Rate of convergence in Wasserstein distance of piecewise-linear Lévy-driven SDEs

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Abstract. In this paper, we study the rate of convergence under the Wasserstein metric of a broad class of multidimensional piecewise Ornstein–Uhlenbeck processes with jumps. These are governed by stochastic differential equations having a piecewise linear drift, and a fairly general class of driving Lévy processes. When the process is irreducible and aperiodic, we identify conditions on the parameters in the drift, the Lévy measure, and/or the covariance function, which result in subexponential or exponential ergodicity under the Wasserstein metric, and in the case of subexponential ergodicity, we exhibit matching lower and upper bounds on the rate of convergence. On the other hand, in the case when the stochastic differential equation might be degenerate, we employ the asymptotic flatness (uniform dissipativity) properties of the drift to establish exponential ergodicity with respect to the Wasserstein metric.

1. Introduction

We study a class of piecewise Ornstein–Uhlenbeck (O–U) processes with jumps governed by a stochastic differential equation (SDE) given in (2.1). Such processes arise as limits of the suitably scaled queueing processes of multiclass many-server queueing networks, or single-class many-server queues with phase-type service times that have heavy-tailed (bursty) arrivals, and/or are subject to asymptotically negligible service interruptions; see Section 4 of [6] for a detailed description. The goal of this paper is to investigate the ergodic properties of such processes. The results apply to a much broader class of SDEs driven by Lévy processes than those arising in these queueing models.

In [6], the subexponential and/or exponential ergodic properties with respect to the total variation distance are studied for this class of SDEs with jumps. The driving Lévy process is either (1) a Brownian motion and a pure-jump Lévy process, or (2) an anisotropic Lévy process with independent one-dimensional symmetric α-stable components, or (3) an anisotropic Lévy process as in (2) and a pure-jump Lévy process. The work in [6] also studies the class of models driven by a subordinate Brownian motion, which contains an isotropic (or rotationally invariant) α-stable Lévy process as a special case. The results on an upper bound of the convergence rate are obtained by using the well-known Lyapunov method (see, e.g., [15,17,23,40]). This method assumes certain structural properties of the process (irreducibility and aperiodicity) which are satisfied if the process is regular enough. For this class of SDEs, this is ensured by non-degeneracy of the diffusion part and/or enough “jump activity” of the Lévy process; see [6, Theorem 3.1] for details. An important, and rather nonstandard result in [6], in the case of subexponential convergence, is a
sharp quantitative characterization of the polynomial rate of convergence via matching upper and lower bounds. An analogous result is reported in this paper (see Theorem 3.2).

It is well known that convergence with respect to the total variation distance does not in general imply convergence with respect to the Wasserstein distance (and vice versa), see, e.g., [47]. On the other hand, it follows directly from the Kantorovich–Rubinstein theorem that $V$-ergodicity, with $V(x) \geq 1 \vee |x|^p$ for $p \geq 1$, implies convergence to the invariant measure with respect to the 1-Wasserstein metric $W_1$ (see (2.4) for its definition). However, this result does not tell us anything about convergence with respect to the $p$-Wasserstein metric $W_p$ for $p > 1$. Our focus in this paper is to investigate the rate of convergence under the Wasserstein metric. In addition to the cases satisfying irreducibility and aperiodicity, we also study the rate of convergence without such structural properties.

1.1. **Summary of the results.** The main results in the case that the processes are irreducible and aperiodic are contained in Theorems 3.1 and 3.3 and Corollary 3.1. These build upon corresponding results under the total variation metric which are established in Theorems 3.2 and 3.5 and Corollary 5.2 of [6], respectively. Two important parameters help us to classify the different cases we consider. One is the value of $\Gamma v$, where in the queueing context, $\Gamma$ is the diagonal matrix of abandonment rates, and $v$ is a constant probability vector. Thus, when $\Gamma v = 0$ the classes of jobs with no abandonment have the lowest priority under the policy $v$. The second parameter is the heaviness of the tail of the Lévy measure. Theorem 3.1 examines the case when $\Gamma v = 0$, in which we also assume that the Lévy measure has finite mean, and that the spare capacity (safety staffing) is positive. We study both cases of subexponential and exponential rate of convergence, in analogy to [6, Theorem 3.2], which depend on the heaviness of the tail of the Lévy measure. We use the $V$-ergodicity property stated in [6, Theorem 3.2], and we apply the Kantorovich-Rubinstein theorem and [15, Theorem 3.2] to compute a rate of convergence with respect to $W_p$ for $p \geq 1$. Theorem 3.2 provides a matching lower bound in the subexponential case, thus rendering this estimate sharp. Theorem 3.3 concerns the case of $\Gamma v \neq 0$, for which we do not need to assume that the spare capacity is positive. Again, under suitable hypotheses on the heaviness of the tail of the Lévy measure, we establish an exponential rate of convergence with respect to $W_1$, and a polynomial rate of convergence with respect to $W_p$ for $p \geq 1$. These results are extended to models with non-constant stationary Markov control in Subsection 3.3, models with general drifts in Corollary 3.1, and general Markov processes in Theorem 3.5.

In Theorem 3.4, for the case of $\Gamma v \neq 0$, we assert an exponential rate of convergence under $W_p$ for $p \geq 1$, without assuming irreducibility and aperiodicity, and this result is extended to general drifts in Corollary 3.2. The proof Theorem 3.4 relies on the property of asymptotic flatness (uniform dissipativity) established in Lemma 4.3. Asymptotic flatness was used in [9] to study the stability in distribution of degenerate diffusions, in particular, the uniqueness of the invariant measure. The results in Theorem 3.4 and Corollary 3.2 allow for degenerate SDEs.

1.2. **Literature review.** Our work contributes to the understanding of the rate of convergence of Markov processes. Most of the existing literature focuses on characterizing the exponential or subexponential rate of convergence under the total variation norm, see, e.g., [17, 26, 39] and references therein. However, there have been some recent developments for Markov processes under the Wasserstein metric. Butkovsky [11] established subgeometric bounds on convergence rates of general Markov processes (both discrete and continuous time) in the Wasserstein metric, extending the results in [15,16,22,23]. Durmus et al. [18] provided sufficient conditions for subgeometric rates of convergence for general state-space Markov chains that are (possibly) not irreducible. In [11], the exponential rate of convergence for a class of stochastic delay equations under the Wasserstein metric is established. In [31], for SDEs of McKean–Vlasov type with Lévy noises, exponential ergodicity is established in the $W_1$ metric, by combining ideas of coupling and Lyapunov functions; see also [32] on the coupling approach. Our work, focusing on a specific class of piecewise linear
Levy-driven SDEs, studies both exponential and subexponential rates of convergence under the Wasserstein distance, and this can be established by following the results with respect to the Wasserstein metric. Observe that if the drift is linear, then asymptotic flatness implies an exponential rate of convergence. All these results with linear and piecewise linear drifts are also proved under the total variation controls are established in [5]; see also the uniform exponential ergodicity for controlled diffusions in [4]. For piecewise O–U drifts in diffusions arise naturally in many-server queueing (network) models [7, 8, 13, 27, 28, 41–43]. For piecewise O–U diffusions arising from many-server queues with phase-type service times, exponential ergodicity is established in [14]. For a broad class of piecewise linear Levy-driven SDEs, exponential and subexponential ergodic properties of such controlled SDEs over any Lipschitz continuous drifts in diffusions is studied in [6]. Uniform exponential and subexponential ergodicity under the Wasserstein metric contribute to this active research topic.

Organization of the paper. In Section 2, we describe the class of SDEs with jumps in detail, and review some basic structural properties. Most of the notation used is also summarized here.

(A1) the function $b: \mathbb{R}^d \to \mathbb{R}^d$ is given by

$$b(x) = \ell - M(x - \langle e, x \rangle^+ v) - \langle e, x \rangle^+ \Gamma v = \begin{cases} \ell - (M + (\Gamma - M)ve^\prime)x, & e^\prime x > 0, \\ \ell - Mx, & e^\prime x \leq 0, \end{cases}$$

where $\ell \in \mathbb{R}^d$, $v \in \mathbb{R}^d_+$ satisfies $\langle e, v \rangle = e^\prime v = 1$ with $e = (1, \ldots, 1)^\prime \in \mathbb{R}^d$, $M \in \mathbb{R}^{d \times d}$ is a nonsingular M-matrix such that the vector $e' M$ has nonnegative components, and $\Gamma = \text{diag}(\gamma_1, \ldots, \gamma_d)$ with $\gamma_i \in \mathbb{R}_+$, $i = 1, \ldots, d$;

(A2) $\{W(t)\}_{t \geq 0}$ is a standard $n$-dimensional Brownian motion, and the covariance function $\sigma: \mathbb{R}^d \to \mathbb{R}^{d \times n}$ is locally Lipschitz and satisfies, for some constant $\kappa > 0$,

$$\|\sigma(x)\|^2 \leq \kappa(1 + |x|^2), \quad x \in \mathbb{R}^d;$$

2. A class of piecewise linear Levy-driven SDEs

We consider a $d$-dimensional stochastic differential equation (SDE) of the form

$$dX(t) = b(X(t)) \, dt + \sigma(X(t)) \, dW(t) + dL(t), \quad X(0) = x \in \mathbb{R}^d. \tag{2.1}$$

Here,
(A3) \( \{L(t)\}_{t \geq 0} \) is a \( d \)-dimensional pure-jump Lévy process specified by a drift \( \vartheta \in \mathbb{R}^d \) and Lévy measure \( \nu \).

In the above, the symbol \( \langle \cdot, \cdot \rangle \) stands for the inner product on \( \mathbb{R}^d \), and \( \|A\| := (\text{Tr} \ A A^\top)^{1/2} \) denotes the Hilbert-Schmidt norm of a \( d \times n \) matrix \( A \). Throughout the paper, \( \text{Tr} S \) stands for the trace of a square matrix \( S \), and for a vector and a matrix \( A \), \( x' \) and \( A' \) stand for their transposes, respectively. A \( d \times d \) matrix \( M \) is called an M-matrix if it can be expressed as \( M = sI - N \) for some \( s > 0 \) and some nonnegative \( d \times d \) matrix \( N \) with the property that \( \rho(N) \leq s \), where \( I \) and \( \rho(N) \) denote the \( d \times d \) identity matrix and the spectral radius of \( N \), respectively. Clearly, the matrix \( M \) is nonsingular if \( \rho(N) < s \). Recall that a Lévy measure \( \nu \) is a \( \sigma \)-finite measure on \( \mathbb{R}^d \setminus \{0\} \) satisfying \( \int_{\mathbb{R}^d} (1 \land |y|^2) \, \nu(dy) < \infty \). We define

\[
\Theta_c := \left\{ \theta \geq 0 : \int_{B_c} |y|^\theta \, \nu(dy) < \infty \right\},
\]

In the case of \( \alpha \)-stable jumps for \( \alpha \in (1, 2) \), we have \( \Theta_c = (0, \alpha) \). If bounded, \( \Theta_c \) is an open or left-open interval by definition. If \( 1 \in \Theta \), then we define

\[
\ell := \ell + \vartheta + \int_{B_c} y \nu(dy).
\]

