**V: Identity and Entanglement**

1. **Identity and symmetry:** Throughout classical physics it was always assumed that in any collection of objects, however similar the members may be, it would, *in principle*, always be possible to distinguish one member from another. In principle, mind you, not necessarily in practice with available technology! This assumption had been self consciously elevated to the status of a basic philosophical principle in the 17th century (The Age of Reason) by the German polymath, G. W. Leibniz. It is still referred to as the **Principle of the Identity of Indiscernibles**: *If nothing* distinguishes one object from another, then there is just one object, and, consequently, if there really are two (or more) objects, then *something* must distinguish between them. There must be some way of telling one object from another.

The quantum world *seems* to violate this principle!

It's not universally agreed that this rule is violated in the quantum world. But it is clear that the quantum world forces one to question the rule and to understand it in a rather subtle way if it is not to be discarded. Many people just discard the rule in the face of the quantum world. Why?

Consider two electrons and assume them to be in the two-electron state, $\Psi$. As we have suggested and as can be proved within QM, *everything* quantum theory says about such a state can be put into the form of a probability assertion:

The probability that the property, A, of a system in the state $\Psi$, has the value, a, will be denoted by $P(a / A, \Psi)$.

Now if $\Psi$ is a two-electron state, *any* mathematical description of $\Psi$ must refer to the 'first' electron and the 'second' electron, or to electron #1 and electron #2, or to e1 and e2, or *something* like that. Let's make that mathematical fact explicit by writing, $\Psi_{e1,e2}$, for the quantum state. Then the sense in which *two* electrons are said to be *indistinguishable* is expressed by the equation,

$$P(a / A, \Psi_{e1,e2}) = P(a / A, \Psi_{e2,e1}),$$

for *any* real, observable, property, A, whatsoever!
For example let's suppose that, speaking roughly, the property A has the value, 1, if there is one electron with z-spin up in the North half of the room and another electron with z-spin down moving East. Then the equality between the probabilities,

\[ P(1 / A, \Psi_{e_1, e_2}) \quad \text{and} \quad P(1 / A, \Psi_{e_2, e_1}) \]

makes it impossible to say which electron, e1 or e2, has z-spin up in the North half of the room or spin down moving East.

Similarly, if A denotes one electron bound to an atom in the tip of my nose and one electron flying wildly in the atmosphere of the sun, we can never say whether e1 or e2 is in the tip of my nose.

Now because of the way probabilities are calculated from state vectors representing quantum states (see the last few pages of IV) this feature of the probabilities can be achieved by having

\[ |\psi_{e_1, e_2}\rangle = |\psi_{e_2, e_1}\rangle \quad \text{or} \quad |\psi_{e_1, e_2}\rangle = -|\psi_{e_2, e_1}\rangle. \]

In the first case we say the state vector is symmetric under the interchange of the electrons. In the second case we say the state vector is anti-symmetric.

Because of the Pauli Exclusion Principle (see Part I and below) electrons avail themselves of the anti-symmetric way. This guarantees that one will never find two electrons in exactly the same single electron state.

For example, suppose that \(|\phi\rangle\) and \(|\rho\rangle\) are two single electron state vectors and we want to describe a two electron state in which one electron is in the state \(\phi\) and the other is in the state \(\rho\). The state, \(\phi\), might be the bound state in the tip of my nose while \(\rho\) has the electron in the sun. We must then write,

\[ |\psi_{e_1, e_2}\rangle = |\phi\rangle|\rho\rangle - |\rho\rangle|\phi\rangle = -(|\rho\rangle|\phi\rangle - |\phi\rangle|\rho\rangle) = -|\psi_{e_2, e_1}\rangle, \]

where the first factor in each product refers to electron 1 and the second factor to electron 2. Pauli’s Exclusion Principle is implemented by the fact that if \(|\phi\rangle = |\rho\rangle\), then \(|\psi_{e_1, e_2}\rangle = 0\), i.e., no such state can exist.
The generalization of the anti-symmetry rule to states for an arbitrary number of electrons is,

$$|\psi_{e_1, e_2, \ldots, e_n, \ldots, e_m, \ldots} > = - |\psi_{e_1, e_2, \ldots, e_m, \ldots, e_n, \ldots} > .$$

Quantons that have only anti-symmetric states, like electrons, are called **fermions**, after Enrico Fermi, who studied the statistical behaviour of large collections of such quantons (Fig. 5.1, bottom). Neutrons and protons are also fermions.

