N}^N(t_1), \hat{N}^N_2(t_1)), \ldots, (\hat{N}^N(t_k), \hat{N}^N_2(t_k)) \right) \Rightarrow \left( (\hat{1}(t_1), \hat{2}(t_1)), \ldots, (\hat{1}(t_k), \hat{2}(t_k)) \right)
\]
in \( \mathbb{R}^{2k} \), where \( (\hat{1}, \hat{2}) \) is given in Theorem 2.1.
Proof. Recall (3.27) and (3.31). Note that the process $\tilde{\mathcal{F}}^N_2(t)$ can be equivalently written as follows using $\bar{Q}$:

$$\tilde{\mathcal{F}}^N_2(t) = \frac{1}{\sqrt{N}} \int_0^t \int_{\mathcal{D}} \int_0^\infty \bar{\lambda}(t-s) 1_{t \leq N\bar{T}(s)} \bar{Q}(ds,d\lambda,du).$$

(3.42)

In order to simplify the notations, we start with the case $k = 2$. For any $\theta_1, \theta_2, \theta'_1, \theta'_2 \in \mathbb{R}$, $t, t' > 0$, we compute the limit as $N \to \infty$ of

$$\mathbb{E}\left[ \exp \left( i\theta_1 \tilde{\mathcal{F}}^N_1(t) + i\theta_2 \tilde{\mathcal{F}}^N_2(t) + i\theta'_1 \tilde{\mathcal{F}}^N_1(t') + i\theta'_2 \tilde{\mathcal{F}}^N_2(t') \right) \right].$$

We apply Lemma 3.2 to the particular case $E = \mathbb{R}_+ \times \mathcal{D} \times \mathbb{R}_+$, $Q = \bar{Q}$ and

$$f(s, \lambda, u) = iN^{-1/2} \left\{ [\theta_1 \bar{\lambda}(t-s) + \theta_2 \bar{\lambda}(t-s)] 1_{s \leq t} + [\theta'_1 \bar{\lambda}(t'-s) + \theta'_2 \bar{\lambda}(t'-s)] 1_{s \leq t'} \right\} 1_{t \leq N\bar{T}(s)},$$

from which we easily deduce that

$$\lim_{N \to \infty} \mathbb{E}\left[ \exp \left( i\theta_1 \tilde{\mathcal{F}}^N_1(t) + i\theta_2 \tilde{\mathcal{F}}^N_2(t) + i\theta'_1 \tilde{\mathcal{F}}^N_1(t') + i\theta'_2 \tilde{\mathcal{F}}^N_2(t') \right) \right] = \exp \left( -\frac{(\theta_1)^2}{2} \int_0^t \mathbb{E}[\bar{\lambda}^2(t-s)] \bar{Y}(s)ds - \frac{(\theta_2)^2}{2} \int_0^t \bar{\lambda}^2(t-s) \bar{Y}(s)ds 
\right.

$$

$$- \frac{(\theta'_1)^2}{2} \int_0^{t'} \mathbb{E}[\bar{\lambda}^2(t'-s)] \bar{Y}(s)ds - \frac{(\theta'_2)^2}{2} \int_0^{t'} \bar{\lambda}^2(t'-s) \bar{Y}(s)ds 

- \theta_1 \theta'_1 \int_0^{t \wedge t'} \mathbb{E}[\bar{\lambda}(t-s)\bar{\lambda}(t'-s)] \bar{Y}(s)ds - \theta_2 \theta'_2 \int_0^{t \wedge t'} \bar{\lambda}(t-s)\bar{\lambda}(t'-s) \bar{Y}(s)ds \right).$$

(3.43)

We have proved that the two dimensional distributions of $(\tilde{\mathcal{F}}^N_1, \tilde{\mathcal{F}}^N_2)$ converge to those of a centered Gaussian process, whose covariances are the ones of $(\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2)$ as given in Theorem 2.1. Note that it is immediate that the $k$ dimensional distributions also converge to those of a centered Gaussian process, and the covariances are fully determined by the above computation, hence the result. □

**Lemma 3.7.** Under Assumptions 2.1(i) and (iii) and 2.2,

$$\tilde{\mathcal{F}}^N_1 - \tilde{\mathcal{F}}^N_1 \to 0$$

in probability in $\mathcal{D}$ as $N \to \infty$.

**Proof.** Recall the expression of $\tilde{\mathcal{F}}^N_1(t) - \tilde{\mathcal{F}}^N_1(t)$ in (3.39). It is straightforward that $\mathbb{E}[\tilde{\mathcal{F}}^N_1(t) - \tilde{\mathcal{F}}^N_1(t)] = 0$, and

$$\mathbb{E}\left[ (\tilde{\mathcal{F}}^N_1(t) - \tilde{\mathcal{F}}^N_1(t))^2 \right] = \mathbb{E}\left[ \int_0^t \bar{\lambda}(t-s)^2|\bar{Y}^N(s) - \bar{Y}(s)|ds \right]$$

$$\leq (\lambda^*)^2 \int_0^t \mathbb{E}[|\bar{Y}^N(s) - \bar{Y}(s)|] ds \to 0 \text{ as } N \to \infty.$$

Here the convergence follows from

$$\mathbb{E}[|\bar{Y}^N(s) - \bar{Y}(s)|] \to 0 \text{ as } N \to \infty,$$

(3.44)

which holds by (3.12) and the dominated convergence theorem. It then suffices to show the tightness of $(\tilde{\mathcal{F}}^N_1 - \tilde{\mathcal{F}}^N_1 : N \in \mathbb{N})$. By the expression in (3.39), it suffices to show the tightness of the processes $(\Xi^N : N \in \mathbb{N})$ defined by

$$\Xi^N(t) := \frac{1}{\sqrt{N}} \int_0^t \int_{\mathcal{D}} \int_{N(\bar{Y}^N(s) \wedge \bar{Y}(s))} |\bar{\lambda}(t-s)| \bar{Q}(ds,d\lambda,du).$$
By Lemma 3.1, it suffices to show that
\[
\limsup_{N \to \infty} \frac{1}{\delta} \mathbb{P}\left( \sup_{v \in [0, \delta]} |\Xi^N(t + v) - \Xi^N(t)| > \epsilon \right) \to 0 \quad \text{as} \quad \delta \to 0. \tag{3.45}
\]

We have
\[
|\Xi^N(t + v) - \Xi^N(t)| \\
\leq \frac{1}{\sqrt{N}} \int_t^{t+\delta} \int_D \int_{N(T^N(s) \cup \tilde{T}(s))} |\hat{X}(t + v - s)| \mathcal{Q}(ds, d\lambda, du) \\
+ \frac{1}{\sqrt{N}} \int_0^t \int_D \int_{N(T^N(s) \cup \tilde{T}(s))} |\hat{X}(t + v - s) - \hat{X}(t - s)| \mathcal{Q}(ds, d\lambda, du) \\
\leq \frac{\lambda^*}{\sqrt{N}} \int_t^{t+\delta} \int_{N(T^N(s) \cup \tilde{T}(s))} \mathcal{Q}(ds, du) \\
+ \frac{1}{\sqrt{N}} \int_0^t \int_D \int_{N(T^N(s) \cup \tilde{T}(s))} \left( 2\varphi(v) + \lambda^* \sum_{j=1}^k 1_{t-s < \xi_j \leq t+v-s} \right) \mathcal{Q}(ds, d\lambda, du),
\]
where the second inequality follows from Lemma 3.4. It is clear that the above upper bound is increasing in \(v\). Thus, we obtain that for any \(\epsilon > 0\),
\[
\mathbb{P}\left( \sup_{v \in [0, \delta]} |\Xi^N(t + v) - \Xi^N(t)| > \epsilon \right) \\
\leq \mathbb{P}\left( \frac{\lambda^*}{\sqrt{N}} \int_t^{t+\delta} \int_{N(T^N(s) \cup \tilde{T}(s))} \mathcal{Q}(ds, du) > \epsilon/4 \right) \\
+ \mathbb{P}\left( \frac{2\varphi(\delta)}{\sqrt{N}} \int_0^t \int_D \int_{N(T^N(s) \cup \tilde{T}(s))} \mathcal{Q}(ds, du) > \epsilon/4 \right) \\
+ \mathbb{P}\left( \frac{1}{\sqrt{N}} \int_0^t \int_D \int_{N(T^N(s) \cup \tilde{T}(s))} \left( \lambda^* \sum_{j=1}^k 1_{t-s < \xi_j \leq t+\delta-s} \right) \mathcal{Q}(ds, d\lambda, du) > \epsilon/4 \right) \\
+ \mathbb{P}\left( \frac{1}{\sqrt{N}} \int_0^t \int_D \int_{N(T^N(s) \cup \tilde{T}(s))} \left( \lambda^* \sum_{j=1}^k (F_j(t + \delta - s) - F_j(t - s)) \right) \mathcal{Q}(ds, d\lambda, du) > \epsilon/4 \right). \tag{3.46}
\]
The first term is bounded by \(\frac{16}{\epsilon^2}\) times
\[
\mathbb{E}\left[ \left( \frac{\lambda^*}{\sqrt{N}} \int_t^{t+\delta} \int_{N(T^N(s) \cup \tilde{T}(s))} \mathcal{Q}(ds, du) \right)^2 \right] \\
\leq 2 \left\{ \mathbb{E} \left[ \int_t^{t+\delta} |\tilde{T}(s) - \tilde{T}(s)| ds + \left( \mathbb{E} \left[ \int_t^{t+\delta} |\tilde{T}(s)| ds \right] \right)^2 \right]\right\} \\
\leq 2 \left\{ \delta \sup_{s \leq T} \mathbb{E}[|\tilde{T}(s) - \tilde{T}(s)|] + \delta^2 \sup_{s \leq T} \mathbb{E}\left( |\tilde{T}(s)|^2 \right) \right\}, \tag{3.47}
\]
where the first inequality follows from $Q(ds, du) = \overline{Q}(ds, du) + ds \times du$. We note that the first term on the right of (3.47) tends to 0 as $N \to \infty$, while the $\limsup_N$ of the second term multiplied by $\delta^{-1}$ tends to 0, as $\delta \to 0$, by the moment property of $\hat{\Upsilon}^N$ in Lemma 3.5, which is exactly what we want.