Such an SDE is often called a piecewise Ornstein–Uhlenbeck (O–U) process with jumps. It is well-known that the SDE (2.1) admits a unique nonexplosive (conservative) strong solution \( \{X(t)\}_{t \geq 0} \) which is a strong Markov process. We let \( P_t(x, \cdot) := \mathbb{P}^x(X(t) \in \cdot) \) and \( P_1 f(x) := \int_{\mathbb{R}^d} f(y) P_t(x, dy) \), with \( t \geq 0 \), \( x \in \mathbb{R}^d \), and \( f \in \mathcal{B}(\mathbb{R}^d) \), where \( \mathcal{B}(\mathbb{R}^d) \) denotes the class of Borel measurable functions on \( \mathbb{R}^d \). Since \( \{X(t)\}_{t \geq 0} \) is nonexplosive, \( P_t(x, \cdot) \) is a probability measure for each \( t \geq 0 \) and \( x \in \mathbb{R}^d \). Also, \( \{X(t)\}_{t \geq 0} \) satisfies the \( C_b \)-Feller property, that is, \( P_t(C_b(\mathbb{R}^d)) \subseteq C_b(\mathbb{R}^d) \) for all \( t \geq 0 \) (see [1, Theorem 3.1, and Propositions 4.2 and 4.3]). Here, \( C_b(\mathbb{R}^d) \) denotes the space of bounded continuous functions. In addition, in the same reference, it is shown that the infinitesimal generator \( (A, \mathcal{D}_A) \) of \( \{X(t)\}_{t \geq 0} \) (with respect to the Banach space \( (\mathcal{B}(\mathbb{R}^d), \|\cdot\|_{\infty}) \)) satisfies \( C^2(\mathbb{R}^d) \subseteq \mathcal{D}_A \) and

\[
A|_{C^2_b(\mathbb{R}^d)} f(x) = \frac{1}{2} \text{Tr}(a(x) \nabla^2 f(x)) + \langle b(x) + \vartheta, \nabla f(x) \rangle + \int_{\mathbb{R}^d} \theta_1 f(x; y) \nu(dy),
\]

with \( \nabla^2 f \) denoting the Hessian of \( f \). Here, \( \mathcal{D}_A, \mathcal{B}_b(\mathbb{R}^d) \) and \( C^2_b(\mathbb{R}^d) \) denote the domain of \( A \), the space of bounded Borel measurable functions and the space of twice continuously differentiable functions with compact support, respectively. In (2.2) we use the notation \( a(x) = (a^{ij}(x))_{1 \leq i, j \leq d} := \sigma(x)\sigma(x)' \), and

\[
\theta_1 f(x; y) := f(x + y) - f(x) - \mathbb{1}_B(y) y \nabla f(x), \quad f \in C^1(\mathbb{R}^d),
\]

where \( B \) denotes the unit ball in \( \mathbb{R}^d \) centered at 0, and \( \mathbb{1}_B \) its indicator function. Since \( \{X(t)\}_{t \geq 0} \) is a Markov process, \( \{P_t\}_{t \geq 0} \) is a semigroup of linear operators on the Banach space \( (\mathcal{B}_b(\mathbb{R}^d), \|\cdot\|_{\infty}) \), that is, \( P_s \circ P_t = P_{s+t} \) for all \( s, t \geq 0 \), and \( P_0 f = f \). Here, \( \|\cdot\|_{\infty} \) denotes the supremum norm on the space of bounded Borel measurable functions \( \mathcal{B}_b(\mathbb{R}^d) \). Recall that the infinitesimal generator \( (A, \mathcal{D}_A) \) of the semigroup \( \{P_t\}_{t \geq 0} \) of \( \{X(t)\}_{t \geq 0} \) is a linear operator \( A : \mathcal{D}_A \rightarrow \mathcal{B}_b(\mathbb{R}^d) \) defined by

\[
Af := \lim_{t \to 0} \frac{P_t f - f}{t}, \quad f \in \mathcal{D}_A := \left\{ f \in \mathcal{B}_b(\mathbb{R}^d) : \lim_{t \to 0} \frac{P_t f - f}{t} \text{ exists in } \|\cdot\|_{\infty} \right\}.
\]

Also, the extended domain of \( \{X(t)\}_{t \geq 0} \), denoted by \( \mathcal{D}_A \), is defined as the set of all \( f \in \mathcal{B}(\mathbb{R}^d) \) such that \( f(X(t)) - f(X(0)) - \int_0^t g(X(s)) \, ds \) is a local \( \mathbb{P}^\vartheta \)-martingale for some \( g \in \mathcal{B}(\mathbb{R}^d) \) and all \( x \in \mathbb{R}^d \). Let us remark that, in general, the function \( g \) does not have to be unique (see [21, p. 24]).
For \( f \in \mathcal{D}_A \),
\[
\hat{A}f := \left\{ g \in \mathcal{B}(\mathbb{R}^d) : f(X(t)) - f(X(0)) - \int_0^t g(X(s)) \, ds \text{ is a local } \mathbb{P}^x\text{-martingale} \right\}.
\]

We call \( \hat{A} \) the extended generator of \( \{X(t)\}_{t \geq 0} \). A function \( g \in \hat{A}f \) is usually abbreviated by \( \hat{A}f(x) := g(x) \). A well-known fact is that \( \mathcal{D}_A \subseteq \mathcal{D}_\hat{A} \), and for \( f \in \mathcal{D}_A \), the function \( \hat{A}f \) is contained in \( \hat{A}f \) (see [21, Proposition IV.1.7]). Also, it has been shown in [37, Lemma 3.7; 38] that
\[
\mathcal{D} := \left\{ f \in C^2(\mathbb{R}^d) : x \mapsto \left| \int_{\mathbb{R}^d} f(x + y) \, \nu(dy) \right| \text{ is locally bounded} \right\} \subseteq \mathcal{D}_A,
\]
and on this set, for the function \( \hat{A}f(x) \) we can take exactly \( Af(x) \), where \( A \) is given by (2.2).

Finally, we recall some standard concepts and results from the ergodic theory of Markov processes. Let \( \mathcal{B}(\mathbb{R}^d) \) denote the Borel \( \sigma \)-algebra on \( \mathbb{R}^d \). We say that the process \( \{X(t)\}_{t \geq 0} \) is

(i) \( \varphi \)-irreducible if there exists a \( \sigma \)-finite measure \( \varphi \) on \( \mathcal{B}(\mathbb{R}^d) \) such that, whenever \( \varphi(B) > 0 \), we have \( \int_0^\infty P_t(x, B) \, dt > 0 \) for all \( x \in \mathbb{R}^d \);

(ii) transient if it is \( \varphi \)-irreducible, and if there exists a collection of nonnegative constants \( \{c_j\}_{j \in \mathbb{N}} \) and a countable covering \( \{B_j\}_{j \in \mathbb{N}} \subseteq \mathcal{B}(\mathbb{R}^d) \) of \( \mathbb{R}^d \), such that \( \int_0^\infty P_t^B(x, B) \, dt \leq c_j \) for all \( x \in \mathbb{R}^d \) and \( j \in \mathbb{N} \);

(iii) recurrent if it is \( \varphi \)-irreducible, and \( \varphi(B) > 0 \) implies that \( \int_0^\infty P_t^B(x, B) \, dt = \infty \) for all \( x \in \mathbb{R}^d \);

(iv) aperiodic if it admits an irreducible skeleton chain, or in other words, if there exist a constant \( t_0 > 0 \) and a \( \sigma \)-finite measure \( \phi \) on \( \mathcal{B}(\mathbb{R}^d) \), such that \( \phi(B) > 0 \) implies \( \sum_{n=0}^\infty P_{nt_0}(x, B) > 0 \) for all \( x \in \mathbb{R}^d \).

Let us remark that if \( \{X(t)\}_{t \geq 0} \) is \( \varphi \)-irreducible, then it is either transient or recurrent (see [46, Theorem 2.3]). Denote by \( \mathcal{P}(\mathbb{R}^d) \) the class of Borel probability measures on \( \mathbb{R}^d \). We adopt the usual notation \( \pi P_t(\cdot, \cdot) = \int_{\mathbb{R}^d} \pi(dx) P_t(x, \cdot) \) and \( \pi f = \int_{\mathbb{R}^d} f(x) \pi(dx) \) for \( t \geq 0, \pi \in \mathcal{P}(\mathbb{R}^d) \), and \( f \in \mathcal{B}(\mathbb{R}^d) \). Therefore, with \( \delta_x \) denoting the Dirac measure concentrated at \( x \in \mathbb{R}^d \), we have \( \delta_x P_t(\cdot) = P_t(x, \cdot) \). Recall that a probability measure \( \pi \in \mathcal{P}(\mathbb{R}^d) \) is called invariant for \( \{X(t)\}_{t \geq 0} \) if \( \pi P_t(\cdot) = \pi(\cdot) \) for all \( t > 0 \). It is well known that if \( \{X(t)\}_{t \geq 0} \) is recurrent, then it possesses a unique (up to constant multiples) invariant measure \( \pi \) (see [46, Theorem 2.6]). If the invariant measure is finite, then it may be normalized to a probability measure. If \( \{X(t)\}_{t \geq 0} \) is recurrent with finite invariant measure, then \( \{X(t)\}_{t \geq 0} \) is called positive recurrent; otherwise it is called null recurrent. Note that a transient Markov process cannot have a finite invariant measure.

In [6, Theorem 3.1], we have shown that \( \{X(t)\}_{t \geq 0} \) in (2.1) is irreducible and aperiodic if one of the following four conditions holds:

(i) \( \nu(\mathbb{R}^d) < \infty \), and for every \( R > 0 \) there exists \( c_R > 0 \) such that
\[
(y, a(x)y) \geq c_R |y|^2, \quad x, y \in \mathbb{R}^d, \quad |x|, |y| \leq R.
\]

(ii) \( \nu(O) > 0 \) for any non-empty open set \( O \subseteq \mathcal{B} \), \( \sigma : \mathbb{R}^d \to \mathbb{R}^{d \times d} \) is Lipschitz continuous and invertible for any \( x \in \mathbb{R}^d \), and
\[
\sup_{x \in \mathbb{R}^d} \|\sigma^{-1}(x)\| > 0.
\]

(iii) \( \sigma(x) \equiv \sigma \) and \( \{L(t)\}_{t \geq 0} \) is of the form \( L(t) = L_1(t) + L_2(t), t \geq 0 \), where \( \{L_1(t)\}_{t \geq 0} \) and \( \{L_2(t)\}_{t \geq 0} \) are independent \( d \)-dimensional pure-jump Lévy processes, such that \( \{L_1(t)\}_{t \geq 0} \) is a subordinate Brownian motion.

(iv) \( \sigma(x) \equiv 0 \) and \( \{L(t)\}_{t \geq 0} \) is of the form \( L(t) = L_1(t) + L_2(t), t \geq 0 \), where \( \{L_1(t)\}_{t \geq 0} \) and \( \{L_2(t)\}_{t \geq 0} \) are independent \( d \)-dimensional pure-jump Lévy processes, such that \( \{L_1(t)\}_{t \geq 0} \) is an anisotropic Lévy process with independent symmetric one-dimensional \( \alpha \)-stable components for \( \alpha \in (0, 2) \), and \( \{L_2(t)\}_{t \geq 0} \) is a compound Poisson process.
We introduce some notation we need in the sequel.

**Notation 2.1.** For a vector \( z \in \mathbb{R}^d \) we write \( z \geq 0 \ (z > 0) \) to indicate that all the components of \( z \) are nonnegative (positive), and analogously for a matrix in \( \mathbb{R}^{d \times d} \). We let

\[
\Delta := \{ v \in \mathbb{R}^d : v \geq 0, \langle e, v \rangle = 1 \}.
\]

Throughout the paper, \( v \) denotes an element of \( \Delta \), unless indicated otherwise. For a symmetric matrix \( S \in \mathbb{R}^{d \times d} \) we write \( S \succeq 0 \ (S > 0) \) to indicate that it is positive semidefinite (positive definite), and we let \( \mathcal{M}_+ \) denote the class of positive definite symmetric matrices in \( \mathbb{R}^{d \times d} \). For \( Q \in \mathcal{M}_+ \), we let \( \| x \|_Q := \langle x, Q x \rangle^{1/2} \) for \( x \in \mathbb{R}^d \). Let \( \phi(x) \) be some fixed nonnegative, convex smooth function which agrees with \( \| x \|_Q \) on the complement of the unit ball centered at 0 in \( \mathbb{R}^d \). For \( \delta > 0 \) we define \( V_{Q, \delta} := (\phi(x))^\delta \) and \( \widetilde{V}_{Q, \delta} := e^{\delta \phi(x)} \), and by \( \mathcal{P}_p(\mathbb{R}^d), \ p > 0 \), we denote the subset of \( \mathcal{P}(\mathbb{R}^d) \) containing all probability measures \( \mu \) with the property that \( \int_{\mathbb{R}^d} |x|^p \mu(dx) < \infty \).

Finally, recall that the \( p \)-Wasserstein metric on \( \mathcal{P}_p(\mathbb{R}^d), \ p \geq 1 \), is defined by

\[
W_p(\mu_1, \mu_2) := \inf_{\Pi \in \mathcal{C}(\mu_1, \mu_2)} \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p \Pi(dx, dy) \right)^{1/p},
\]

where \( \mathcal{C}(\mu_1, \mu_2) \) is the family of couplings of \( \mu_1 \) and \( \mu_2 \), that is, \( \Pi \in \mathcal{C}(\mu_1, \mu_2) \) if, and only if, \( \Pi \) is a measure in \( \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) \) having \( \mu_1 \) and \( \mu_2 \) as its marginals. Note that under the metric \( W_p \), the space \( \mathcal{P}_p(\mathbb{R}^d) \) becomes a complete separable metric space (see [47, Theorem 6.18]). The topology generated by \( W_p \) on \( \mathcal{P}_p(\mathbb{R}^d) \) is finer than the one induced by the Prokhorov topology, that is, the topology of weak convergence.

### 3. Main Results

We are now in position to state the main results of this paper. In all these results, \( \{X(t)\}_{t \geq 0} \) refers to a solution of (2.1).

#### 3.1. The case \( \Gamma v = 0 \)

We first discuss the case when \( \Gamma v = 0 \).

**Theorem 3.1.** We assume that \( \{X(t)\}_{t \geq 0} \) is irreducible and aperiodic, \( \Gamma v = 0 \), \( 1 \in \Theta_c \), and \( \langle e, M^{-1} \tilde{\ell} \rangle < 0 \).