The *other* way, the symmetric way of building multi-quanton states applies to photons. Thus for two photons (one entering my eye, $\phi$, and the other just leaving the sun, $\rho$, say) we would have

$$|\psi_{\gamma_1, \gamma_2} > = |\phi >|\rho > + |\rho >|\phi > = |\rho >|\phi > + |\phi >|\rho > = |\psi_{\gamma_2, \gamma_1} > ,$$

and, again, we couldn't tell which is which. And for many-photon states we have,

$$|\psi_{\gamma_1, \gamma_2, \ldots, \gamma_n, \ldots} > = |\psi_{\gamma_1, \gamma_2, \ldots, \gamma_n, \ldots} > .$$

Such quantons are called **bosons** after S. Bose who studied their statistics (Fig. 5.1). Unlike fermions of a given type, no two of which can be in the same state, bosons of a given type behave as if they preferred to be in the same state. There are other fundamental bosons in nature but they are rather exotic for our purposes. Remember the neutral pions from last week? They are bosons. If one doesn't look too closely, Helium atoms and Helium nuclei behave like bosons.

A general rule called the **spin-statistics theorem** is that quantons with integral spin are always bosons and quantons with half odd integral spin are always fermions.

All quantons appear to be either bosons or fermions and so, in any collection of quantons of a *single type*, with properties distributed among them, one can never say, and there seems to be no meaning in saying, which ones have which properties. They seem to be, in violation of Leibniz’s principle, *non-identical indiscernables*!
Fig. 5.1: Quantum Statistics. Top: Classical and quantum boson statistics illustrated. Bottom: Statistical proportionality factors for classical particles, quantum fermions and quantum bosons.

Consider a large collection of classical particles in thermal equilibrium at a definite absolute temperature, T. There will be many different states the particles can be in and statistical analysis shows that of any subclass of states differing only in energy the average number of particles at a given energy is proportional to the Boltzmann factor, \( B(E, T) \), where, \( 0 \leq B(E, T) = B(0, T) \frac{B(\frac{E}{n}, mT)}{B(0, mT)} = \exp\left[ -\frac{(E - \mu)}{kT} \right] \).

For quantum fermions which, remember, can never have more than one particle of a given type in one quantum state, the corresponding proportionality factor is \( f(E, T) = B(E, T) / (1 + B(E, T)) \leq 1 \).

This is approximately the same as \( B(E, T) \) when \( B(E, T) \) is much smaller than 1.

For quantum bosons the corresponding proportionality factor is \( b(E, T) = \frac{B(E, T)}{1 - B(E, T)} \), and this can get arbitrarily large when \( B(E, T) \) approaches 1 from below. That's what happens when Helium becomes superfluid at very low T. But again, \( b(E, T) \) is approximately the same as \( B(E, T) \) when the latter is small. Small Boltzmann factor defines the realm in which classical statistics are adequate.
2. Effects of quantum statistics: The fermionic character of electrons is a primary ingredient in our understanding of the chemical properties of atoms (Fig. 5.2) as reflected in the periodic table of the elements (Fig. 5.3). To a good first approximation the binding energy of the electrons surrounding nuclei to make up neutral atoms is determined by the “shell” of a concentric shell like structure to which the electron belongs. The innermost “shell”, designated 1s and having the strongest binding energy, admits only electrons with zero orbital angular momentum. Consequently no more than two electrons can occupy it – one with spin “up” and one with spin “down”. The next shell allows orbital angular momentum of magnitude zero and \( (2)^{1/2} \hbar \). These cases are labeled as the subshells 2s and 2p, respectively. As with 1s, the 2s subshell can accommodate no more than two electrons, but the 2p subshell can accommodate 6 electrons since the orbital angular momentum can have three orientations with respect to any fixed direction. The third shell has three subshells, 3s, 3p and 3d, which can hold as many as 2, 6 and 10 electrons, respectively. And so on and so on.