The second term is bounded by $\frac{16}{\sqrt{T}}$ times
\[
E \left[ \left( \frac{2}{\sqrt{N}} \varphi(\delta) \int_0^t \int_{N(\hat{\Upsilon}^N(s) \cup \Upsilon(s))} Q(ds, du) \right)^2 \right]
\]
\[
\leq 2(2\varphi(\delta))^2 \int_0^t E|\hat{\Upsilon}^N(s) - \Upsilon(s)| ds + 2(2\varphi(\delta))^2 T E \sup_{s \in [0,T]} |\hat{\Upsilon}^N(s)|^2.
\]

By (2.12), the first term converges to zero as $N \to \infty$, while, thanks to our assumption (2.8) and Lemma 3.5, the $\limsup_N$ of the second term multiplied by $\delta^{-1}$ tends to 0, as $\delta \to 0$, which again is exactly what we want.

The third term on the right hand side of (3.46) is bounded by $\frac{16}{\sqrt{T}}$ times
\[
E \left[ \left( \frac{1}{\sqrt{N}} \int_0^t \int_D D_{N(\hat{\Upsilon}^N(s) \cup \Upsilon(s))} \left( \lambda^* \sum_{j=1}^k \mathbf{1}_{t-s < \xi_j \leq t+\delta-s} \right) \tilde{Q}(ds, d\lambda, du) \right)^2 \right]
\]
\[
\leq 2E \left[ \left( \frac{1}{\sqrt{N}} \int_0^t \int_D D_{N(\hat{\Upsilon}^N(s) \cup \Upsilon(s))} \left( \lambda^* \sum_{j=1}^k \mathbf{1}_{t-s < \xi_j \leq t+\delta-s} \right) \tilde{Q}(ds, d\lambda, du) \right)^2 \right]
\]
\[
+ 2(\lambda^*)^2 E \left[ \left( \int_0^t \left( \sum_{j=1}^k (F_j(t+\delta-s) - F_j(t-s)) \right) |\hat{\Upsilon}^N(s)| ds \right)^2 \right].
\]

Here the first term is equal to twice
\[
\int_0^t E \left[ \left( \lambda^* \sum_{j=1}^k \mathbf{1}_{t-s < \xi_j \leq t+\delta-s} \right)^2 |\hat{\Upsilon}^N(s) - \Upsilon(s)| \right] ds
\]
\[
\leq (\lambda^*)^2 k^2 \int_0^t E[|\hat{\Upsilon}^N(s) - \Upsilon(s)|] ds,
\]
which converges to zero as $N \to \infty$ by (2.12). The second term satisfies, thanks to (3.26) and Lemma 3.5,
\[
\frac{1}{\delta} E \left[ \left( \int_0^t \left( \sum_{j=1}^k (F_j(t+\delta-s) - F_j(t-s)) \right) |\hat{\Upsilon}^N(s)| ds \right)^2 \right]
\]
\[
\leq \frac{k^2}{\delta} \sum_{j=1}^k \left( \int_0^t (F_j(t+\delta-s) - F_j(t-s)) ds \right)^2 E \sup_{s \in [0,T]} |\hat{\Upsilon}^N(s)|^2
\]
\[
\leq C \delta.
\]

(3.48)

The fourth and last term on the right hand side of (3.46) is bounded by $\frac{16}{\sqrt{T}}$ times
\[
E \left[ \left( \frac{1}{\sqrt{N}} \int_0^t \int_D D_{N(\hat{\Upsilon}^N(s) \cup \Upsilon(s))} \left( \lambda^* \sum_{j=1}^k (F_j(t+\delta-s) - F_j(t-s)) \right) Q(ds, du) \right)^2 \right]
\]
\[\leq 2(\lambda^*)^2 \mathbb{E}\left[\left(\frac{1}{\sqrt{N}} \int_0^t \int_{\mathcal{N}(\hat{\mathcal{Y}}^N(s) \cup \mathcal{Y}(s))} \left(\sum_{j=1}^k (F_j(t + \delta - s) - F_j(t - s))\right) \mathcal{Q}(ds, du)\right)^2\right] \]

\[\quad + 2(\lambda^*)^2 \mathbb{E}\left[\left(\int_0^t \left(\sum_{j=1}^k (F_j(t + \delta - s) - F_j(t))\right) |\hat{\mathcal{Y}}^N(s)| ds\right)^2\right] \]

\[\leq 2(\lambda^*)^2 \int_0^t \left(\sum_{j=1}^k (F_j(t + \delta - s) - F_j(t - s))\right)^2 \mathbb{E}[|\hat{\mathcal{Y}}^N(s) - \mathcal{Y}(s)|] ds \]

\[+ 2(\lambda^*)^2 \mathbb{E}\left[\left(\int_0^t \left(\sum_{j=1}^k (F_j(t + \delta - s) - F_j(t))\right) |\hat{\mathcal{Y}}^N(s)| ds\right)^2\right].\]

Here the first term converges to zero as \(N \to \infty\) by (3.44). The second term also satisfies (3.48).

It is then clear that (3.45) holds for \(\Xi^N\). This completes the proof. \(\square\)

**Lemma 3.8.** Under Assumption 2.1(i) and (iii) and Assumption 2.2,

\[\hat{\mathcal{Y}}^N_2 - \bar{\mathcal{Y}}^N_2 \to 0\]

in probability in \(\mathcal{D}\) as \(N \to \infty\).

**Proof.** Recall the expression of \(\hat{\mathcal{Y}}^N_2 - \bar{\mathcal{Y}}^N_2\) in (3.40). It is clear that

\[\mathbb{E}[\hat{\mathcal{Y}}^N_2(t) - \bar{\mathcal{Y}}^N_2(t)] = 0,\]

and

\[\mathbb{E}[\left(\hat{\mathcal{Y}}^N_2(t) - \bar{\mathcal{Y}}^N_2(t)\right)^2] = \int_0^t \bar{\lambda}(t - s)^2 \mathbb{E}[|\hat{\mathcal{Y}}^N(s) - \mathcal{Y}(s)|] ds \to 0 \quad \text{as} \quad N \to \infty,\]

where the convergence follows from the bounded convergence theorem and (2.12). It then suffices to show tightness of the sequence \(\{\hat{\mathcal{Y}}^N_2 - \bar{\mathcal{Y}}^N_2; N \in \mathbb{N}\}\). By the expression in (3.40), tightness of the processes \(\{\hat{\mathcal{Y}}^N_2 - \bar{\mathcal{Y}}^N_2; N \in \mathbb{N}\}\) can be deduced from the tightness of the following two processes

\[\Xi^N_1(t) := \frac{1}{\sqrt{N}} \int_0^t \int_{\mathcal{N}(\hat{\mathcal{Y}}^N(s) \cup \mathcal{Y}(s))} \lambda(t - s)Q(ds, du),\]

\[\Xi^N_2(t) := \int_0^t \bar{\lambda}(t - s)|\hat{\mathcal{Y}}^N(s)| ds.\]