(a) If

\[
\limsup_{|x| \to \infty} \frac{\| a(x) \|}{|x|} < \infty,
\]

then \( \{X(t)\}_{t \geq 0} \) admits unique invariant measure \( \pi \in \mathcal{P}_{\theta-1}(\mathbb{R}^d) \) for all \( \theta > 1, \ \theta \in \Theta_c \). Further, if \( 2 \in \Theta_c \), then for any \( \theta > 2, \ \theta \in \Theta_c \), the following hold.

(i) There exists \( C_\theta > 0 \) such that

\[
(1 \vee t)^{\theta-2} W_1(\delta_x P_t, \pi) \leq C_\theta (1 + |x|^\theta), \quad x \in \mathbb{R}^d, \ t \geq 0,
\]

\[
\int_0^\infty (1 \vee t)^{\theta-2} W_1(\delta_x P_t, \delta_x P_t) \, dt \leq C_\theta (1 + |x|^\theta + |y|^\theta), \quad x, y \in \mathbb{R}^d,
\]

and, provided \( \theta \geq 3 \), we have

\[
\int_0^\infty (1 \vee t)^{\theta-3} W_1(\delta_x P_t, \pi) \, dt \leq C_\theta (1 + |x|^\theta), \quad x \in \mathbb{R}^d.
\]

(ii) Let

\[
\overline{m}_p := \int_{\mathbb{R}^d} |x|^p \pi(dx), \quad p \geq 0.
\]

For each \( p \in [1, \theta - 1] \), with \( \theta \in \Theta_c \), there exists \( C_p > 0 \) such that

\[
(1 \vee t)^{\theta-1+p} W_p(\delta_x P_t, \pi) \leq C_p \overline{m}_{\theta-1} + |x|^\theta \right)^{1/p}, \quad x \in \mathbb{R}^d, \ t \geq 0.
\]
In addition,
\[ \lim_{t \to \infty} W_{\theta-1}(\pi P_t, \pi) = 0, \quad \pi \in \mathcal{P}(\mathbb{R}^d). \] (3.7)

(b) If \( a(x) \) is bounded and
\[ \int_{\mathbb{R}^d} e^{\theta|y|} \nu(\mathrm{d}y) < \infty \] for some \( \theta > 0 \), then \( \{X(t)\}_{t \geq 0} \) admits a unique invariant measure \( \pi \in \mathcal{P}(\mathbb{R}^d) \), and there exists \( Q \in \mathcal{M}_+ \) such that \( \int_{\mathbb{R}^d} e^{\theta|y|} q \pi(\mathrm{d}y) < \infty \) for any \( 0 < r < \theta \|Q\|^{-1/2} \). In addition, the following hold.

(i) For any \( 0 < r < \theta \|Q\|^{-1/2} \) there exist positive constants \( \gamma \) and \( C_\gamma \), such that
\[ W_1(\delta_x P_t, \pi) \leq C_\gamma e^{r\|x\|} e^{-\gamma t}, \quad x \in \mathbb{R}^d, \quad t \geq 0. \] (3.9)

(ii) For any \( 0 < r < \theta \|Q\|^{-1/2} \) and any \( q \geq p \geq 1 \) there exists \( C_{r,p,q} > 0 \) such that
\[ (1 \vee t)^{q-p} W_p(\delta_x P_t, \pi) \leq C_{r,p,q} \left(e^{r\|x\|} q + 1\right)^{1/p}, \quad x \in \mathbb{R}^d, \quad t \geq 0. \] (3.10)

In addition, for any \( p \geq 1 \), we have
\[ \lim_{t \to \infty} W_p(\pi P_t, \pi) = 0, \] for all \( \pi \in \mathcal{P}(\mathbb{R}^d) \) such that \( \int_{\mathbb{R}^d} e^{r\|x\|} q \pi(\mathrm{d}x) < \infty \) for some \( 0 < r < \theta \|Q\|^{-1/2} \).

Remark 3.1. In [6, Theorem 3.3 (b) and Lemma 5.7] it has been shown that the assumptions \( 1 \in \Theta_c \) and \( \langle e, M^{-1} \tilde{e} \rangle < 0 \) are both necessary for the existence of an invariant probability measure of \( \{X(t)\}_{t \geq 0} \). Using this, we can exhibit an example where we have convergence in total variation but no convergence in \( W_1 \). Suppose that \( \Gamma v = 0, \langle e, M^{-1} \tilde{e} \rangle < 0 \), and \( \{L(t)\}_{t \geq 0} \) is an \( \alpha \)-stable process, or is an anisotropic Lévy process with independent symmetric one-dimensional \( \alpha \)-stable components for \( \alpha \in (1, 2) \). Then [6, Theorem 3.1 (i)] shows that
\[ \lim_{t \to \infty} t^{\alpha-1-\epsilon} \|\delta_x P_t(\cdot) - \pi(\cdot)\|_{TV} = 0, \quad \forall x \in \mathbb{R}^d. \]
for all \( \epsilon > 0 \). However \( \int_{\mathbb{R}^d} |x| \pi(\mathrm{d}x) = \infty \) by [6, Theorem 3.4 (b)], so we cannot have convergence in \( W_1 \).

We next exhibit a lower bound on the polynomial rate of convergence in Theorem 3.1 (a), which is analogous to [6, Theorem 3.4]. Note that the lower bound in (3.11) below matches the upper bound in (3.2). We let \( \theta_c := \sup \{ \theta \in \Theta_c \} \), and
\[ \tilde{\theta}_c := \sup \left\{ \theta : \int_{\mathbb{R}^d} \left(\langle e, M^{-1} x \rangle^+ \right)^{\theta} \nu(\mathrm{d}x) < \infty \right\}. \] Note that, in general, \( \tilde{\theta}_c \geq \theta_c \).

Remark 3.2. In [5,6] it is assumed that \( \{L(t)\}_{t \geq 0} \) is a compound Poisson process with drift \( \vartheta \), and Lévy measure \( \nu \) which is supported on a half-line of the form \( \{tw : t \in [0, \infty)\} \) with \( \langle e, M^{-1} w \rangle > 0 \). This implies that \( \tilde{\theta}_c = \theta_c \), and subsequently, this equality is used in the proof of [6, Lemma 5.7 (b)] to establish that, provided \( \Gamma v = 0 \), \( \int_{\mathbb{R}^d} \left(\langle e, M^{-1} x \rangle^+ \right)^{p-1} \pi(\mathrm{d}x) < \infty \) implies \( p \in \Theta_c \). We use this fact, namely that the conclusions of [6, Lemma 5.7 (b)] hold under the weaker assumption that \( \tilde{\theta}_c = \theta_c \) in the proof of Theorem 3.2 which follows.

Theorem 3.2. We assume that \( \{X(t)\}_{t \geq 0} \) is irreducible and aperiodic, \( \langle e, M^{-1} \tilde{e} \rangle < 0 \), \( \Gamma v = 0 \), and that \( \tilde{\theta}_c = \theta_c \in (2, \infty) \). Then for each \( p \in [1, \theta_c - 1) \), \( \epsilon \in (0, 1/\alpha) \), and \( x \in \mathbb{R}^d \), there exist a positive constant \( C \) and a diverging increasing sequence \( \{t_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R} \), depending on these parameters, such that
\[ W_p(\delta_x P_{t_n}, \pi) \geq C(t_n + |x|^{\theta_c - \epsilon})^{-\frac{\theta_n - p - 1 + \epsilon}{(1 - \epsilon)/p}}, \] (3.11)
3.2. The case $\Gamma v \neq 0$. We now discuss the case when $\Gamma v \neq 0$.

**Theorem 3.3.** Suppose that $\{X(t)\}_{t \geq 0}$ is irreducible and aperiodic,

$$\limsup_{|x| \to \infty} \frac{|a(x)|}{|x|^2} = 0, \quad (3.12)$$

and that one of the following holds:

(a) $Mv \geq \Gamma v \geq 0$;

(b) $M = \text{diag}(m_1, \ldots, m_d)$ with $m_i > 0$, $i = 1, \ldots, d$, and $\Gamma v \neq 0$.

Then $\{X(t)\}_{t \geq 0}$ admits a unique invariant probability measure $\pi \in \mathcal{P}_\rho(\mathbb{R}^d)$, for any $\theta \in \Theta_c$. Furthermore, the following hold.

(i) If $1 \in \Theta_c$, then for any $\theta \geq 1$, $\theta \in \Theta_c$, there exist positive constants $\gamma$ and $C_\gamma$, such that

$$\mathcal{W}_1(\delta_x P_t, \pi) \leq C_\gamma|x|^\theta e^{-\gamma t}, \quad x \in \mathbb{R}^d, \ t \geq 0. \quad (3.13)$$

(ii) For any $p \in [1, \theta] \subset \Theta_c$ there exists a constant $C_p > 0$ such that

$$(1 + t)^{\frac{\theta - 1}{p}} \mathcal{W}_p(\delta_x P_t, \pi) \leq C_p(\mathbb{E} |x|^\theta)^{1/p}, \quad x \in \mathbb{R}^d, \ t \geq 0. \quad (3.14)$$

In addition,

$$\lim_{t \to \infty} \mathcal{W}_p(\pi P_t, \pi) = 0, \quad \forall \pi \in \mathcal{P}_\rho(\mathbb{R}^d).$$

**Remark 3.3.** The results in (3.6), (3.9), (3.10), (3.13), and (3.14) should be compared to equation (2.5) in [11, Theorem 2.4]. See also [18, Theorem 3(ii)], which is the analogous result in the discrete-time case. The Wasserstein distance $\mathcal{W}_\rho$ with respect to a bounded metric $\rho$ on the state space is considered in [11]. The starting point is a Foster-Lyapunov condition of the form

$$\mathbb{E}_x[V(X_t)] - V(x) \leq \kappa t - \int_0^t \mathbb{E}_x[\phi(V(X_s))] \, ds, \quad (3.15)$$

where $\kappa > 0$, $V : \mathbb{R}^d \to [0, \infty)$ is measurable, and $\phi : [0, \infty) \to [0, \infty)$ is continuously differentiable, concave, nondecreasing, and vanishing at zero. It is shown that if $\rho$ is contracting, and the sublevel sets are $\rho$-small (see (3) and (4) in [11, Theorem 2.4]), then an analogous estimate to (3.6) holds for $\mathcal{W}_\rho$. For the model studied in this paper, the aforementioned Foster–Lyapunov condition is given by

$$\mathcal{A}V(x) \leq c_0 - c_1 \phi(V(x)), \quad (3.16)$$

for some $c_0, c_1 > 0$, where (see [6, Theorems 3.2 and 3.5])

(i) $\phi(t) = t^{\theta - 1/\theta}$ and $V = V_{Q, \theta}$ in the case of Theorem 3.1 (a);

(ii) $\phi(t) = t$ and $V = V_{Q, x}$ in the case of Theorem 3.1 (b);

(iii) $\phi(t) = t$ and $V = V_{Q, \theta}$ in the case of Theorem 3.3.

Let us remark that (3.16) implies (3.15) (see [15, Theorem 3.4]). We derive the estimate in (3.6) for $\theta \geq 2$ and in (3.14) for $\theta \geq p$, since otherwise it is not in general the case that $\pi \in \mathcal{P}_1(\mathbb{R}^d)$ and $\pi \in \mathcal{P}_p(\mathbb{R}^d)$, respectively. Let us also remark that under the assumptions of Theorems 3.1 and 3.3 (or [6, Theorems 3.2 and 3.5]), [47, Theorem 6.15] implies polynomial and/or exponential ergodicity of $\{X(t)\}_{t \geq 0}$ with respect to $\mathcal{W}_\rho$ for any bounded metric $\rho$ on $\mathbb{R}^d$, which is an analogous result to the one obtained in [11, Theorem 2.4], but without assuming either contraction properties of $\rho$ or $\rho$-smallness of the sublevel sets. Observing that the proofs of Theorems 3.1 and 3.3 can be applied to more general settings, we extend these results to irreducible and aperiodic $\mathbb{R}^d$-valued càdlàg strong Markov processes in Theorem 3.5 in Subsection 3.5.

In the case when $\Gamma v \neq 0$ the dynamics are contractive in the metric $\mathcal{W}_\rho$. This is shown by establishing an asymptotic flatness (uniform dissipativity) property for $\{X(t)\}_{t \geq 0}$ (see Lemma 4.3). As a consequence, we assert exponential ergodicity of $\{X(t)\}_{t \geq 0}$ with respect to $\mathcal{W}_\rho$, without
assuming irreducibility and aperiodicity, that is, we allow the SDE to be degenerate. In fact, as in [9], if the covariance matrix $\sigma$ is present, we assume that $\sigma$ is Lipschitz continuous, but allow $a = \sigma \sigma'$ to be singular. See also Remarks 3.5 and 3.7.