The chemical properties of an atom are largely determined by the nature and percentage occupation of its outermost occupied shell. The most chemically reactive atoms are those with just one electron or just one missing electron from an outermost s-shell. The least reactive atoms are those with the outermost shell exactly full of its electron allotment (Fig. 5.2).

The bosonic character of photons and helium atoms is a primary ingredient in our understanding of lasers and superfluid helium, respectively. Unlike identical fermions, no two of which can occupy the same quantum state, identical bosons display a tendency (which some anthropomorphize into a “preference”) towards multiple occupancy of quantum states.

Thus lasers are possible because atoms or molecules that can emit photons of a given frequency, polarization, direction and phase have an enhanced likelihood of doing so if such photons are already present in their vicinity. And the enhancement is proportional to the number of such photons present (Fig. 5.4).

Similarly, liquid He \(^4\) becomes a superfluid, losing most of its viscosity and heat capacity at very low temperatures, because a significant fraction of its atoms collapse into the same lowest energy state possible (Fig. 5.5).
**Fig. 5.2:** A first approximation to the electron shell structure of the first eleven atoms of the periodic table of the elements (**Fig. 5.3**).
Fig. 5.2 (cont’d): The individual electrons are “in shells” labeled by an integer, n = 1, 2, 3, - - -, in the order of decreasing binding energy, and subshells labeled by a letter, s, p, d, - - - in the order of increasing orbital angular momentum. The most chemically reactive atoms are those that have, in the outermost occupied shell, just one electron missing from a complete closed shell. Above, these are H, Li, F and Na. The least reactive are those atoms whose electrons exactly fill or close the shells they occupy. These are He, and Ne. The atoms, C, N and O are increasingly reactive and participate in a vast and important array of molecular varieties and structures.
**Fig. 5.3: The periodic table of the chemical elements.** The “periodic” in the periodic table refers to the approximate recycling of chemical properties within a vertical column. Thus, in the left column, the chemistry of H is similar to that of Li and Na, etc., while in the last two columns on the right the chemistry of F is similar to that of Cl and Br, etc., and that of He is similar to that of Ne and Ar, etc. This periodicity of chemical properties was recognized by Dimitri Mendeleyev in the 19th century, long before any understanding of the atomic structure of matter existed.
Fig. 5.4: Light Amplification by Stimulated Emission of Radiation (LASER). The total emission rate from atoms or molecules for photons of any particular type is proportional to, \( N(1 + p) \), where \( N \) is the number of atoms in the proper excited state to emit and \( p \) is the number of photons of the type in question already present. So inside a partially reflecting chamber and holding \( N \) constant with \( p = 0 \), let \( t_0 \) be the average time for one photon to be emitted, \( p = 0 \Rightarrow p = 1 \). Now the emission rate is doubled and in the next \( t_0 \) interval two photons are emitted and, \( p = 1 \Rightarrow p = 3 \). Now the emission rate is quadrupled and in the next \( t_0 \) interval four photons are emitted and, \( p = 3 \Rightarrow p = 7 \). The process continues and in the nth \( t_0 \) interval we find \( p = 2^{n-1} - 1 \Rightarrow p = 2^n - 1 \). If \( t_0 \) is very small the cascade process is very fast and very quickly large numbers of identical, in-step, photons break free of the partially reflecting chamber to form a laser beam.
Superfluid helium flowing to the inside

Fig. 5.5: Superfluid Helium. Helium liquefies at the very low temperature of 4.2K. If one cools it even further, it does not solidify (except under a pressure of 25 atm. or more) but becomes a superfluid below 2.2K. The word indicates the presence of such features as a heat conductivity so large that the liquid can’t boil and a viscosity so low that circular flows, once started, never stop. Superfluid Helium also displays the bizarre pictured phenomena of creeping film. If one container of the superfluid is inside another and the bulk fluid levels are different in the two containers, a film of the superfluid will flow over the walls of the inside container until all the bulk fluid is at the same level.