By Lemma 3.1, it suffices to show that for \(\ell = 1, 2,\)

\[\limsup_{N \to \infty} \frac{1}{\delta} \mathbb{P}\left(\sup_{v \in [0, \delta]} |\Xi^N_\ell(t + v) - \Xi^N_\ell(t)| > \epsilon\right) \to 0 \quad \text{as} \quad \delta \to 0. \quad (3.49)\]

For the process \(\Xi^N_1(t),\) we have

\[|\Xi^N_1(t + v) - \Xi^N_1(t)| \leq \frac{\lambda^*}{\sqrt{N}} \int_t^{t+v} \int_{\mathcal{N}(\hat{\mathcal{Y}}^N(s) \cup \mathcal{Y}(s))} Q(ds, du) \]

\[+ \frac{1}{\sqrt{N}} \int_0^t \int_{\mathcal{N}(\hat{\mathcal{Y}}^N(s) \cup \mathcal{Y}(s))} |\bar{\lambda}(t + v - s) - \bar{\lambda}(t - s)| Q(ds, du).\]
We already know how to treat the first term, see (3.47). By Lemma 3.4, the second term on the right hand side is bounded by

\[
\frac{1}{\sqrt{N}} \int_0^t \int_{N(\tilde{T}(s) \land \tilde{T}(s))} (\varphi(t) + \lambda^* \sum_{j=1}^k (F_j(t + v - s) - F_j(t - s))) Q(ds, du),
\]

which is nondecreasing in \(v\). Thus, we obtain that for any \(\epsilon > 0\),

\[
P \left( \sup_{v \in [0, \delta]} |\Xi_1^N(t + v) - \Xi_1^N(t)| > \epsilon \right) \leq \mathbb{P} \left( \frac{\lambda^*}{\sqrt{N}} \int_0^{t+\delta} \int_{N(\tilde{T}(s) \lor \tilde{T}(s))} Q(ds, du) > \epsilon/3 \right)
\]

\[+ \mathbb{P} \left( \frac{1}{\sqrt{N}} \varphi(\delta) \int_0^t \int_{N(\tilde{T}(s) \lor \tilde{T}(s))} Q(ds, du) > \epsilon/3 \right)
\]

\[+ \mathbb{P} \left( \frac{1}{\sqrt{N}} \int_0^t \int_{N(\tilde{T}(s) \lor \tilde{T}(s))} \left( \lambda^* \sum_{j=1}^k (F_j(t + \delta - s) - F_j(t - s)) \right) Q(ds, du) > \epsilon/3 \right).
\]

The first term is bounded as in (3.47). The second term is bounded by \(\frac{9}{\epsilon^2}\) times

\[
\mathbb{E} \left[ \left( \frac{1}{\sqrt{N}} \varphi(\delta) \int_0^t \int_{N(\tilde{T}(s) \lor \tilde{T}(s))} Q(ds, du) \right)^2 \right] \leq 2\mathbb{E} \left[ \left( \frac{1}{\sqrt{N}} \varphi(\delta) \int_0^t \int_{N(\tilde{T}(s) \lor \tilde{T}(s))} Q(ds, du) \right)^2 \right] + 2\mathbb{E} \left[ (\varphi(\delta) \int_0^t |\tilde{T}(s)|ds)^2 \right]
\]

\[\leq 2\varphi(\delta)^2 \int_0^t \mathbb{E} |\tilde{T}(s) - \tilde{T}(s)|ds + 2\varphi^2(\delta)^2 T \mathbb{E} \left[ \sup_{s \in [0, T]} |\tilde{T}(s)|^2 \right].
\]

This upper bound satisfies the proper bound (3.49), by the same argument as already used in the proof of Lemma 3.7.

The third term on the right hand side of (3.50) is bounded by \(\frac{9}{\epsilon^2}\) times

\[
\mathbb{E} \left[ \left( \frac{1}{\sqrt{N}} \int_0^t \int_{N(\tilde{T}(s) \lor \tilde{T}(s))} \left( \lambda^* \sum_{j=1}^k (F_j(t + \delta - s) - F_j(t - s)) \right) Q(ds, du) \right)^2 \right]
\]

\[\leq 2(\lambda^*)^2 \mathbb{E} \left[ \left( \frac{1}{\sqrt{N}} \int_0^t \int_{N(\tilde{T}(s) \lor \tilde{T}(s))} \left( \sum_{j=1}^k (F_j(t + \delta - s) - F_j(t - s)) \right) |\tilde{T}(s)|ds \right)^2 \right]
\]

\[+ 2(\lambda^*)^2 \mathbb{E} \left[ \left( \int_0^t \left( \sum_{j=1}^k (F_j(t + \delta - s) - F_j(t - s)) \right) |\tilde{T}(s)|ds \right)^2 \right]
\]

\[\leq 2(\lambda^*)^2 \int_0^t \left( \sum_{j=1}^k (F_j(t + \delta - s) - F_j(t - s)) \right)^2 \mathbb{E} [|\tilde{T}(s) - \tilde{T}(s)|^2]ds
\]

\[+ 2(\lambda^*)^2 \mathbb{E} \left[ \left( \int_0^t \left( \sum_{j=1}^k (F_j(t + \delta - s) - F_j(t - s)) \right) |\tilde{T}(s)|ds \right)^2 \right].
\]
Here the first term converges to zero as $N \to \infty$ by (2.12). The second term satisfies (3.48).

Next for the process $\Xi_2^N(t)$, we have
\[
|\Xi_2^N(t+v) - \Xi_2^N(t)| = \left| \int_0^{t+v} \hat{\lambda}(t+v-s) |\hat{Y}^N(s)| ds - \int_0^t \hat{\lambda}(t-s) |\hat{Y}^N(s)| ds \right| \\
\leq \int_t^{t+v} \hat{\lambda}(t+v-s) |\hat{Y}^N(s)| ds + \int_0^t |\hat{\lambda}(t+v-s) - \hat{\lambda}(t-s)| |\hat{Y}^N(s)| ds.
\]

The first term is bounded from above by
\[
\lambda^* \int_t^{t+v} |\hat{Y}^N(s)| ds,
\]
while the second term on the right hand side can be bounded by
\[
\int_0^t \left( \varphi(v) + \lambda^* \sum_{j=1}^k (F_j(t+v-s) - F_j(t-s)) \right) ds \left( \sup_{s \in [0,T]} |\hat{Y}^N(s)| \right).
\]

Those two upper bounds are nondecreasing in $v$, and by already used arguments, we easily establish that $\Xi_2^N$ satisfies (3.49). This completes the proof of the lemma. \qed

3.6. Completing the proof of Theorem 2.1. We are now ready to complete the proof of Theorem 2.1 for the joint convergence of $(\hat{S}^N, \hat{N}^N)$.

Proof of Theorem 2.1. We first prove the joint convergence
\[
(I^n(0), M_A^n, \hat{N}_0^n, \bar{N}_1^n, \bar{N}_2^n) \Rightarrow (I(0), \bar{M}_A, \bar{N}_0, \bar{N}_1, \bar{N}_2) \quad \text{in} \quad \mathbb{R} \times \mathbf{D}^4 \quad \text{as} \quad N \to \infty. \tag{3.51}
\]

By the independence of the variables associated with the initially and newly infected individuals, it suffices to show the joint convergences
\[
(I^n(0), \hat{N}_0^n) \Rightarrow (I(0), \hat{N}_0) \quad \text{in} \quad \mathbb{R} \times \mathbf{D} \quad \text{as} \quad N \to \infty,
\]
and
\[
(M_A^n, \bar{N}_1^n, \bar{N}_2^n) \Rightarrow (\bar{M}_A, \bar{N}_1, \bar{N}_2) \quad \text{in} \quad \mathbf{D}^3 \quad \text{as} \quad N \to \infty. \tag{3.52}
\]

The convergence of $(\hat{I}^N(0), \hat{N}_0^N)$ is straightforward. We focus on the convergence of $(\tilde{M}_A^N, \bar{N}_1^N, \bar{N}_2^N)$. Recall the compensated PRM $\tilde{Q}(ds, d\lambda, du)$ on $\mathbb{R}_+ \times \mathbf{D} \times \mathbb{R}_+$. Define an auxiliary process
\[
\tilde{M}_A^N(t) := \frac{1}{\sqrt{N}} \int_0^t \int_0^\infty \int_0^\infty \mathbf{1}_{u \leq N \hat{Y}(s)} \tilde{Q}(ds, d\lambda, du). \tag{3.53}
\]