**Theorem 3.4.** Suppose that $\theta \in \Theta_c$, $\theta \geq 1$, $\sigma$ is Lipschitz continuous, and that one of the following holds:

(a) $Mv \geq \Gamma v \geq 0$;
(b) $M = \text{diag}(m_1, \ldots, m_d)$ with $m_i > 0$, $i = 1, \ldots, d$, and $\Gamma v \neq 0$.

Then there exists $Q \in M_+$ such that

$$MQ + QM > 0, \quad \text{and} \quad (M - ev'(M - \Gamma))Q + Q(M - (M - \Gamma)e\nu') \succ 0. \quad (3.17)$$

Let $\kappa$ denote the smallest eigenvalue of the positive definite matrices in (3.17), and $\lambda_Q$ ($\lambda_Q$) denote the largest (smallest) eigenvalue of $Q$. For $p \geq 0$, let

$$c_p := \frac{p}{2} \left( \frac{(\kappa - \text{Lip}^2(\sigma \sqrt{Q})) - (p - 1) \text{Lip}^2(\sigma Q)}{\lambda_Q^2} \right),$$

where Lip$(\sigma \sqrt{Q})$ and Lip$(\sigma Q)$ are the Lipschitz constants of $\sigma \sqrt{Q}$ and $\sigma Q$ with respect to the Hilbert–Schmidt norm, respectively, and suppose that the constant $c_p$ is positive. Then for any $p \in [1, \theta]$ we have

$$\mathcal{W}_p(\delta_x P_t, \delta_y P_t) \leq \left( \frac{\kappa}{\lambda_Q} \right)^{1/2} |x - y| e^{-c_p t/\theta}, \quad x, y \in \mathbb{R}^d, \quad t \geq 0. \quad (3.18)$$

In addition, $\{X(t)\}_{t \geq 0}$ admits a unique invariant probability measure $\pi \in \mathcal{P}_p(\mathbb{R}^d)$, and

$$\mathcal{W}_p(\pi P_t, \pi) \leq \left( \frac{\kappa}{\lambda_Q} \right)^{1/2} \mathcal{W}_p(\pi, \pi) e^{-c_p t/\theta} \quad \text{for all} \ t \geq 0 \quad \text{and} \ \pi \in \mathcal{P}_p(\mathbb{R}^d). \quad (3.19)$$

**Remark 3.4.** The hypothesis in Theorem 3.4 that $c_p > 0$ is, of course, always true if $\sigma$ is a constant matrix, in which case we have $c_p = \frac{\nu}{2X_0}$. This is the scenario for multiclass queueing models with service interruptions described in [6, Section 4.2].

**Remark 3.5.** Some examples of degenerate SDEs for which Theorem 3.4 is applicable are the following.

(i) $\{L(t)\}_{t \geq 0}$ is given by $L(t) = R \tilde{L}(t)$, where $R \in \mathbb{R}^{d \times m}$ has rank smaller than $\min\{d, m\}$ and $\tilde{L}(t)$ is an $m$-dimensional Lévy process. As a special case $\tilde{L}(t)$ may be composed of mutually independent $\alpha$-stable processes. This is the case in the queueing example in Remark 3.7.

(ii) $\{L(t)\}_{t \geq 0}$ is a degenerate subordinate Brownian motion, as studied in [52].

**Remark 3.6.** We remark here that without assuming irreducibility and aperiodicity, establishing subgeometric ergodicity in the case $\Gamma v = 0$ is difficult. In order to see this, consider the following example. Let $d = 1$, $L(t) \equiv 0$, $t \geq 0$, and

$$b(x) = \begin{cases} -1, & x \geq 0, \\ -1 - x, & x \leq 0. \end{cases}$$

Clearly, $b(x)$ satisfies all the assumptions in [6], and

$$X^x(t) = x + \int_0^t b(X^x(s)) \, ds, \quad t \geq 0, \ x \in \mathbb{R}, \ X^x(0) = x.$$
A straightforward calculation shows that

\[ X^x(t) = \begin{cases} 
  x - t, & 0 \leq t \leq x, \\
  e^{-t} - 1, & t \geq x, \\
  -1 + e^{-t} + xe^{-t}, & x \leq 0.
\end{cases} \]

Let

\[ \rho(x, y) := \frac{|x - y|}{1 + |x - y|}. \]

Then it is easy to see that the conditions (1)-(3) in [11] hold true. However, condition (4) in [11] Theorem 3.4 is applicable. Consider a two class GI/M/n queue

Remark 3.7. The following is an example of a degenerate SDE that arises in applications for which Theorem 3.4 is applicable. Consider a two class GI/M/n + M queue with class-1 jobs having a Poisson process, and class-2 jobs having a heavy-tailed renewal arrival process. Service and patience times are exponentially distributed with rates \( \mu_i \) and \( \gamma_i \) for \( i = 1, 2 \), respectively. Assume that the arrival, service and abandonment processes are mutually independent, and that the number of servers is \( n \). Consider a sequence of such models indexed by \( n \), operating in the critically loaded asymptotic modified Halfin–Whitt regime as \( n \to \infty \). Let \( \{A^n_i(t)\}_{t \geq 0} \) denote the arrival process for class \( i \), with arrival rates \( \lambda^n_i \). Assume that \( \mu_i \) and \( \gamma_i \), \( i = 1, 2 \), are independent of \( n \), and that \( \lambda^n_i \to \lambda_i > 0 \) as \( n \to \infty \), for \( i = 1, 2 \). Suppose that the arrival process \( \{A^n_i(t)\}_{t \geq 0} \) satisfies a functional central limit theorem (FCLT) with a Brownian motion (BM) limit \( \{A_1(t)\}_{t \geq 0} = \{\sqrt{\lambda_1}B_1(t)\}_{t \geq 0} \), where \( \{B_1(t)\}_{t \geq 0} \) is a standard BM, that is, \( \{A^n_1(t)\}_{t \geq 0} = \{n^{-1/2}(A^n_1(t) - \lambda_1^n t)\}_{t \geq 0} \overset{\text{d}}{=} \{\hat{A}_1(t)\}_{t \geq 0} \) as \( n \to \infty \). Here, \( \overset{\text{d}}{=} \) denotes the convergence in the space \( D = D([0, \infty), \mathbb{R}_+) \) of càdlàg functions endowed with the Skorokhod \( J_1 \) topology. We assume that the arrival process \( \{A^n_2(t)\}_{t \geq 0} \) satisfies an FCLT with a symmetric \( \alpha \)-stable Lévy process \( \{\hat{A}_2(t)\}_{t \geq 0}, \alpha \in (1, 2) \), that is,

\[ \{\hat{A}_2^n(t)\}_{t \geq 0} = \{n^{-1/\alpha}(\hat{A}_2^n(t) - \lambda^n_2 t)\}_{t \geq 0} \overset{\text{M}1}{\Rightarrow} \{\hat{A}_2(t)\}_{t \geq 0} \]

as \( n \to \infty \). Here, \( \overset{\text{M}1}{\Rightarrow} \) denotes the convergence in the space \( D \) with the \( M_1 \) topology. Let \( \rho^n_i = \frac{\lambda^n_i}{\mu_i} \) and \( \rho_i = \frac{\lambda_i}{\mu_i} \) for \( i = 1, 2 \). The modified Halfin-Whitt regime requires the parameters satisfy

\[ n^{1-1/\alpha}(1 - \sum_{i=1}^{2} \rho^n_i) \xrightarrow{n \to \infty} \hat{\rho} \in \mathbb{R}, \quad \text{and} \quad \sum_{i=1}^{2} \rho_i = 1. \]

In addition, we assume that \( n^{-\alpha}(\lambda^n_i - n\lambda_i) \to \hat{\lambda}_i \) as \( n \to \infty \) for \( i = 1, 2 \).

Let \( \{X^n_i(t)\}_{t \geq 0} \) denote the number of class-\( i \) jobs in the system. Define the scaled processes

\[ \hat{X}^n_i(t) = n^{-1/\alpha}(X^n_i(t) - n\rho_i t), \quad t \geq 0. \]

Let \( \{U^n_i(t)\}_{t \geq 0} \) be the scheduling control process, representing allocations of service capacity to class \( i \). Let \( \hat{X}^n(t) = (\hat{X}^n_1(t), \hat{X}^n_2(t))' \) and \( U^n(t) = (U^n_1(t), U^n_2(t))' \), \( t \geq 0 \). We consider work conserving and preemptive scheduling policies resulting in constant controls in the limit, that is, \( \{U^n(t)\}_{t \geq 0} \overset{\text{d}}{=} \{V(t)\}_{t \geq 0} \) as \( n \to \infty \), where \( V(t) = v \) for \( t \geq 0 \) and \( v = (v_1, v_2)' \in \mathbb{R}_+^2 \) satisfies \( \langle e, v \rangle = 1 \). Then, as in [6, Theorem 4.1], it can be shown that

\[ \{\hat{X}^n(t)\}_{t \geq 0} \overset{\text{M}1}{\Rightarrow} \{\hat{X}(t)\}_{t \geq 0} \text{ as } n \to \infty, \]

where the limit process \( \{\hat{X}(t)\}_{t \geq 0} \) is a two-dimensional degenerate \( \alpha \)-stable driven SDE satisfying:

\[
\begin{align*}
\frac{d\hat{X}_1(t)}{dt} &= \left(\hat{\ell}_1 - \mu_1(\hat{X}_1(t) - \langle e, \hat{X}(t) \rangle^+ v_1) - \gamma_1 \langle e, \hat{X}(t) \rangle^+ v_1\right)dt, \\
\frac{d\hat{X}_2(t)}{dt} &= \left(\hat{\ell}_2 - \mu_2(\hat{X}_1(t) - \langle e, \hat{X}(t) \rangle^+ v_2) - \gamma_2 \langle e, \hat{X}(t) \rangle^+ v_2\right)dt + d\hat{A}_2(t).
\end{align*}
\]

Observe that the process \( \{\hat{X}(t)\}_{t \geq 0} \) does not fall into any of the four categories in [6, Theorem 3.1].
In fact, one can consider multiple classes of jobs with all heavy-tailed arrival processes that have different scaling parameters \( \alpha_i \)'s, \( i = 1, \ldots, d \), in their corresponding FCLTs. The centered queueing process should be scaled as \( n^{-1/\alpha} \), where \( \alpha := \min_{i=1, \ldots, d} \{ \alpha_i \} \), and the limit process has the components \( \{ \tilde{X}_i(t) \}_{t \geq 0} \) driven by independent \( \alpha \)-stable processes if the arrival process of class \( i \) has the parameter \( \alpha_i \) equal to the minimum \( \alpha \), and the other components are degenerate without stochastic driving terms.

3.3. Results for nonconstant Markov controls. Observe that in Theorems 3.1, 3.3, and 3.4 we consider a model with a constant control, that is, with the vector \( v \in \Delta \) being constant and fixed. Recently, in [5] the authors have studied ergodic properties with respect to the total variation distance for a class of piecewise Ornstein-Uhlenbeck processes with jumps under a stationary Markov control. More precisely, in this scenario, the drift function is of the form

\[
b_v(x) = \ell - M(x - \langle e, x \rangle + v(x)) - \langle e, x \rangle + \Gamma v(x),
\]

where \( M = \text{diag}(m_1, \ldots, m_d) \) with \( m_i > 0, i = 1, \ldots, d \), and \( v: \mathbb{R}^d \to \Delta \) is measurable, and such that \( b_v(x) \) is locally Lipschitz continuous. We let \( \tilde{U}_{sm} \) denote the class of such Markov controls. Based on [5, Theorems 3 and 5], and using exactly the same computations as in the proofs of Theorems 3.1 and 3.3, we have the following results for the above model.

(i) Assume that \( \Gamma > 0, \sigma \equiv 0 \), and \( \{ L(t) \}_{t \geq 0} \) is either a rotationally invariant \( \alpha \)-stable Lévy process, or an anisotropic Lévy process consisting of independent one-dimensional symmetric \( \alpha \)-stable components with \( \alpha \in (1, 2) \). Then \( \{ X(t) \}_{t \geq 0} \) admits a unique invariant probability measure \( \tilde{\pi}_v \in \mathcal{P}_\theta(\mathbb{R}^d) \) for any \( v \in \tilde{U}_{sm} \) and \( \theta \in [1, \alpha) \), and for any such \( \theta \in [1, \alpha) \) there exist positive constants \( \gamma \) and \( C_\gamma \) such that

\[
W_1(\delta_x P_t, \tilde{\pi}_v) \leq C_\gamma |x|^\theta e^{-\gamma t} \quad \forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d, \forall v \in \tilde{U}_{sm}.
\]  

Also, for any \( p \in [1, \theta) \), \( 1 \leq \theta < \alpha \), there exists a positive constant \( C_p \) such that

\[
(1 \vee t)^{\theta p - 1} W_p(\delta_x P_t, \tilde{\pi}_v) \leq C_p(\mu_\theta + |x|^\theta)^{1/p} \quad \forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d,
\]

and all \( v \in \tilde{U}_{sm} \). In addition,

\[
\lim_{t \to \infty} W_p(\pi P_t, \tilde{\pi}_v) = 0 \quad \forall \pi \in \mathcal{P}_\theta(\mathbb{R}^d), \forall v \in \tilde{U}_{sm}.
\]

(ii) Assume that \( \langle e, M^{-1/\ell} \rangle < 0, \sigma \) is a constant nonsingular diagonal matrix, and \( \{ L(t) \}_{t \geq 0} \) is a compound Poisson process with drift \( \theta \), and a finite Lévy measure \( \nu \) satisfying \( 1 \in \Theta_c \), and which is supported on a half-line of the form \( \{ tw : t \in [0, \infty) \} \) with \( \langle e, M^{-1} w \rangle > 0 \). Then \( \{ X(t) \}_{t \geq 0} \) admits a unique invariant probability measure \( \tilde{\pi}_v \in \mathcal{P}_{\theta-1}(\mathbb{R}^d) \) for any \( \theta \in \Theta_c \cap [1, \infty) \). Further, if \( 2 \in \Theta_c \), then for any \( \theta \in \Theta_c \cap [2, \infty) \), there exists a constant \( C_\theta > 0 \) such that

\[
(1 \vee t)^{\theta - 2} W_1(\delta_x P_t, \tilde{\pi}_v) \leq C_\theta(1 + |x|^\theta) \quad \forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d, \forall v \in \tilde{U}_{sm},
\]

and for each \( p \in [1, \theta - 1] \), there exists \( C_p > 0 \) such that

\[
(1 \vee t)^{\theta - \frac{1}{p}} W_p(\delta_x P_t, \tilde{\pi}_v) \leq C_p(\mu_{\theta-1} + |x|^\theta)^{1/p} \quad \forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d, \forall v \in \tilde{U}_{sm}.
\]

In addition,

\[
\lim_{t \to \infty} W_{\theta - 1}(\pi P_t, \tilde{\pi}_v) = 0 \quad \forall \pi \in \mathcal{P}_\theta(\mathbb{R}^d), \forall v \in \tilde{U}_{sm}.
\]

(iii) Assume that \( \Gamma > 0, \sigma \) is a constant nonsingular diagonal matrix, and \( \{ L(t) \}_{t \geq 0} \) is as in part (ii). Then \( \{ X(t) \}_{t \geq 0} \) admits a unique invariant probability measure \( \tilde{\pi}_v \in \mathcal{P}_\theta(\mathbb{R}^d) \) for any \( \theta \in \Theta_c \), and for any such \( \theta \) there exist positive constants \( \gamma \) and \( C_\gamma \) such that (3.20) holds. Also, for any \( p \in \Theta_c \cap [1, \theta] \), there exists a positive constant \( C_p \) such that (3.21) and
(3.22) hold. Observe that in all three cases above, according to [6, Theorem 3.1], \( \{X(t)\}_{t \geq 0} \) is irreducible and aperiodic.

Remark 3.8. Part (ii) holds true under the more general hypotheses on the Lévy process in Theorem 3.2. Also the lower bound asserted in Theorem 3.2 applies for this part.

3.4. Models with general drifts. In this section, we discuss ergodic properties of the solution to (2.1) for more general drift functions \( b \). We assume that \( b(x) \) is locally Lipschitz continuous, and there exists \( \kappa > 0 \) such that \( \langle x, b(x) \rangle \leq \kappa (1 + |x|^2) \) for all \( x \in \mathbb{R}^d \). Then, (2.1) admits a unique nonexplosive strong solution \( \{X(t)\}_{t \geq 0} \) which is a strong Markov process and it satisfies the \( C_b \)-Feller property (see [1, Theorem 3.1, and Propositions 4.2 and 4.3]). Moreover, its infinitesimal generator \( (\mathcal{A}, \mathcal{D}_A) \) satisfies \( C^2_\kappa(\mathbb{R}^d) \subseteq \mathcal{D}_A, \mathcal{A}|_{C^2_\kappa(\mathbb{R}^d)} \) takes the form in (2.2), the corresponding extended domain contains the set \( \mathcal{D} \), and on this set, instead of \( \bar{A}f(x) \) we can again use \( A f(x) \). Irreducibility and aperiodicity of \( \{X(t)\}_{t \geq 0} \) with this general drift can be established as in [6, Theorem 3.1]. Based on [6, Corollary 5.2], using exactly the same reasoning as in the proofs of Theorems 3.1 and 3.3 we conclude the following.

Corollary 3.1. Suppose that \( \{X(t)\}_{t \geq 0} \) is irreducible and aperiodic, and \( \theta \in \Theta_C, \theta \geq 1 \). Then the following hold.

(i) If \( a(x) \) satisfies (3.1), and there exists \( Q \in \mathcal{M}_+ \) such that
\[
\limsup_{|x| \to \infty} \frac{\langle b(x) + r + \int_{\mathcal{B}^c} yv(dy), Qx \rangle}{|x|} < 0,
\]
then the conclusions of Theorem 3.1 (a) hold true.

(ii) If \( a(x) \) satisfies (3.12), and \( \limsup_{|x| \to \infty} \frac{\langle b(x), Qx \rangle}{|x|^2} < 0 \) for some \( Q \in \mathcal{M}_+ \), then the conclusion of Theorem 3.3 holds true.

(iv) Suppose that \( \sigma(x) \) is bounded, and there exist \( \theta > 0 \) and \( Q \in \mathcal{M}_+ \) such that (3.8) holds and
\[
\limsup_{|x| \to \infty} \frac{\langle b(x) + r + \int_{\mathcal{B}^c} yv(dy), Qx \rangle}{|x|} < 0.
\]
Then the conclusion of Theorem 3.1 (b) follows.

Analogously to Theorem 3.4, without assuming irreducibility and aperiodicity of the underlying process, we conclude the following.

Corollary 3.2. Assume that there are \( Q \in \mathcal{M}_+ \) and \( \kappa > 0 \), such that
\[
\langle x - y, Q(b(x) - b(y)) \rangle \leq -\frac{1}{2}\kappa |x - y|^2 , \quad x, y \in \mathbb{R}^d.
\]
Then, using the same notation, the conclusion of Theorem 3.4 holds true.

3.5. Results on general Markov processes. It is interesting to note that neither contraction properties, nor the particular structure of the drift are used for the proof of Theorems 3.1 and 3.3. Therefore the results are generic for systems satisfying (3.15), provided that there is enough coercivity so that the map \( x \mapsto (1 + |x|)^{-\eta} \phi(V(x)) \) is bounded below away from 0 for some \( \eta > 1 \). Then the analogous estimate to (3.6) holds for any \( p \in [1, \eta) \). These results are stated in the following theorem.

Theorem 3.5. Let \( \{X(t)\}_{t \geq 0} \) be a irreducible and aperiodic \( \mathbb{R}^d \)-valued càdlàg strong Markov process, and suppose that it satisfies the Foster–Lyapunov condition
\[
E_x[V(X_t)] - V(x) \leq b \int_0^t E_x[1_{C}(X_s)] \, ds - \int_0^t E_x[\phi(V(X_s))] \, ds,
\]
or, equivalently, $AV(x) \leq b \mathbb{1}_C(x) - \phi(V(x))$, for some continuous $V : \mathbb{R}^d \to [1, \infty)$, a constant $b > 0$, a nondecreasing differentiable concave function $\phi : [1, \infty) \to (0, \infty)$, and a closed petite set $C \in \mathcal{B}(\mathbb{R}^d)$ (see [15] for details). Also, assume that $\sup_{x \in C} V(x) < \infty$, and

$$\inf_{x \in \mathbb{R}^d} \frac{\phi(V(x))}{(1 + |x|^q)^q} > 0$$

for some $\eta > 1$. Then \{X(t)\}_{t \geq 0}$ admits a unique invariant measure $\pi \in \mathcal{P}(\mathbb{R}^d)$ such that $\pi(\phi(V)) < \infty$. In particular, $\pi \in \mathcal{P}_\eta(\mathbb{R}^d)$. In addition, with $H_\phi(t) := \int_1^t \frac{ds}{\phi(s)}$ and $r_\phi(t) := \phi(H_\phi^{-1}(t))$, the following hold.

(i) If $\lim_{t \to \infty} \phi'(t) = 0$, then for some constant $C_\eta > 0$, we have

$$\left(1 \lor r_\phi(t)^{(q-1)/q}\right) W_1(\delta_x P_t, \pi) \leq C_\eta V(x),$$

and

$$\int_0^\infty \left(1 \lor r_\phi(t)^{(q-1)/q}\right) W_1(\delta_x P_t, \delta_y P_t) dt \leq C_\eta (V(x) + V(y)), \quad \text{and if, in addition,}$$

$$\lim_{t \to \infty} \frac{\log(r_\phi(t))}{t} = \lim_{t \to \infty} \frac{\log(r_\phi'(t))}{t} = 0,$$

then

$$\int_0^\infty \left(1 \lor r_\phi(t)^{(q-1)/q}\right)'(t) W_1(\delta_x P_t, \pi) dt \leq C_\eta V(x).$$

(ii) For any $p \in [1, \eta]$ there is $C_p > 0$ such that

$$\left(1 \lor \left(t^{\frac{\eta-p}{p'}} \wedge t^{\frac{1-p}{p'}}\right) r_\phi(t)^{(q-1)/pq}\right) W_p(\delta_x P_t, \pi) \leq C_p (V(x) + m_\eta)$$

for all $x \in \mathbb{R}^d$ and $t \geq 0$.

(iii) If $\phi(t) = t$, then there exist positive constants $C_V$ and $\gamma$ such that

$$e^{\gamma t} W_1(\delta_x P_t, \pi) \leq C_V V(x).$$

Also, for any $p \in [1, \eta]$ there exists a constant $C_p > 0$ such that

$$\left(1 \lor t^{\gamma(p-1)}\right) W_p(\delta_x P_t, \pi) \leq C_p (m_\eta + V(x))^{1/p}.$$

4. Proofs

4.1. Proofs of Theorems 3.1 to 3.3 and 3.5. We start with the following lemma.

Lemma 4.1. For any $\theta \in \Theta_c$ there exists a constant $C > 0$ such that

$$\mathbb{E}_x [\|X_t\|^\theta] \leq (|x|^\theta + 1)e^{Ct}, \quad t \geq 0, \quad x \in \mathbb{R}^d.$$

Proof. Let $\varphi \in C^2(\mathbb{R}^d)$ be such that $\varphi(x) \geq 0$, $\varphi(x) \leq |x|^\theta$, and $\varphi(x) = |x|^\theta$ for $x \in \mathbb{B}^c$. Further, for $n \in \mathbb{N}$, let $\varphi_n \in C^2_0(\mathbb{R}^d)$ be such that $\varphi_n(x) \geq 0$, $\varphi_n(x) = \varphi|_{\mathbb{B}_{n+1}}(x)$, and $\varphi_n(x) \to \varphi(x)$, and $\tau_n := \inf\{t \geq 0 : X_t \in \mathbb{B}_n^c\}$. Then, according to Itô’s formula (see [1, Remark 2.2]), we have

$$\mathbb{E}_x [\varphi_n(X_t \wedge \tau_n)] \leq \varphi_n(x) + C(t \wedge \tau_n) + C_n \mathbb{E}_x \left[ \int_0^{t \wedge \tau_n} \varphi_n(X_s) ds \right]$$

$$\leq \varphi_n(x) + C_n t + C_n \int_0^t \mathbb{E}_x [\varphi_n(X_{s \wedge \tau_n})] ds, \quad n \in \mathbb{N}, \ t \geq 0, \ x \in \mathbb{R}^d,$$
where the constants $C_n$ depend on $\theta, b(x), \sigma(x)$, and the quantities
\[
\int_B |g|^2 \nu(dy), \quad \nu(B^c), \quad \sup_{x \in \mathbb{R}^d} |\nabla \varphi_n(x)|, \quad \text{and} \quad \sup_{x \in \mathbb{R}^d} |\nabla^2 \varphi_n(x)|
\]
for $R > 0$ large enough. Clearly, the functions $\varphi_n(x)$ can be chosen such that $C := \sup_{n \in \mathbb{N}} C_n < \infty$. Now, since the function $t \mapsto \mathbb{E}_x[\varphi_n(X_{t+\tau_n})]$ is bounded and càdlàg, Gronwall’s lemma implies that
\[
\mathbb{E}_x[\varphi_n(X_{t+\tau_n})] \leq (\varphi_n(x) + 1)e^{Ct} - 1, \quad n \in \mathbb{N}, \quad t \geq 0, \quad x \in \mathbb{R}^d.
\]
By letting $n \to \infty$ monotone convergence theorem and nonexplosivity of $\{X(t)\}_{t \geq 0}$ imply that
\[
\mathbb{E}_x[\varphi(X_t)] \leq (\varphi(x) + 1)e^{Ct} - 1, \quad t \geq 0, \quad x \in \mathbb{R}^d.
\]
Finally, we have that
\[
\mathbb{E}_x[|X_t^0|] \leq \mathbb{E}_x[\varphi(X_t)] + 1 \leq (\varphi(x) + 1)e^{Ct} \leq (|x|^\theta + 1)e^{Ct}, \quad t \geq 0, \quad x \in \mathbb{R}^d. \tag*{\square}
\]

**Proof of Theorem 3.1 (a).** First, under the assumptions of the theorem, it has been shown in [6, Theorem 3.2] that $\{X(t)\}_{t \geq 0}$ admits a unique invariant probability measure $\pi \in \mathcal{P}(\mathbb{R}^d)$, and there exist $Q \in \mathcal{M}_+$, depending on $v$, and positive constants $c_0 = c_0(\theta)$ and $c_1$, such that for any $\theta \in \Theta_e$, $\theta \geq 2$, we have
\[
\mathcal{A}V_{Q,\theta}(x) \leq c_0(\theta) - c_1V_{Q,\theta-1}(x), \quad x \in \mathbb{R}^d. \tag{4.1}
\]
Now, the fact that $\pi \in \mathcal{P}_{\theta-1}(\mathbb{R}^d)$ follows from [40, Theorem 4.3].