Recall the process $\tilde{N}_1^N(t)$ defined in (3.27), where $\tilde{Q}$ can be replaced by $\tilde{Q}$. Also, recall the process $\tilde{N}_2^N(t)$ in (3.42) using $Q$. A minor extension of the computation done in Lemma 3.6 yields the joint convergence of the finite dimensional distributions of the processes $(\tilde{M}_A^N, \tilde{N}_1^N, \tilde{N}_2^N)$. Note that $\tilde{M}_A^N$ being a martingale, its tightness is easily established. Hence we deduce the joint convergence
\[
(\tilde{M}_A^N, \tilde{N}_1^N, \tilde{N}_2^N) \Rightarrow (\bar{M}_A, \bar{N}_1, \bar{N}_2) \quad \text{in} \quad \mathbf{D}^3, \quad \text{as} \quad N \to \infty. \tag{3.54}
\]

A simplified version of the proofs in Lemmas 3.7 and 3.8 yields that $\tilde{M}_A^N - \tilde{M}_A^N \to 0$ in probability in $\mathbf{D}$, as $N \to \infty$. Hence
\[
(\tilde{M}_A^N - \bar{M}_A^N, \tilde{N}_1^N - \bar{N}_1^N, \tilde{N}_2^N - \bar{N}_2^N) \to 0
\]
in probability in $\mathbf{D}^3$, as $N \to \infty$. Combined with (3.54), this establishes (3.52).

Observe that the equations (3.14) and (3.7) coupled with (3.13) define uniquely the processes $(\hat{S}^N, \hat{N}^N)$ as the solution of a two-dimensional integral equation driven by $(I^N(0), M_A^N, \hat{N}_0^N, \bar{N}_1^N, \bar{N}_2^N)$ and the fixed functions $\hat{\lambda}^0(t)$, $\hat{\lambda}(t)$, $\hat{S}(t)$ and $\bar{N}(t)$. The mapping which to those data associates the solution is continuous in the Skorohod $J_1$ topology, see Lemma 8.1 in [18]. Thus, by the joint convergence in (3.51), we apply the continuous mapping theorem to conclude (2.16).
Finally we show that the limit processes \( \hat{\mathcal{L}}_1 \) and \( \hat{\mathcal{L}}_2 \) have a continuous version in \( \mathcal{C} \), given their consistent finite dimensional distributions. Since its increments of the processes are Gaussian and centered, it suffices to show the continuity of the covariance functions. It is easy to calculate that
\[
\mathbb{E} [ |\hat{\mathcal{L}}_1(t + \delta) - \hat{\mathcal{L}}_1(t)|^2 ] = \int_t^{t + \delta} \text{Var}(\lambda(t + \delta - s)) \tilde{\mathcal{Y}}(s) ds + \int_0^t \text{Var}((\lambda(t + \delta - s) - \lambda(t - s))^2) \tilde{\mathcal{Y}}(s) ds ,
\]
and
\[
\mathbb{E} [ |\hat{\mathcal{L}}_2(t + \delta) - \hat{\mathcal{L}}_2(t)|^2 ] = \int_t^{t + \delta} \tilde{\lambda}(t + \delta - s)^2 \tilde{\mathcal{Y}}(s) ds + \int_0^t (\tilde{\lambda}(t + \delta - s) - \tilde{\lambda}(t - s))^2 \tilde{\mathcal{Y}}(s) ds .
\]
Then we can verify the continuity property under the conditions in Assumption 2.1 (iii). \( \square \)

4. Convergence of \((\hat{\mathcal{E}}^N, \hat{\mathcal{I}}^N, \hat{\mathcal{R}}^N)\)

The proof of Theorem 2.2 on the convergence of \((\hat{\mathcal{E}}^N, \hat{\mathcal{I}}^N, \hat{\mathcal{R}}^N)\) follows essentially the same arguments as that of Theorem 3.2 in [18], for which we only highlight the differences. We have the following representations for the processes \((\hat{\mathcal{E}}^N, \hat{\mathcal{I}}^N, \hat{\mathcal{R}}^N)\):
\[
\hat{\mathcal{E}}^N(t) = \hat{\mathcal{E}}^N(0)G_0^N(t) + \hat{\mathcal{E}}_0^N(t) + \hat{\mathcal{E}}_1^N(t) + \int_0^t G^c(t - s) \hat{\mathcal{Y}}^N(s) ds , \tag{4.1}
\]
\[
\hat{\mathcal{I}}^N(t) = \hat{\mathcal{I}}^N(0)F_0^c(t) + \hat{\mathcal{I}}_0^N(t) + \hat{\mathcal{I}}_1^N(t) + \int_0^t \Psi(t - s) \hat{\mathcal{Y}}^N(s) ds , \tag{4.2}
\]
\[
\hat{\mathcal{R}}^N(t) = \hat{\mathcal{R}}^N(0)F_0^c(t) + \hat{\mathcal{R}}_0^N(t) + \hat{\mathcal{R}}_1^N(t) + \int_0^t \Phi(t - s) \hat{\mathcal{Y}}^N(s) ds , \tag{4.3}
\]
where
\[
\hat{\mathcal{E}}_0^N(t) := \frac{1}{\sqrt{N}} \sum_{j=1}^{\hat{E}^{N}(0)} (1_{\xi_j > t} - G_0^N(t)),
\]
\[
\hat{\mathcal{I}}_{0,1}^N(t) := \frac{1}{\sqrt{N}} \sum_{k=1}^{\hat{I}^{N}(0)} (1_{\eta_k^0 > t} - F_{0,1}^c(t)), \quad \hat{\mathcal{I}}_{0,2}^N(t) := \frac{1}{\sqrt{N}} \sum_{j=1}^{\hat{I}^{N}(0)} (1_{\xi_j < 0} (1_{\xi_j > t} - \Psi(t)),
\]
\[
\hat{\mathcal{R}}_{0,1}^N(t) := \frac{1}{\sqrt{N}} \sum_{k=1}^{\hat{R}^{N}(0)} (1_{\eta_k^0 < t} - F_{0,1}^c(t)), \quad \hat{\mathcal{R}}_{0,2}^N(t) := \frac{1}{\sqrt{N}} \sum_{j=1}^{\hat{R}^{N}(0)} (1_{\xi_j + \eta_j < t} - \Phi(t)),
\]
and
\[
\hat{\mathcal{E}}_1^N(t) := \frac{1}{\sqrt{N}} \sum_{i=1}^{\hat{E}^{N}(t)} (1_{\tau_i^N + \zeta > t} - \sqrt{N} \int_0^t G^c(t - s) \tilde{S}^N(s) \tilde{\mathcal{Y}}^N(s) ds ,
\]
\[
\hat{\mathcal{I}}_1^N(t) := \frac{1}{\sqrt{N}} \sum_{i=1}^{\hat{I}^{N}(t)} (1_{\tau_i^N + \zeta > t} - \sqrt{N} \int_0^t \Phi(t - s) \tilde{S}^N(s) \tilde{\mathcal{Y}}^N(s) ds,
\]
\[
\hat{\mathcal{R}}_1^N(t) := \frac{1}{\sqrt{N}} \sum_{i=1}^{\hat{R}^{N}(t)} (1_{\tau_i^N + \zeta + \eta > t} - \sqrt{N} \int_0^t \Phi(t - s) \tilde{S}^N(s) \tilde{\mathcal{Y}}^N(s) ds.
\]
Lemma 4.1. Under Assumptions 2.1 and 2.2,
\[ (E_0^N, I_0^N, R_0^N, \tilde{E}_0^N, \tilde{I}_0^N, \tilde{R}_0^N) \Rightarrow (\hat{E}_0, \hat{I}_0, \hat{R}_0) \] in \( D^5 \) as \( N \to \infty \),
jointly with the convergence \( (\hat{E}^N_{0,1}, \hat{I}^N_{0,1}, \hat{R}^N_{0,1}) \Rightarrow (\hat{E}_{0,1}, \hat{I}_{0,1}, \hat{R}_{0,1}) \) in (3.15), where \( (\hat{E}_0, \hat{I}_0, \hat{R}_0) \) is as given in Theorem 2.2.

Proof. By the independence of the sequences \( \{\lambda^0_{j,1}\}_{j \geq 1} \) and \( \{\lambda^0_{k,1}\}_{k \geq 1} \), it suffices to prove the joint convergence of \( (\hat{E}^N_{0,1}, \hat{I}^N_{0,1}, \hat{R}^N_{0,1}) \) and \( (\hat{E}^N_{0,2}, \hat{I}^N_{0,2}, \hat{R}^N_{0,2}) \) separately.