We continue now with the proof of part (i). By the Kantorovich–Rubinstein theorem we have
\[
\mathcal{W}_1(\mu_1, \mu_2) = \sup_{f : \text{Lip}(f) \leq 1} \left| \int_{\mathbb{R}^d} f(x)(\mu_1(dx) - \mu_2(dx)) \right|, \quad \mu_1, \mu_2 \in \mathcal{P}_1(\mathbb{R}^d).
\]
We apply [15, Theorem 3.2], with $V(x) = 1 + V_{Q,\theta}(x)$, $r_+(t) = t^{\theta-1}$, $f_+(x) = |x|^\theta - 1$, $\mathcal{W}_1(\mu, \nu) = \int_{\mathbb{R}^d} \frac{1}{\eta} d\nu(x)$, and $\mathcal{W}_2(\mu, \nu) = \int_{\mathbb{R}^d} \mu(x) d\nu(x)$. Note that if $g$ is such that $|\text{Lip}(g)| \leq 1$ and $g(0) = 0$, then $|g(x)| \leq |x| = \mathcal{W}_2(\mu_1(x), \nu_1(x)).$

Thus
\[
\sup_{f : \text{Lip}(f) \leq 1} \left| \int_{\mathbb{R}^d} f(x)(\mu_1(dx) - \mu_2(dx)) \right| \leq \sup_{|g| \leq \mathcal{W}_2(\mu_1(x), \nu_1(x))} \left| \int_{\mathbb{R}^d} g(x)(\mu_1(dx) - \mu_2(dx)) \right| = \|\mu_1 - \mu_2\|_{\mathcal{W}_2(\mu_1(x), \nu_1(x))},
\]
where for a signed measure $\mu$ on $\mathcal{B}(\mathbb{R}^d)$ and a function $h : \mathbb{R}^d \to [1, \infty]$ we define
\[
\|\mu\|_h := \sup_{g \in \mathcal{B}(\mathbb{R}^d), |g| \leq h} |\mu(g)|.
\]
Now, from [15, (3.5), (3.6) and (3.7)] we have
\[
(\Psi_1(r_+(t)) \vee 1)\mathcal{W}_1(\delta_x \mathcal{P}_t, \pi) \leq (\Psi_1(r_+(t)) \vee 1)\left| \delta_x \mathcal{P}_t - \pi \|_{\mathcal{W}_2(\mu_1(x), \nu_1(x))} \leq C_\theta V(x),
\]
\[
\int_0^\infty (\Psi_1(r_+(t)) \vee 1)\mathcal{W}_1(\delta_x \mathcal{P}_t, \mathcal{P}_t)dt \leq \int_0^\infty (\Psi_1(r_+(t)) \vee 1)\left| \delta_x \mathcal{P}_t - \pi \|_{\mathcal{W}_2(\mu_1(x), \nu_1(x))}dt \leq C_\theta (V(x) + V(y)),
\]
and
\[
\int_0^\infty (\Psi_1(r_+(t)) \vee 1)\mathcal{W}_1(\delta_x \mathcal{P}_t, \mathcal{P}_t)dt \leq \int_0^\infty (\Psi_1(r_+(t)) \vee 1)\left| \delta_x \mathcal{P}_t - \pi \|_{\mathcal{W}_2(\mu_1(x), \nu_1(x))}dt \leq C_\theta (V(x),
\]
for some $C_\theta > 0$, which proves (3.2), (3.3), and (3.4), respectively.
We continue with part (ii). Here \( p \in [1, \theta - 1] \). Applying \([15, \text{Theorem 3.2 (3.5)}]\) with \( \Psi_1(z) = 1 \), and \( \Psi_2(z) = z \), we obtain \( \mathbb{E}_x \left[ |X_t|^\theta - 1 \right] \leq \bar{m}_{\theta - 1} + \kappa \left| x \right|^\theta \), for some \( \kappa > 0 \), and all \( x \in \mathbb{R}^d \) and \( t \geq 0 \). Hence
\[
\mathbb{E}_x \left[ |X_t|^p \right] \leq t^{p - \theta + 1} \left( \bar{m}_{\theta - 1} + \kappa \left| x \right|^\theta \right), \quad x \in \mathbb{R}^d, \quad t \geq 0.
\]
For \( t \geq 0 \), and \( \Pi \in \mathcal{C}(\delta_{x_0}, \Pi, \pi) \), we have
\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p \Pi(dx, dy) = \int_{\mathbb{B}_1} |x - y|^p \Pi(dx, dy) + \int_{(\mathbb{B}_1)^c} |x - y|^p \Pi(dx, dy)
\]
\[
\leq (2t)^{p-1} \int_{\mathbb{R}^d} |x - y|^p \Pi(dx, dy) + 2^{p-1} \int_{\mathbb{B}_1^c} |x|^p \left[ \delta_{x_0} P_t(dx) + \pi(dx) \right].
\]
Therefore, using (4.2), and the bound \( \int_{\mathbb{B}_1^c} |x|^p \pi(dx) \leq t^{p-\theta+1} \bar{m}_{\theta - 1} \), which follows by (3.5), we have
\[
\mathcal{W}_p^p(\delta_{x_0} P_t, \pi) \leq (2t)^{p-1} \mathcal{W}_1(\delta_{x_0} P_t, \pi) + 2^{p-1} t^{p-\theta+1} \left( 2\bar{m}_{\theta - 1} + \kappa \left| x \right|^\theta \right), \quad x \in \mathbb{R}^d, \quad t \geq 0,
\]
and combining this with (3.2) we obtain
\[
(1 \vee t^{\theta-1-p}) \mathcal{W}_p^p(\delta_{x_0} P_t, \pi) \leq 2^{p-1} C_\theta \left| x \right|^\theta + 2^{p-1} \left( 2\bar{m}_{\theta - 1} + \kappa \left| x \right|^\theta \right), \quad x \in \mathbb{R}^d, \quad t \geq 0,
\]
from which (3.6) follows with \( C_p = 2^{p-1/\ell} \max\{2, C_\theta, \kappa\}^{1/\ell} \).

Finally, as shown in \([47, \text{Theorem 7.12}]\), if \( \{\mu_n\}_{n \in \mathbb{N}} \) is a sequence of probability measures in \( \mathcal{P}_p(\mathbb{R}^d) \), and \( \mu \in \mathcal{P}(\mathbb{R}^d) \), then the following statements are equivalent.

1. \( \mu_n \mathop{\rightharpoonup}^{w} \mu \) as \( n \to \infty \);
2. \( \mu_n \mathop{\rightharpoonup}^{w} \mu \) as \( n \to \infty \), and \( \int_{\mathbb{R}^d} |x|^p \mu_n(dx) \xrightarrow{n \to \infty} \int_{\mathbb{R}^d} |x|^p \mu(dx) \),

where \( \xrightarrow{w} \) denotes weak convergence of probability measures. This equivalence together with Lemma 4.1 and (3.4) in \([15]\) implies (3.7). This completes the proof. \( \square \)

Proof of Theorem 3.1 (b). First, under the assumptions of the theorem, it has been shown in \([6, \text{Theorem 3.2}]\) that \( \{X(t)\}_{t \geq 0} \) admits a unique invariant probability measure \( \pi \in \mathcal{P}(\mathbb{R}^d) \), and there exist \( Q \in \mathcal{M}_+ \), depending on \( v \), and positive constants \( c_0 \) and \( c_1 \), such that
\[
A\tilde{V}_{Q,r}(x) \leq c_0 - c_1 \tilde{V}_{Q,r}(x), \quad x \in \mathbb{R}^d,
\]
where \( 0 < r < \theta \|Q\|^{-1/2} \). Now, the fact that \( \int_{\mathbb{R}^d} e^{r \|y\|Q} \pi(dy) \) for any \( 0 < r < \theta \|Q\|^{-1/2} \) follows again from \([40, \text{Theorem 4.3}]\).

Next, the proof of part (i) follows again by the Kantorovich–Rubinstein theorem. Namely, as in part (a), we obtain
\[
\sup_{f : \text{Lip}(f) \leq 1} \left| \int_{\mathbb{R}^d} f(x)(\mu_1(dx) - \mu_2(dx)) \right| \leq \|\mu_1 - \mu_2\|_{\Psi_2} \|1\|_1 \leq \|\mu_1 - \mu_2\|_{\tilde{V}_{Q,r}}(x),
\]
which together with \([6, \text{Theorem 3.2}]\) proves the assertion.

We continue with part (ii). For \( 0 < r < \theta \|Q\|^{-1/2} \), let \( \bar{m}_{r} := \int_{\mathbb{R}^d} e^{r \|x\|Q} \pi(dx) \). Applying \([6, \text{Theorem 3.2}]\), for any \( 0 < r < \theta \|Q\|^{-1/2} \) there exists \( \kappa > 0 \) such that
\[
\mathbb{E}_x \left[ e^{r \|X_t\|Q} \right] \leq \kappa e^{r \|x\|Q} + \bar{m}_{r}, \quad x \in \mathbb{R}^d, \quad t \geq 0.
\]
This in particular means that for any \( 0 < r < \theta \|Q\|^{-1/2} \) and \( q > 0 \) there exists \( \kappa' > 0 \) such that
\[
\mathbb{E}_x \left[ |X_t|^q \right] \leq \kappa' \left( e^{r \|x\|Q} + 1 \right), \quad x \in \mathbb{R}^d, \quad t \geq 0.
\]
Hence, for any $0 < p < q$ we have that
\[
\mathbb{E}_x \left[ |X_t|^p 1_{B(t)}(X_t) \right] \leq t^{p-q} \kappa' \left( e^{r||x||q} + 1 \right), \quad x \in \mathbb{R}^d, \quad t \geq 0.
\] (4.4)

By (4.3) and (4.4), we have
\[
\mathcal{W}_p^p(\delta_x P_t, \pi) \leq (2t)^{p-1} \mathcal{W}_1(\delta_x P_t, \pi) + 2^{p-1} t^{p-q} \tilde{\kappa} \left( e^{r||x||q} + 1 \right), \quad x \in \mathbb{R}^d, \quad t \geq 0,
\]
for some constant $\tilde{\kappa} > 0$. Combining this with (3.9) we obtain
\[
(1 \vee t^{q-p}) \mathcal{W}_p^p(\delta_x P_t, \pi) \leq 2^{p-1} \tilde{\kappa} \left( e^{r||x||q} + 1 \right), \quad x \in \mathbb{R}^d, \quad t \geq 0,
\]
for some $\tilde{\kappa} > 0$, from which (3.10) follows with $C_{r,p,q} = 2^{(p-1)/p} \tilde{\kappa}^{1/p}$.

Finally, the last assertion follows as in the proof of Theorem 3.1. This completes the proof. \(\square\)

**Proof of Theorem 3.2.** Fix some $x_0 \in \mathbb{R}^d$, $p \in [1, \theta_c - 1)$, and $\epsilon \in (0, 1/4)$. For $s > 0$ define $f_s : \mathbb{R}^d \to \mathbb{R}$, by $f_s(x) = 0$ if $\langle e, M^{-1}x \rangle \leq \frac{s}{2}$, and $f_s(x) = \langle e, M^{-1}x \rangle - \frac{s}{2}$ if $\langle e, M^{-1}x \rangle > \frac{s}{2}$. We have
\[
\int_{\mathbb{R}^d} (f_s(x))^p \pi(dx) \geq \left( \frac{s^p}{2} \right) \pi(\{x : \langle \tilde{w}, x \rangle > s_n\}). \tag{4.5}
\]

Since $\int_{\mathbb{R}^d} (\langle e, M^{-1}x \rangle)^{\theta - 1 + \epsilon} \pi(dx) = \infty$ by [6, Lemma 5.7 (b)] and Remark 3.2, there exists an increasing diverging sequence $\{s_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$ such that
\[
\left( \frac{s_n}{2} \right) \pi(\{x : \langle e, M^{-1}x \rangle > s_n\}) \geq 2^p s_n^{1+\theta - \theta_c - \epsilon}. \tag{4.6}
\]

Note also that there exists a positive constant $\tilde{C}$ such that $f_s^p(x) \leq \tilde{C} s^{\theta + \epsilon - \theta_c} V_{Q, \theta_c - \epsilon}(x)$ for all $s > 1$ and $x \in \mathbb{R}^d$. Thus, by the Foster–Lyapunov equation (4.1) (see [40, Theorem 1.1]), we obtain
\[
\int_{\mathbb{R}^d} (f_s(x))^p \delta_{x_0} P_t(dx) \leq \tilde{C} s^{\theta + \epsilon - \theta_c} (c_0 t + V_{Q, \theta_c - \epsilon}(x_0)). \tag{4.7}
\]

with $c_0 = c_0(\theta - \epsilon)$. Select a sequence $\{t_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$ such that
\[
s_n^{1-2\epsilon} = \tilde{C}(c_0 t_n + V_{Q, \theta_c - \epsilon}(x_0)). \tag{4.8}
\]

Combining (4.5)–(4.8), above we have

\[
\left( \int_{\mathbb{R}^d} (f_{s_n}(x))^p \pi(dx) \right)^{1/p} - \left( \int_{\mathbb{R}^d} (f_{s_n}(x))^p \delta_{x_0} P_{t_n}(dx) \right)^{1/p} \geq s_n^{1+\theta - \theta_c - \epsilon} \geq \left( \tilde{C}(c_0 t_n + V_{Q, \theta_c - \epsilon}(x_0)) \right)^{- \frac{\theta - \theta_c - \epsilon}{1 - \epsilon}}.
\]

The result then follows by [47, Proposition 7.29]. \(\square\)

**Proof of Theorem 3.3.** We proceed as in the proof of Theorem 3.1. Under the assumptions of the theorem, it has been shown in [6, Theorem 3.5] that $\{X(t)\}_{t \geq 0}$ admits a unique invariant probability measure $\pi \in \mathcal{P}(\mathbb{R}^d)$, and there exist $Q \in \mathcal{M}_+$, depending on $v$, and positive constants $c_0$ and $c_1$, such that
\[
\mathbf{AV}_{Q, \theta}(x) \leq c_0 - c_1 V_{Q, \theta}(x), \quad x \in \mathbb{R}^d.
\]

Now, the fact that $\pi \in \mathcal{P}_Q(\mathbb{R}^d)$ follows again from [40, Theorem 4.3].

By employing the Kantorovich–Rubinstein theorem again, we obtain
\[
\sup_{f : \text{Lip}(f) \leq 1} \left| \int_{\mathbb{R}^d} f(x) (\mu_1(dx) - \mu_2(dx)) \right| \leq \|\mu_1 - \mu_2\|_{\psi_2(f) \vee 1} \leq \|\mu_1 - \mu_2\|_{[\psi_2(f)]^\theta \vee 1}.
\]

Part (i) now follows from [6, Theorem 3.5] (recall that by assumption $\theta \in \Theta_c$ and $\theta \geq 1$).
We continue with part (ii). Here \( p \in [1, \theta] \). Applying [40, Theorem 6.1], we obtain
\[
\mathbb{E}_x \left[ |X_t|^\theta \right] \leq \bar{m}_\theta + \kappa |x|^\theta,
\]
for some \( \kappa > 0 \), and all \( x \in \mathbb{R}^d \) and \( t \geq 0 \). Hence
\[
\mathbb{E}_x \left[ |X_t|^p 1_{B^c_r}(X_t) \right] \leq t^{p-\theta} \left( \bar{m}_\theta + \kappa |x|^\theta \right), \quad x \in \mathbb{R}^d, \quad t \geq 0.
\] (4.9)
By (4.3) and (4.9), we have
\[
W_p^p(\delta_x P, \pi) \leq (2t)^{p-1} W_1(\delta_x P, \pi) + 2^{p-1} t^{p-\theta} \left( 2\bar{m}_\theta + \kappa |x|^\theta \right), \quad x \in \mathbb{R}^d, \quad t \geq 0,
\]
and combining this with (3.13) we obtain
\[
(1 \vee t^{\theta-p}) W_p^p(\delta_x P, \pi) \leq 2^{p-1} C_\theta |x|^\theta + 2^{p-1} \left( 2\bar{m}_\theta + \kappa |x|^\theta \right), \quad x \in \mathbb{R}^d, \quad t \geq 0,
\]
from which (3.14) follows again with \( C_p = 2^{(p-1)/p} \max\{2, C_\theta, \kappa\}^{1/p} \).

Finally, the last assertion follows as in the proof of Theorem 3.1. This completes the proof. \( \square \)

**Proof of Theorem 3.5.** Part (i) follows by the same argument used in the proof of (i) of Theorem 3.1 (a), by selecting \( f_s(x) = \phi(V(x)), r_s(t) = \phi(H^{-1}_s)(t) \), \( \Psi_1(z) = z^{(\eta-1)/\eta} \), \( \Psi_2(z) = z^{1/\eta} \), and using (3.23).

We next prove part (ii). Applying (3.23) and [15, Theorem 3.2 (3.5)] with \( \Psi_1(z) = 1 \), and \( \Psi_2(z) = z \), we obtain
\[
\mathbb{E}_x \left[ |X_t|^\eta \right] \leq \bar{m}_\eta + \kappa V(x), \quad x \in \mathbb{R}^d, \quad t \geq 0.
\]
Equation (3.8) now follows from the Kantorovich–Rubinstein theorem and (3.23). Let \( p \in [1, \eta] \). First, from (4.11) we obtain
\[
\mathbb{E}_x \left[ |X_t|^\eta \right] \leq \bar{m}_\eta + \kappa V(x), \quad t \geq 0.
\]
Hence
\[
\mathbb{E}_x \left[ |X_t|^p 1_{B^c_r}(X_t) \right] \leq t^{\eta-p} \left( \bar{m}_\eta + \kappa V(x) \right), \quad x \in \mathbb{R}^d, \quad t \geq 0.
\] (4.12)
By (4.3) and (4.12), we have
\[
W_p^p(\delta_x P, \pi) \leq (2t)^{p-1} W_1(\delta_x P, \pi) + 2^{p-1} t^{p-\eta} \left( 2\bar{m}_\eta + \kappa V(x) \right), \quad x \in \mathbb{R}^d, \quad t \geq 0,
\]
and combining this with (3.26) we obtain
\[
(1 \vee t^{\eta-p}) W_p^p(\delta_x P, \pi) \leq 2^{p-1} C_V V(x) + 2^{p-1} \left( 2\bar{m}_\eta + \kappa V(x) \right), \quad x \in \mathbb{R}^d, \quad t \geq 0,
\]
from which (3.14) follows again with \( C_p = 2^{(p-1)/p} \max\{2, C_\eta, \kappa\}^{1/p} \). This completes the proof. \( \square \)
4.2. Proof of Theorem 3.4. We start with the following result on the drift.

**Lemma 4.2.** Under the assumptions of Theorem 3.4 we have that
\[
F(x, y) := \langle x - y, Q(b(x) - b(y)) \rangle \leq -\frac{1}{2} \kappa |x - y|^2, \quad x, y \in \mathbb{R}^d.
\]

*Proof.* With \( \hat{v} = -M^{-1}(Mv - \Gamma v) \), we have \( b(x) = \ell - M(x + \langle e, x \rangle^+ \hat{v}) \). If both \( x \) and \( y \) are on the same half-space, that is, \( e'x \geq 0 \) and \( e'y \geq 0 \), or the opposite, then \( F(x, y) \leq -\frac{1}{2} \kappa |x - y|^2 \). So suppose, without loss of generality, that \( e'x \geq 0 \) and \( e'y \leq 0 \). Then we have
\[
\langle x - y, Qb(x) \rangle = (x - y)'Q \ell - \langle x - y, QMx \rangle - \langle x - y, QM \hat{v}e'x \rangle
\]
\[
\langle x - y, Qb(y) \rangle = (x - y)'Q \ell - \langle x - y, QMy \rangle.
\]

We distinguish two cases.

1. \( \langle x - y, QM \hat{v}e'x \rangle \geq 0 \). Then of course subtracting (4.13b) from (4.13a), we obtain
\[
F(x, y) = -\langle x - y, QM(x - y) \rangle - \langle x - y, QM \hat{v}e'x \rangle \\
\leq -\langle x - y, QM(x - y) \rangle \leq -\frac{1}{2} \kappa |x - y|^2.
\]

2. \( \langle x - y, QM \hat{v}e'x \rangle < 0 \). Since \( e'x \geq 0 \), we must have \( \langle x - y, QM \hat{v} \rangle < 0 \). This in turn implies, since \( e'y \leq 0 \), that
\[
0 \leq \langle x - y, QM \hat{v}e'y \rangle.
\]

Adding (4.13a) and (4.14) and subtracting (4.13b) from the sum, we obtain
\[
F(x, y) \leq -\langle x - y, QM(x - y) \rangle - \langle x - y, QM \hat{v}e'(x - y) \rangle \\
\leq -\langle x - y, QM(\ell + \hat{v}e')(x - y) \rangle \leq -\frac{1}{2} \kappa |x - y|^2,
\]
thus completing the proof. \( \square \)

For \( x, z \) in \( \mathbb{R}^d \) define
\[
\Delta_{2} b(x) := b(x + z) - b(x), \quad \Delta_{2} \sigma(x) := \sigma(x + z) - \sigma(x),
\]
and \( \hat{a}(x; z) := \Delta_{2} \sigma(x) \Delta_{2} \sigma'(x) \). If \( \sigma(x) \equiv \sigma \), then of course \( \Delta_{2} \sigma(x) \) and \( \hat{a}(x; z) \) are equal to zero. We next show the following result on the asymptotic flatness (uniform dissipativity) in the \( p \)th mean.

**Lemma 4.3.** Let \( \{X^{x+z}(t)\}_{t \geq 0} \) and \( \{X^x(t)\}_{t \geq 0} \) denote the solutions of (2.1) starting at \( x + z \) and \( x \), respectively. Under the assumptions of Theorem 3.4 it holds that
\[
\mathbb{E}[\|X^{x+z}_t - X^x_t\|_Q^p] \leq \|z\|_Q^p e^{-\phi/p}, \quad x, z \in \mathbb{R}^d, \quad t \geq 0, \quad p \in [0, \theta].
\]

*Proof.* We adapt the proof of [3, Lemma 7.3.4], where an analogous result is shown for \( p = 1 \). Define
\[
V_{\varepsilon, p}(x) := \frac{\|x\|_Q^{p+1}}{(\varepsilon + \|x\|_Q^2)^{\frac{p}{2}}}, \quad \varepsilon > 0, \quad p \in (0, \theta],
\]
and
\[
\tilde{L} f(x; z) := \sum_{i=1}^{d} \Delta_{2} b'(x) \frac{\partial f}{\partial x_i}(z) + \sum_{i,j=1}^{d} \hat{a}_{ij}(x; z) \frac{\partial^2 f}{\partial x_i \partial x_j}(z).
\]
By Lemma 4.2 we have \( 2\langle Qz, \Delta_{2} b(x) \rangle \leq -\kappa |z|^2 \). Thus,
\[
2\langle Qz, \Delta_{2} b(x) \rangle + \text{Tr} \left( \hat{a}(x; z)Q \right) \leq -\kappa |z|^2 + \text{Lip}^2(\sigma \sqrt{Q}) |z|^2.
\]
Calculating \( \tilde{L} V_{\varepsilon, p}(x; z) \), using (4.16), we obtain
\[
\tilde{L} V_{\varepsilon, p}(x; z) = \frac{((p + 1)\varepsilon + p\|z\|_Q^2)}{2(\varepsilon + \|z\|_Q^2)^{\frac{p}{2}}} \left[ 2\langle Qz, \Delta_{2} b(x) \rangle + \text{Tr} \left( \hat{a}(x; z)Q \right) \right]
\]
\[+ \frac{\varepsilon^2(p-1)(p+1)||z||_Q^{-p+3} + 2\varepsilon(p+1)(p-2)||z||_Q^{-p+1} + p(p-2)||z||_Q^{-p+1} |\Delta x \sigma'(x) Qz|^2}{2(\varepsilon + ||z||_Q^2)^{5/2}} \]
\[
\leq - (\kappa - \text{Lip}^2(\sigma\sqrt{Q})) \frac{(p+1)||z||_Q^{-2} + p}{2(\varepsilon + ||z||_Q^2)^2} |z|^2 \mathcal{V}_{\varepsilon,p}(z) + \text{Lip}^2(\sigma Q) \frac{\varepsilon^2(p-1)(p+1)||z||_Q^{-p} + 2\varepsilon(p+1)(p-2)||z||_Q^{-p+2} + p(p-2)||z||_Q^{-p}}{2(\varepsilon + ||z||_Q^2)^2} |z|^4 \mathcal{V}_{\varepsilon,p}(z) \]
\[
\leq - \frac{\kappa - \text{Lip}^2(\sigma\sqrt{Q})}{\lambda_Q} \frac{\varepsilon^2(p+1)}{2(\varepsilon + ||z||_Q^2)^2} + \text{Lip}^2(\sigma Q) \frac{\varepsilon^2(p-1)(p+1) + 2\varepsilon(p+1)(p-2)||z||_Q^{-p} + p(p-2)||z||_Q^{-p}}{2(\varepsilon + ||z||_Q^2)^2} \mathcal{V}_{\varepsilon,p}(z) \]
\[
= - \left( \frac{(p+1)(\kappa - \text{Lip}^2(\sigma\sqrt{Q}))}{2\lambda_Q} - \text{Lip}^2(\sigma Q)(p-1)(p+1) \right) \frac{\varepsilon^2}{(\varepsilon + ||z||_Q^2)^2} \mathcal{V}_{\varepsilon,p}(z)
\]
\[
- \left( \frac{(2p+1)(\kappa - \text{Lip}^2(\sigma\sqrt{Q}))}{4\lambda_Q} - \frac{2(p+1)(p-2)\text{Lip}^2(\sigma Q)}{4\lambda_Q} \right) \frac{2\varepsilon||z||_Q^2}{(\varepsilon + ||z||_Q^2)^2} \mathcal{V}_{\varepsilon,p}(z)
\]
\[
- \left( \frac{p(\kappa - \text{Lip}^2(\sigma\sqrt{Q}))}{2\lambda_Q} - \frac{p(p-2)\text{Lip}^2(\sigma Q)}{2\lambda_Q} \right) \frac{||z||_Q^4}{(\varepsilon + ||z||_Q^2)^2} \mathcal{V}_{\varepsilon,p}(z). \]