Recall the processes \( \hat{E}^N_{0,1} \) and \( \tilde{E}^N_{0,2} \) defined in (3.16) and (3.17), respectively. Similarly we define \( (\tilde{E}^N_0, \tilde{I}^N_0, \tilde{R}^N_0, \tilde{E}^N_1, \tilde{I}^N_1, \tilde{R}^N_1, \tilde{E}^N_2, \tilde{I}^N_2, \tilde{R}^N_2) \) by replacing \( E^N(0) \) and \( I^N(0) \) by \( N \tilde{E}(0) \) and \( N \tilde{I}(0) \), respectively. By the FCLT for random elements in \( D \) (see Theorem 2 in [10], applied to the processes \( \hat{E}^N_{0,1} \) and \( \tilde{E}^N_{0,2} \) under Assumption 2.1(i) (a) and (b)) and the FCLT for empirical processes (see Theorem 14.3 in [4], applied to the processes \( (\tilde{E}^N_0, \tilde{I}^N_0, \tilde{R}^N_0, \tilde{E}^N_1, \tilde{I}^N_1, \tilde{R}^N_1, \tilde{E}^N_2, \tilde{I}^N_2, \tilde{R}^N_2) \)), and by the definitions in (2.2) and (2.3), we obtain the joint convergences
\[ (\hat{E}^N_{0,1}, \hat{I}^N_{0,1}, \hat{R}^N_{0,1}) \Rightarrow (\hat{E}_{0,1}, \hat{I}_{0,1}, \hat{R}_{0,1}) \] in \( D^3 \) as \( N \to \infty \),
\[ (\hat{E}^N_{0,2}, \hat{I}^N_{0,2}, \hat{R}^N_{0,2}) \Rightarrow (\hat{E}_{0,2}, \hat{I}_{0,2}, \hat{R}_{0,2}) \] in \( D^4 \) as \( N \to \infty \).

It then suffices to show that \( (\hat{E}^N_{0,1} - \hat{E}^N_{0,2} - \hat{I}^N_{0,1} - \hat{R}^N_{0,1} - \hat{E}^N_{0,2} - \hat{I}^N_{0,2} - \hat{R}^N_{0,2}, \hat{E}^N_0 - \hat{E}^N_1 - \hat{E}^N_2, \hat{I}^N_0 - \hat{I}^N_1 - \hat{I}^N_2, \hat{R}^N_0 - \hat{R}^N_1 - \hat{R}^N_2) \to 0 \) in probability in \( D^7 \), as \( N \to \infty \). The convergence for \( (\hat{E}^N_{0,1} - \hat{E}^N_{0,2} - \hat{I}^N_{0,1} - \hat{R}^N_{0,1} - \hat{E}^N_{0,2} - \hat{I}^N_{0,2} - \hat{R}^N_{0,2}) \to 0 \) in probability in \( D^2 \) is shown in (3.19). For the other process, the convergence follows similar arguments as already used above. See also the proofs of Lemmas 6.1 and 8.1 in [18]. This completes the proof. \( \square \)

Lemma 4.2. Under Assumptions 2.1 and 2.2,
\[ (E_1^N, I_1^N, \tilde{R}_1^N) \Rightarrow (\hat{E}_1, \hat{I}_1, \hat{R}_1) \] in \( D^3 \) as \( N \to \infty \),
jointly with the convergence of \( (\hat{E}^N_1, \hat{I}^N_1, \hat{R}^N_1) \Rightarrow (\hat{E}_1, \hat{I}_1, \hat{R}_1) \) where \( (\hat{E}_1, \hat{I}_1, \hat{R}_1) \) is given in Theorem 2.1 and \( (\hat{E}_1, \hat{I}_1, \hat{R}_1) \) is as given in Theorem 2.2.

Proof. The convergence of \( (E_1^N, I_1^N, \tilde{R}_1^N) \) follows from the same argument as in the proofs of Lemmas 8.3 and 8.4 in [18]. Let us sketch the argument, and in particular the coupling between \( (E_1^N, I_1^N, \tilde{R}_1^N) \) and \( (\hat{E}_1^N, \hat{I}_1^N, \hat{R}_1^N) \), referring the reader to [18] for the technical details. We have shown the joint convergence of \( (\hat{E}_1^N, \hat{I}_1^N) \) in (3.52) using the processes \( (\hat{E}_1^N, \hat{I}_1^N) \) which are defined via the PRM \( \tilde{Q}(ds, d\lambda, du) \). Recall the definition of the pair \( (\zeta, \eta) \) as a function of \( \lambda \) in (2.1). This defines a map \( \phi : D \mapsto \mathbb{R}_+^2 \) such that \( (\zeta, \eta) = \phi(\lambda) \). Recall that \( \zeta \) is the duration of the exposed period, and \( \eta \) the duration of the infectious period. We now define \( \Lambda : \mathbb{R}_+ \times D \times \mathbb{R}_+ \mapsto \mathbb{R}_+^4 \) by
\[ \Lambda(s, \lambda, u) = (s, \phi(s)(\lambda), u), \quad \text{where} \quad \phi(s)(\lambda) = (s + \phi_1(\lambda), s + \phi_1(\lambda) + \phi_2(\lambda)). \]
The intuition behind this definition is that if \( s \) is the time of infection, \( \phi_1(s) \) stands for the end of the exposed period (transition from exposed to infectious), and \( \phi_2(s) \) the end of the infectious period (transition from infectious to recovered). We finally define a PRM \( Q_1 \) on \( \mathbb{R}_+^4 \), which is the image of \( \tilde{Q} \) by the mapping \( \Lambda \), i.e., for any Borel subset \( A \subset \mathbb{R}_+^4 \),
\[ Q_1(A) = \tilde{Q}(\Lambda^{-1}(A)). \]
This PRM $Q_1$ plays the role of $M_1$ in [18] (see Definition 8.1 there). Let $\tilde{Q}_1$ denote its associated compensated measure. We have

$$E^N(t) = \frac{1}{\sqrt{N}} \int_0^t \int_0^\infty \int_0^\infty \int_0^\infty 1_{u \leq Y^N(s^-)} \tilde{Q}_1(ds,dy,dz,du),$$

$$I^N(t) = \frac{1}{\sqrt{N}} \int_0^t \int_0^\infty \int_0^\infty \int_0^\infty 1_{u \leq Y^N(s^-)} \tilde{Q}_1(ds,dy,dz,du),$$

$$R^N(t) = \frac{1}{\sqrt{N}} \int_0^t \int_0^\infty \int_0^\infty \int_0^\infty 1_{u \leq Y^N(s^-)} \tilde{Q}_1(ds,dy,dz,du).$$

Recall that

$$M^N_A(t) = \frac{1}{\sqrt{N}} \int_0^t \int_0^\infty \int_0^\infty \int_0^\infty 1_{u \leq Y^N(s^-)} \tilde{Q}_1(ds,dy,dz,du),$$

and define the auxiliary process

$$L^N(t) = \frac{1}{\sqrt{N}} \int_0^t \int_0^\infty \int_0^\infty \int_0^\infty 1_{u \leq Y^N(s^-)} \tilde{Q}_1(ds,dy,dz,du).$$

We define $(\tilde{E}^N_1, \tilde{I}^N_1, \tilde{R}^N_1, \tilde{M}^N_A, \tilde{L}^N_1)$ by replacing $1_{u \leq Y^N(s^-)}$ by $1_{u \leq N\tilde{Y}(s^-)}$ in the above integrals. It is not hard to show that the three processes $\tilde{M}^N_A(t), \tilde{L}^N_1(t)$ and $\tilde{R}^N_1(t)$ are martingales (with respect to three different filtrations), whose tightness is easy to establish. Moreover, we have

$$\tilde{E}^N_1(t) = \tilde{M}^N_A(t) - \tilde{L}^N_1(t), \quad \tilde{I}^N_1(t) = \tilde{L}^N_1(t) - \tilde{R}^N_1(t).$$

(4.4)