Now, it is easy to see that \(c_p\) is a lower bound of the three terms in parenthesis, and they are positive if and only if \(c_p > 0\). Thus,
\[\hat{L} \mathcal{V}_{\varepsilon,p}(x; z) \leq -c_p \mathcal{V}_{\varepsilon,p}(z), \quad x, z \in \mathbb{R}^d.\]

Next, let \(\tau = \inf\{t \geq 0: X_t^{x+z} = X_t^x\}\) (possibly +\(\infty\)). By Itô’s formula, combined with the fact that the Lévy noise does not depend on the state, we obtain
\[\mathbb{E} \left[ \mathcal{V}_{\varepsilon,p}(X_{t+\tau}^{x+z} - X_{t+\tau}^x) \right] - \mathcal{V}_{\varepsilon,p}(z) = \mathbb{E} \left[ \int_0^{t+\tau} \hat{L} \mathcal{V}_{\varepsilon,p}(X_s^{x+z}, X_s^{x+z} - X_s^x) \, ds \right]
\]
\[
= \mathbb{E} \left[ \int_0^t \hat{L} \mathcal{V}_{\varepsilon,p}(X_{s+\tau}^x, X_{s+\tau}^{x+z} - X_{s+\tau}^z) \, ds \right]
\]
\[
= \int_0^t \mathbb{E} \left[ \hat{L} \mathcal{V}_{\varepsilon,p}(X_{s+\tau}^x, X_{s+\tau}^{x+z} - X_{s+\tau}^z) \right] \, ds,
\]

since, for \(t \geq \tau\), \(X_t^{x+z} = X_t^x\) a.s. by the pathwise uniqueness of the solution of (2.1). From this and Lemma 4.1 we conclude that the function \(t \mapsto \mathbb{E} \left[ \mathcal{V}_{\varepsilon,p}(X_{t+\tau}^{x+z} - X_{t+\tau}^x) \right] \) is differentiable a.e. on \((0, \infty)\). Note that \(|\hat{L} \mathcal{V}_{\varepsilon,p}(x; z)| \leq c|z|^p\) for some nonnegative constant \(c\). We conclude now that
\[\frac{d}{dt} \mathbb{E} \left[ \mathcal{V}_{\varepsilon,p}(X_{t+\tau}^{x+z} - X_{t+\tau}^x) \right] = \mathbb{E} \left[ \hat{L} \mathcal{V}_{\varepsilon,p}(X_{s+\tau}^x, X_{s+\tau}^{x+z} - X_{s+\tau}^z) \right]
\]
\[
\leq -c(p) \mathbb{E} \left[ \mathcal{V}_{\varepsilon,p}(X_{t+\tau}^{x+z} - X_{t+\tau}^x) \right], \quad \text{a.e.} \ t > 0.
\]

Thus by Gronwall’s lemma it follows that
\[\mathbb{E} \left[ \mathcal{V}_{\varepsilon,p}(X_{t}^{x+z} - X_t^x) \right] = \mathbb{E} \left[ \mathcal{V}_{\varepsilon,p}(X_{0}^{x+z} - X_0^x) \right] \leq \mathcal{V}_{\varepsilon,p}(z) e^{-c_p t}
\]
for all \(x, z \in \mathbb{R}^d\) and \(t > 0\). Taking limits as \(\varepsilon \to 0\) in (4.17), and using monotone convergence, we obtain (4.15). This completes the proof. \(\square\)
Proof of Theorem 3.4. We use Lemma 4.3, and the bound \( \lambda_Q |z|^2 \leq \|z\|_Q^2 \leq \lambda_Q |z|^2 \), to obtain
\[
\mathbb{E}[|X_t^{x,z} - X_t^z|^p] \leq (\lambda_Q)^{-p/2} \mathbb{E}[\|X_t^{x,z} - X_t^z\|_Q^p] \\
\leq (\lambda_Q)^{-p/2} (\lambda_Q)^{p/2} |z|^p e^{-cp^t},
\]
thus establishing (3.18).

Finally, in order to conclude (3.19), we follow the idea from [34, Proof of Corollary 1.8] or [30, Proof of Theorem 2.1]. Observe first that, according to Lemma 4.1, for any \( \pi \in \mathcal{P}_p(\mathbb{R}^d) \), \( \pi P_t \in \mathcal{P}_p(\mathbb{R}^d) \) for all \( t \geq 0 \). Next, let \( \pi_1, \pi_2 \in \mathcal{P}_p \) be arbitrary. According to Theorem 3.4, we have
\[
\mathcal{W}_p(\pi_1 P_t, \pi_2 P_t) \leq \left( \frac{\lambda_Q}{\lambda_Q^2} \right)^{1/2} \mathcal{W}_p(\pi_1, \pi_2) e^{-c_{\pi} t}, \quad t \geq 0.
\]
Fix \( t_0 \geq 0 \) such that
\[
\left( \frac{\lambda_Q}{\lambda_Q^2} \right)^{1/2} e^{-c_{\pi} t_0} < 1.
\]
Then, the mapping \( \pi \mapsto \pi P_{t_0} \) is a contraction on \( \mathcal{P}_p(\mathbb{R}^d) \). Thus, since \( \langle \mathcal{P}_p(\mathbb{R}^d), \mathcal{W}_p \rangle \) is a complete metric space, the Banach fixed point theorem entails that there exists a unique \( \pi_{t_0} P_{t_0} = \pi_{t_0} \). Further, by defining \( \pi := t_0^{-1} \int_0^{t_0} \pi_0 P_t ds \), we can easily see that \( \pi P_t = \pi \) for all \( t \geq 0 \), that is, \( \pi \) is an invariant probability measure for \( \{X(t)\}_{t \geq 0} \). By employing Lemma 4.1 again, we also see that \( \pi \in \mathcal{P}_p(\mathbb{R}^d) \). Finally, for any \( \pi \in \mathcal{P}_p(\mathbb{R}^d) \) we have
\[
\mathcal{W}_p(\pi P_t, \pi) = \mathcal{W}_p(\pi P_{t_0} P_t, \pi P_{t_0}) \leq \left( \frac{\lambda_Q}{\lambda_Q^2} \right)^{1/2} \mathcal{W}_p(\pi, \pi) e^{-c_{\pi} t_0}, \quad t \geq 0,
\]
which also proves uniqueness of \( \pi \). \( \square \)

Remark 4.1. In what follows we give an alternative proof of Theorem 3.4 in the case when \( \sigma \) is constant. Grant the assumptions of Theorem 3.4. Let \( \tilde{X}(t) := \tilde{Q}^{1/2} X(t), \ t \geq 0 \). Clearly, \( \{\tilde{X}(t)\}_{t \geq 0} \) is again a nonexplosive strong Markov process which satisfies the \( C_0 \)-Feller property. Next, the corresponding transition probability satisfies
\[
\tilde{P}_t(x, dy) = \tilde{P}_x(\tilde{X}(t) \in dy) = \mathbb{P}_{Q^{-1/2} x}(X(t) \in Q^{-1/2} dy) = P_t(Q^{-1/2} x, Q^{-1/2} dy), \quad x \in \mathbb{R}^d, \ t \geq 0.
\]
Also, \( \{\tilde{X}(t)\}_{t \geq 0} \) satisfies the following SDE
\[
d\tilde{X}(t) = \tilde{X}_0 + \tilde{Q}^{1/2} \sigma(\tilde{Q}^{-1/2} \tilde{X}(t)) d\tilde{W}(t) + \tilde{Q}^{1/2} dL(t),
\]
with \( \tilde{X}(0) = x \in \mathbb{R}^d \). Now, since the Lévy triplet of the Lévy process \( \{Q^{1/2} L(t)\}_{t \geq 0} \) is given by
\[
\left( Q^{1/2} \phi + \int_{\mathbb{R}^d} Q^{1/2} y \left( 1_{\mathbb{B}}(y) - 1_{\mathbb{B}}(Q^{1/2} y) \right) \nu(dy), 0, \nu(Q^{-1/2} dy) \right),
\]
we conclude that the corresponding infinitesimal generator \( \mathcal{A}^\tilde{X}, \mathcal{D}_{\mathcal{A}^\tilde{X}} \) again satisfies \( C_0^2(\mathbb{R}^d) \subseteq \mathcal{D}_{\mathcal{A}^\tilde{X}} \) and
\[
\mathcal{A}^\tilde{X}|_{C^2(\mathbb{R}^d)} f(x) = \frac{1}{2} \text{Tr}(\tilde{a}(x) \nabla^2 f(x)) + \langle \tilde{b}(x) + \tilde{v}, \nabla f(x) \rangle + \int_{\mathbb{R}^d} \delta_1 f(x; y) \tilde{v}(dy),
\]
where
\[
\tilde{b}(x) = Q^{1/2} \sigma(Q^{-1/2} x), \\
\tilde{a}(x) = Q^{1/2} \sigma(Q^{-1/2} x) \sigma'(Q^{-1/2} x) Q^{1/2}, \\
\tilde{v}(dy) = \nu(Q^{-1/2} dy),
\]
\[ \bar{\vartheta} = Q^{1/2} \vartheta + \int_{\mathbb{R}^d} Q^{1/2} y \left( \mathbb{1}_B(y) - \mathbb{1}_B(Q^{1/2} y) \right) \nu(dy), \]

and $\delta_1$ is as in (2.3). Thus, by Lemma 4.2, we have
\begin{equation}
(x - y, \bar{b}(x) - \bar{b}(y)) = F(Q^{-1/2} x, Q^{-1/2} y) \leq -\frac{\kappa}{2} |Q^{-1/2}(x - y)|^2 \leq -\frac{\kappa}{2 \lambda_{Q}} |x - y|^2 \quad \text{for all } x, y \in \mathbb{R}^d.
\end{equation}

Now, in [10], in the case when $\sigma(x) \equiv \sigma$, it has been shown that (4.18) implies that
\[ \mathcal{W}_p(\delta_x \bar{P}_t, \delta_y \bar{P}_t) \leq |x - y| e^{-\omega/2 \lambda_{Q}}, \quad x, y \in \mathbb{R}^d, \quad t \geq 0. \]

Observe that $c_p = n_2/2 \lambda_{Q}$. Finally we get
\begin{align*}
\mathcal{W}_p(\delta_x \bar{P}_t, \delta_y \bar{P}_t) &= \mathcal{W}_p(\bar{P}_t(Q^{1/2} x, Q^{1/2} y), \bar{P}_t(Q^{1/2} y, Q^{1/2} y)) \\
&\leq (\lambda_{Q})^{-1/2} Q^{1/2}(x - y) |e^{-\omega/2 \lambda_{Q}}| \\
&\leq (\lambda_{Q})^{-1/2} (\lambda_{Q})^{1/2} |x - y| e^{-\omega/2 \lambda_{Q}}, \quad x, y \in \mathbb{R}^d, \quad t \geq 0,
\end{align*}

which is (3.18).

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