Hence the tightness of $\tilde{E}^N_1$ and $\tilde{I}^N_1$. In order to establish the joint convergence $(\tilde{E}^N_1, \tilde{I}^N_1, \tilde{R}^N_1, \tilde{M}^N_A, \tilde{L}^N_1) \Rightarrow (\tilde{E}_1, \tilde{I}_1, \tilde{R}_1, \tilde{M}_A, \tilde{L}_1)$ in $D^5$ as $N \to \infty$, it remains to prove the joint convergence of the finite-dimensional distributions, that is, for any $k \geq 1$, $0 < s_1 < s_2 < \cdots < s_k$, as $N \to \infty$, $((\tilde{E}^N_1(s_1), \tilde{I}^N_1(s_1), \tilde{R}^N_1(s_1)), \ldots, (\tilde{E}^N_1(s_k), \tilde{I}^N_1(s_k), \tilde{R}^N_1(s_k)))$ converges to $((\tilde{E}_1(s_1), \tilde{I}_1(s_1), \tilde{R}_1(s_1)), \ldots, (\tilde{E}_1(s_k), \tilde{I}_1(s_k), \tilde{R}_1(s_k)))$ in distribution in $\mathbb{R}^{5k}$. Noting that an integral w.r.t. the compensated measure $\tilde{Q}_1$ can be considered in fact as an integral w.r.t. the compensated measure $\tilde{Q}$, it is clear that the computation done in the proof of Lemma 3.6 can be extended to this situation, yielding the announced result. We just show one piece of this computation, namely, for $\vartheta, \psi, \vartheta', \psi' \in \mathbb{R}$ and $t, t' > 0$, we get

$$\lim_{N \to \infty} \mathbb{E} \left[ \exp \left( i\vartheta \tilde{E}^N_1(t) + i\vartheta \tilde{I}^N_1(t) + i\vartheta' \tilde{I}^N_1(t') + i\vartheta' \tilde{I}^N_1(t') \right) \right]$$

$$= \exp \left( -\frac{\vartheta^2}{2} \int_0^t \mathbb{E} [\tilde{\chi}^2(t-s)] \tilde{Y}(s)ds - \frac{\vartheta^2}{2} \int_0^t \vartheta \int_0^\vartheta \mathbb{E} [\tilde{\chi}^2(t'-s)] \tilde{Y}(s)ds \right.$$
\[-d'\theta \int_0^{t''t'} \left( \mathbb{E}[\lambda(t' - s)1_{\xi \leq t - s < \xi + \eta}] - \tilde{\lambda}(t' - s)\Psi(t - s) \right) \tilde{\Upsilon}(s)ds \]

\[-d'\theta' \int_0^{t''t'} \left( \mathbb{E}[\lambda(t' - s)1_{\xi \leq t' - s < \xi + \eta}] - \tilde{\lambda}(t' - s)\Psi(t' - s) \right) \tilde{\Upsilon}(s)ds \]

Finally, we need to show that \((E_1^N - \tilde{E}_1^N, I_1^N - \tilde{I}_1^N, \hat{R}_1^N - \tilde{R}_1^N) \to 0\) in \(D^3\) in probability. Since \(E_1^N(t) = \hat{M}_1^N(t) - \tilde{L}_1^N(t)\) and \(I_1^N(t) = \hat{L}_1^N(t) - \tilde{R}_1^N(t)\), given (4.4), it suffices to show that \((\hat{M}_1^N - \tilde{M}_1^N, \hat{L}_1^N - \tilde{L}_1^N, \hat{R}_1^N - \tilde{R}_1^N) \to 0\) in \(D^3\) in probability. We will show that \(\hat{R}_1^N - \tilde{R}_1^N \to 0\) and the other two follow from similar arguments. It is clear that

\[
\mathbb{E} [\| \hat{R}_1^N(t) - \tilde{R}_1^N(t) \|^2] = \int_0^t \Phi(t - s) \mathbb{E} [\| \tilde{\Upsilon}^N(s^-) - \tilde{\Upsilon}(s) \|] ds \to 0 \quad \text{as} \quad N \to \infty.
\]

We have

\[
|\hat{R}_1^N(t) - \tilde{R}_1^N(t)| \leq \frac{1}{\sqrt{N}} \int_0^t \int_0^t \int_0^t \int_{N(\hat{\Upsilon}^N(s^-) \vee \tilde{\Upsilon}(s))}^{N(\hat{\Upsilon}^N(s^-) \wedge \tilde{\Upsilon}(s))} Q_1(ds, dy, dz, du) + \int_0^t \Phi(t - s)|\hat{\Upsilon}^N(s)| ds.
\]

Denote these two terms as \(\mathcal{A}_1^N(t)\) and \(\mathcal{A}_2^N(t)\). We show tightness of these two processes. Observe that both are increasing in \(t\). Thus, we only need to verify the following (see the Corollary on page 83 in [4]): for any \(\epsilon > 0\), and \(i = 1, 2\),

\[
\limsup_{N \to \infty} \frac{1}{\delta} \mathbb{P}(\mathcal{A}_i^N(t + \delta) - \mathcal{A}_i^N(t) \geq \epsilon) \to 0 \quad \text{as} \quad \delta \to 0.
\]

By some direct calculations (see also the similar derivations in the proofs of Lemmas 6.3 and 8.3 in [18] for the models with constant infectivity rate), this condition for both processes \(\mathcal{A}_1^N(t)\) and \(\mathcal{A}_2^N(t)\) reduces to show that

\[
\limsup_{N \to \infty} \frac{1}{\delta} \mathbb{E} \left[ \left( \int_0^{t + \delta} \Phi(t + \delta - s) - \Phi(t - s)|\tilde{\Upsilon}^N(s)| ds \right)^2 \right] \to 0
\]

and

\[
\limsup_{N \to \infty} \frac{1}{\delta} \mathbb{E} \left[ \left( \int_0^t (\Phi(t + \delta - s) - \Phi(t - s))|\tilde{\Upsilon}^N(s)| ds \right)^2 \right] \to 0
\]

as \(\delta \to 0\). It is clear the expectation in (4.6) is bounded by \(\delta^2 \mathbb{E}_{0 \leq s \leq T} [\tilde{\Upsilon}^N(s)^2] \), so the claim follows by (3.37). For (4.7), we first observe that

\[
\Phi(t + \delta - s) - \Phi(t - s) = \int_0^{t + \delta - s} F(t + \delta - s - u|u)dG(u) - \int_0^{t - s} F(t - s - u|u)dG(u)
\]

\[
= \int_0^{t + \delta - s} F(t + \delta - s - u|u)dG(u) + \int_0^{t - s} (F(t + \delta - s - u|u) - F(t - s - u|u))dG(u).
\]

Thus, we have

\[
\mathbb{E} \left[ \left( \int_0^t (\Phi(t + \delta - s) - \Phi(t - s))|\tilde{\Upsilon}^N(s)| ds \right)^2 \right] \leq 2 \mathbb{E} \left[ \sup_{0 \leq s \leq T} |\tilde{\Upsilon}^N(s)|^2 \right] \left( \int_0^t (G(t + \delta - s) - G(t - s)) ds \right)^2 + \left( \int_0^t \int_0^{t - s} (F(t + \delta - s - u|u) - F(t - s - u|u))dG(u)ds \right)^2
\]
\[ \leq 2\delta^2 \mathbb{E} \left[ \sup_{0 \leq s \leq T} |\tilde{\mathcal{Y}}(s)|^2 \right], \]

where the last inequality follows from applying (3.26) (noting that the second integral requires using interchange of the order of integration first). Thus by (3.35), we obtain (4.7). This completes the proof.

The representations of \( E^N(t), I^N(t) \) and \( R^N(t) \) in (4.1), (4.2) and (4.3), give a natural integral mapping from \( (\hat{E}^N(0), \hat{R}^N(0)), (\hat{E}_0^N, \hat{I}_0^N, \hat{R}_0^N), \) \( (\hat{E}_1^N, \hat{I}_1^N, \hat{R}_1^N), (\hat{S}^N, \hat{\mathcal{I}}^N) \) and the fixed functions \( G_0(t), F_0(t), \Psi(t), \lambda(t), S(t) \) and \( \tilde{\mathcal{I}}(t) \) to the processes \( (\hat{E}^N, \hat{I}^N, \hat{R}^N) \). The mapping is continuous in the Skorohod \( J_1 \) topology (by a slight modification of Lemma 8.1 in [18]). We can then apply the continuous mapping theorem to conclude the convergence \( (\hat{E}^N, \hat{I}^N, \hat{R}^N) \Rightarrow (\hat{E}, \hat{I}, \hat{R}) \) in \( \mathbb{D}^2 \). Given their joint convergence with \( (\hat{\mathcal{I}}^N_{0,1}, \hat{\mathcal{I}}^N_{0,2}) \) in Lemma 4.1 and \( (\hat{\mathcal{I}}^N_1, \hat{\mathcal{I}}^N_2) \) in Lemma 4.2, we can also conclude the joint convergence of the processes \( (\hat{E}^N, \hat{I}^N, \hat{R}^N) \) with \( (\hat{S}^N, \hat{\mathcal{I}}^N) \). This completes the proof of Theorem 2.2.

5. On the generalized SIS and SIRS models with varying infectivity

In this section, we discuss how the results can be generalized to the SIS and SIRS models. We state the FCLTs for these models without proofs, since they can be done analogously.

5.1. Generalized SIS models with varying infectivity. In the SIS model, individuals become susceptible immediately after going through the infectious periods. Since \( S^N(t) = N - I^N(t) \), the epidemic dynamics is determined by the two-dimensional processes \( \tilde{\mathcal{I}}^N(t) \) and \( I^N(t) \). As stated in Remark 2.3 of [9], the FLLN limit \( (\tilde{\mathcal{I}}(t), I(t)) \) is determined by the two-dimensional integral equations:

\[
\tilde{\mathcal{I}}(t) = \tilde{I}(0)\tilde{\lambda}^0(t) + \int_0^t \tilde{\lambda}(t-s)(1 - \tilde{I}(s))\tilde{\mathcal{I}}(s)ds, \tag{5.1}
\]

\[
I(t) = I(0)F_0(t) + \int_0^t F^c(t-s)(1 - I(s))\tilde{\mathcal{I}}(s)ds. \tag{5.2}
\]

Here the c.d.f.’s \( F \) and \( F_0 \) denote the distributions of the infectious periods of newly infected individuals and those of initially infectious ones.

**Theorem 5.1.** In the generalized SIS model, under Assumptions 2.1 and 2.2 (with \( E^N(t) \equiv 0 \) and only infectious periods),

\[
(\hat{\mathcal{I}}^N, I^N) \to (\hat{\mathcal{I}}, I) \quad \text{in} \quad \mathbb{D}^2 \quad \text{as} \quad N \to \infty.
\]

The limit processes \( \hat{\mathcal{I}} \) and \( I \) are the unique solution to the following stochastic integral equations:

\[
\hat{\mathcal{I}}(t) = \hat{I}(0)\hat{\lambda}^0(t) + \hat{\mathcal{I}}_0(t) + \hat{\mathcal{I}}_1(t) + \hat{\mathcal{I}}_2(t) + \int_0^t \hat{\lambda}(t-s)\hat{\mathcal{I}}(s)ds,
\]

\[
I(t) = I(0)F_0(t) + \hat{I}(0) + \hat{I}_1(t) + \int_0^t F^c(t-s)\hat{\mathcal{I}}(s)ds,
\]

where

\[
\hat{\mathcal{Y}}(t) = (1 - I(t))\hat{\mathcal{I}}(t) - \hat{\mathcal{I}}(t)I(t),
\]

and \( \hat{\mathcal{I}}(t) \) and \( I(t) \) are given by the unique solutions to the integral equations (5.1) and (5.2). \( \hat{\mathcal{I}}_0, \hat{\mathcal{I}}_1 \) and \( \hat{\mathcal{I}}_2 \) are Gaussian processes with covariance functions: for \( t, t' \geq 0 \),

\[
\text{Cov}(\hat{\mathcal{I}}_0(t), \hat{\mathcal{I}}_0(t')) = \hat{I}(0)\text{Cov}(\lambda^0(t), \lambda^0(t')),
\]
\[ \text{Cov}(\hat{\mathcal{I}}(t), \hat{\mathcal{I}}(t')) = \int_0^{t \wedge t'} \text{Cov}(\lambda(t - s), \lambda(t' - s))(1 - \bar{I}(s))\bar{I}(s)ds, \]
\[ \text{Cov}(\hat{\mathcal{I}}(t), \hat{\mathcal{I}}(t')) = \int_0^{t \wedge t'} \bar{\lambda}(t - s)\bar{\lambda}(t' - s)(1 - \bar{I}(s))\bar{I}(s)ds. \]

\hat{\mathcal{I}}_0 \text{ is independent of } \hat{\mathcal{I}}_1 \text{ and } \hat{\mathcal{I}}_2. \hat{\mathcal{I}}_1 \text{ and } \hat{\mathcal{I}}_2 \text{ have covariance function } \text{Cov}(\hat{\mathcal{I}}_1(t), \hat{\mathcal{I}}_2(t')) = 0 \text{ for } t, t' \geq 0. \hat{I}_0 \text{ and } \hat{I}_1 \text{ are independent Gaussian processes with covariance functions: for } t, t' \geq 0,
\[ \text{Cov}(\hat{I}_0(t), \hat{I}_0(t')) = \bar{I}(0)(F_0^c(t \vee t') - F_{0,0}^c(t)F_{0,0}^c(t')), \]
\[ \text{Cov}(\hat{I}_1(t), \hat{I}_1(t')) = \int_0^{t \wedge t'} F^c(t \vee t' - s)(1 - \bar{I}(s))\bar{I}(s)ds. \]

\hat{\mathcal{I}}_0 \text{ and } \hat{\mathcal{I}}_0 \text{ have covariance function}
\[ \text{Cov}(\hat{\mathcal{I}}_0(t), \hat{\mathcal{I}}_1(t')) = \bar{I}(0)(\mathbb{E}[\lambda^0(t)1_{\eta^0,t > t'}] - \bar{\lambda}^0(t)F^c(t')), \]
and \hat{\mathcal{I}}_1, \hat{\mathcal{I}}_2 \text{ and } \hat{\mathcal{I}}_1 \text{ have covariance functions}
\[ \text{Cov}(\hat{\mathcal{I}}_1(t), \hat{\mathcal{I}}_1(t')) = \int_0^{t \wedge t'} (\mathbb{E}[\lambda(t - s)1_{\eta,t' > s}] - \bar{\lambda}(t - s)F^c(t' - s))(1 - \bar{I}(s))\bar{I}(s)ds, \]
\[ \text{Cov}(\hat{\mathcal{I}}_2(t), \hat{\mathcal{I}}_1(t')) = \int_0^{t \wedge t'} \bar{\lambda}(t - s)F^c(t' - s)(1 - \bar{I}(s))\bar{I}(s)ds. \]

If \( \bar{\lambda}^0(t) \) and \( F_{0,1}^c \) are continuous, then the limits \( \hat{\mathcal{I}} \) and \( \hat{I} \) are continuous.

5.2. Generalized SIRS models with varying infectivity. In the SIRS model, individuals experience the infectious and recovered/immune periods, and then become susceptible. We use \( I^N(t) \) and \( R^N(t) \) to denote the numbers of infectious and recovered/immune individuals, respectively, corresponding to the processes \( E^N(t) \) and \( I^N(t) \) in the SEIR model. Similarly, we use \( \{\lambda^0_j\}_{j \geq 1} \) and \( \{\lambda_i\}_{i \geq 1} \) to denote the infectivity processes of the initially and newly infectious individuals, respectively, and also use \( \{\xi^0_j, \eta^0_j\}_{j \geq 1} \) and \( \{\xi_i, \eta_i\}_{i \geq 1} \) for the infectious and recovered/immune periods for the initially and newly infected individuals, respectively. Denote the remaining immune time of the initially recovered/immune individuals by \( \eta^0_{k,R} \) (changing notation \( \eta^0_{0,R} \) to \( \eta^0_{0,R} \) accordingly). Of course, \( \{\lambda^0_j\}_{j \geq 1} \) and \( \{\lambda_i\}_{i \geq 1} \) only take positive values over the intervals \( [0, \xi^0_j] \) and \( [0, \xi_i] \), respectively. The definitions of the variables \( \xi_i, \eta_i, \xi^0_j, \eta^0_j \) and \( \eta^0_{0,R} \) in (2.1), (2.2) and (2.3) also need to change accordingly in a natural manner. The c.d.f.’s \( G_0, G \) denote the distributions of infectious periods of the initially and newly infectious individuals, and the c.d.f.’s \( F_{0,R} \) and \( F \) denote the distributions of the recovered/immune periods of the initially and newly recovered individuals.

Similarly for the notation \( \Psi_0, \Psi, \Phi_0, \Phi \).

Since \( S^N(t) = N - I^N(t) - R^N(t) \), the epidemic dynamics is described by the three processes \( \bar{S}^N, \bar{I}^N, \bar{R}^N \). As stated in Remark 2.3 in [9], the FLLN limit \( (\bar{S}, \bar{I}, \bar{R}) \) is determined by the three-dimensional integral equations:
\[ \bar{S}(t) = \bar{I}(0)\bar{\lambda}^0(t) + \int_0^t \bar{\lambda}(t - s)(1 - \bar{I}(s) - \bar{R}(s))\bar{I}(s)ds, \]
(5.3)
\[ \bar{I}(t) = \bar{I}(0)G_0^c(t) + \int_0^t G^c(t - s)(1 - \bar{I}(s) - \bar{R}(s))\bar{I}(s)ds, \]
(5.4)
\[ \bar{R}(t) = \bar{R}(0)F_{0,R}^c(t) + \bar{I}(0)\Psi_0(t) + \int_0^t \Psi(t - s)(1 - \bar{I}(s) - \bar{R}(s))\bar{I}(s)ds. \]
(5.5)
We next state the FCLT for these processes.
Theorem 5.2. In the generalized SIRS model, under Assumptions 2.1 and 2.2,

\((\hat{N}, \hat{\Gamma}, \hat{R}) \Rightarrow (\hat{N}, \hat{\Gamma}, \hat{R})\) in \(D^3\) as \(N \to \infty\).

The limit process \((\hat{N}, \hat{\Gamma}, \hat{R})\) is the unique solution to the following system of stochastic integral equations:

\[
\begin{align*}
\hat{N}(t) &= I(0)\lambda_0(t) + \hat{N}_0(t) + \hat{N}_1(t) + \hat{N}_2(t) + \int_0^t \lambda(t-s)\hat{\Gamma}(s)ds, \\
\hat{\Gamma}(t) &= I(0)\gamma_0(t) + \hat{\Gamma}_0(t) + \hat{\Gamma}_1(t) + \int_0^t \gamma(t-s)\hat{\Gamma}(s)ds, \\
\hat{R}(t) &= \hat{R}(0)F_{0,R}(t) + \hat{I}(0)\Psi_0(t) + \hat{R}_{0.1}(t) + \hat{R}_{0.2}(t) + \hat{R}_1(t) + \int_0^t \Psi(t-s)\hat{\Gamma}(s)ds,
\end{align*}
\]

where

\[
\hat{\Gamma}(t) = (1 - \hat{I}(t) - \hat{R}(t))\hat{N}(t) - \hat{\Gamma}(t)(\hat{I}(t) + \hat{R}(t)),
\]

and \(\hat{N}(t), \hat{\Gamma}(t)\) and \(\hat{R}(t)\) are given by the unique solution to the integral equations (5.3)–(5.5). \(\hat{N}_0, \hat{\Gamma}_1\) and \(\hat{\Gamma}_2\) are Gaussian processes with covariance functions: for \(t, t' \geq 0\),

\[
\begin{align*}
\text{Cov}(\hat{N}_0(t), \hat{N}_0(t')) &= I(0)\text{Cov}(\lambda_0(t), \lambda_0(t')), \\
\text{Cov}(\hat{\Gamma}_1(t), \hat{\Gamma}_1(t')) &= \int_0^{t\wedge t'} \text{Cov}(\lambda(t-s), \lambda(t'-s))(1 - \hat{I}(s) - \hat{R}(s))\hat{\Gamma}(s)ds, \\
\text{Cov}(\hat{\Gamma}_2(t), \hat{\Gamma}_2(t')) &= \int_0^{t\wedge t'} \lambda(t-s)\lambda(t'-s)(1 - \hat{I}(s) - \hat{R}(s))\hat{\Gamma}(s)ds.
\end{align*}
\]

\(\hat{N}_0\) is independent of \(\hat{\Gamma}_1\) and \(\hat{\Gamma}_2\). \(\hat{\Gamma}_1\) and \(\hat{\Gamma}_2\) have covariance function \(\text{Cov}(\hat{\Gamma}_1(t), \hat{\Gamma}_2(t')) = 0\) for \(t, t' \geq 0\). \(\hat{I}_0\) and \(\hat{\Gamma}_1\) are independent Gaussian processes with covariance functions: for \(t, t' \geq 0\),

\[
\begin{align*}
\text{Cov}(\hat{I}_0(t), \hat{I}_0(t')) &= I(0)(G_0^c(t \vee t') - G_0^c(t)G_0^c(t')), \\
\text{Cov}(\hat{\Gamma}_1(t), \hat{\Gamma}_1(t')) &= \int_0^{t\wedge t'} G^c(t \vee t' - s)(1 - \hat{I}(s) - \hat{R}(s))\hat{\Gamma}(s)ds.
\end{align*}
\]

\(\hat{R}_{0.1}, \hat{R}_{0.2}\) and \(\hat{R}_1\) are mutually independent Gaussian processes with covariance functions: for \(t, t' \geq 0\),

\[
\begin{align*}
\text{Cov}(\hat{R}_{0.1}(t), \hat{R}_{0.1}(t')) &= \hat{R}(0)(F_{0,R}^c(t \vee t') - F_{0,R}^c(t)F_{0,R}^c(t')), \\
\text{Cov}(\hat{R}_{0.2}(t), \hat{R}_{0.2}(t')) &= \hat{I}(0)\left(\int_0^{t\wedge t'} F_{0,R}^c(t \vee t' - s|s)dsG_0(s) - \Psi_0(t)\Psi_0(t')\right), \\
\text{Cov}(\hat{R}_1(t), \hat{R}_1(t')) &= \int_0^{t\wedge t'} \int_0^{t\wedge t' - s} F_R^c(t \vee t' - u|u)dG(u)(1 - \hat{I}(s) - \hat{R}(s))\hat{\Gamma}(s)ds.
\end{align*}
\]

The processes \(\hat{N}_0, \hat{I}_0, \hat{R}_{0.1}, \hat{R}_{0.2}\) are independent of \(\hat{\Gamma}_1, \hat{\Gamma}_2, \hat{I}_1, \hat{R}_1\). \(\hat{N}_0, \hat{I}_0\) and \(\hat{R}_{0.2}\) are independent of \(\hat{R}_{0.1}\), and have covariance functions

\[
\begin{align*}
\text{Cov}(\hat{N}_0(t), \hat{I}_0(t')) &= \hat{I}(0)(E[\lambda_0(t)1_{\zeta_0 > t'}] - \lambda_0(t)G_0^c(t')), \\
\text{Cov}(\hat{N}_0(t), \hat{R}_{0.2}(t')) &= \hat{I}(0)(E[\lambda_0(t)1_{\zeta_0 \leq t' < \zeta_0 + \eta}] - \lambda_0(t)\Psi(t')), \\
\text{Cov}(\hat{I}_0(t), \hat{R}_{0.2}(t')) &= \hat{I}(0)1(t' \geq t)\left(\int_t^{t'} F_{0,R}^c(t' - s|s)dsG_0(s) - G_0^c(t)\Psi_0(t')\right).
\end{align*}
\]
The processes $\hat{I}_1$, $\hat{I}_2$, $\hat{I}_1$ and $R_1$ have covariance functions: for $t, t' > 0$, $\text{Cov}(\hat{I}_1(t), \hat{I}_2(t')) = 0$, and

$$
\text{Cov}(\hat{I}_1(t), \hat{I}_1(t')) = \int_0^{t \wedge t'} \left( \mathbb{E}[\lambda(t-s) \mathbf{1}_{s \leq t'}] - \lambda(t-s)G^c(t' - s) \right) \left( 1 - I(s) - R(s) \right) \mathcal{F}(s)ds,
$$

$$
\text{Cov}(\hat{I}_1(t), R_1(t')) = \int_0^{t \wedge t'} \left( \mathbb{E}[\lambda(t-s) \mathbf{1}_{s \leq t'}] - \lambda(t-s)\Psi(t' - s) \right) \left( 1 - I(s) - R(s) \right) \mathcal{F}(s)ds,
$$

$$
\text{Cov}(\hat{I}_2(t), \hat{I}_1(t')) = \int_0^{t \wedge t'} \lambda(t-s)G^c(t' - s) \left( 1 - I(s) - R(s) \right) \mathcal{F}(s)ds,
$$

$$
\text{Cov}(\hat{I}_1(t), R_1(t')) = \int_0^{t \wedge t'} \lambda(t-s)\Psi(t' - s) \left( 1 - I(s) - R(s) \right) \mathcal{F}(s)ds,
$$

$$
\text{Cov}(\hat{I}_1(t), R_1(t')) = \mathbf{1}(t' \geq t) \int_0^{t \wedge t'} \int_{t-s}^{t'} F^c_R(t' - s - u) dG(u) \left( 1 - I(s) - R(s) \right) \mathcal{F}(s)ds.
$$

If $\hat{I}^0(t)$, $G_0$ and $F_{0,R}$ are continuous, the limits $\hat{I}$, $\hat{I}$ and $R$ have continuous paths.

REFERENCES


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