The main reason for these striking features is that below the transition temperature a growing fraction of the Helium atoms collapse into the same single lowest energy state, called the Bose-Einstein condensate state. Once that happens, those atoms can no longer, independently, give up or absorb small amounts of energy, the processes that give rise to things like boiling and viscosity. They’re too tightly correlated with each other. Furthermore, it’s not quite right to think that some of the atoms are in the condensate state and some aren’t. Rather it’s more like each atom is in a superposition of the condensate and more ordinary states. In fact, a truly fundamental and thorough understanding of superfluid Helium does not yet exist.
3. **Entanglement:** Entanglement has, in the last two decades, become recognized as the characteristically quantum feature of any quantum system that can be regarded as having at least two parts. For many people entanglement is more bizarre and counterintuitive than anything we have yet discussed!

Using the notation used in **Part IV**, consider the polarization states of two photons or the spin states of two electrons. Not all such states are strange. One could just have two vertically polarized (V) photons and the state would be simply,

\[ \text{VV} , \]

and each photon has a definite polarization. Such states are called **product states**. Similarly (using H, D, R and L for horizontal, 45° diagonal, right circular and left circular polarization, respectively) for HH or DD etc., or even VH or VD or RL etc. One can find strange features in these states, but you have to look for them. But consider the state, \( \Phi \), defined by,

\[ \Phi : = (1/2)^{1/2} ( R L – L R ) . \]

Neither photon has a definite circular polarization. Each has a 50-50 chance of being found as either R or L.

Well what about linear polarization? What does \( \Phi \) look like in terms of V, H, D, and D’? Using the equations from **IV** expressing R and L in terms of V and H and then expressing V and H in terms of D and D', we get,

\[
\begin{align*}
(2)^{1/2} \Phi &= R L – L R \\
&= (1/2)^{1/2} ( V + i H ) (1/2)^{1/2} ( V – i H ) – (1/2)^{1/2} ( V – i H ) (1/2)^{1/2} ( V + i H ) \\
&= i ( HV – VH ) \\
&= i [(1/2)^{1/2} ( D – D')(1/2)^{1/2} ( D + D' ) – (1/2)^{1/2} ( D + D')(1/2)^{1/2} ( D – D')] \\
&= i ( DD' – D'D ) .
\end{align*}
\]

It appears that neither photon has any definite polarization state at all (this would be confirmed upon further analysis) but, instead, has a 50-50 chance
of being found in either one of *any kind* of polarization state you like! In other words, for *either* single photon, we have,

\[ P(R) = P(L) = P(V) = P(H) = P(D) = P(D') = 1/2. \]

If we look for D, half will be D and the other half D'. If we look for V, half will be V and the other half H. If we look for R, half will be R and the other half L. Clearly, what they are, or rather *what they become when we look* depends on *what we look for!*

But what about the *other* photon? What happens to it? Well, using our discussion of measurement and state reduction from IV, if we look to see if the 'first' photon is a V photon and find that it *is,* then the state change is

\[ \Phi - - - - > VH, \]

and the 'second' photon is definitely an H photon. If we look to see if the 'first' photon is a D and find that it *isn't,* then the state reduction is

\[ \Phi - - - - > D'D, \]

and the 'second' photon (which we did not measure) is definitely a D! States like \( \Phi, \) in which each constituent may have few or no definite properties but the possible properties of both constituents are correlated, are called *entangled states.*

I've been referring to 'first' and 'second' photons. That should sound suspicious! How do we tell them apart? Remember what we said previously about the indistinguishability of quantons of the same type? So what am I talking about here?!

We can get away with this kind of talk because we haven't specified everything that can be specified about photons. We've only talked about polarization. What about momentum, for instance. The 'first' photon could be one that's traveling West while the 'second' photon could be traveling East (Fig. 5.6). Then that first state reduction means we found the West moving photon to be a V and, therefore, the East moving photon must be an H. Or, in the second case we found the West moving photon was a D' and, therefore, the East moving photon must be a D.
But imagine how far apart two oppositely moving photons can get in a tiny fraction of a second?! So OK, we choose to examine the West moving photon to see if it's V or H, thereby guaranteeing that the East moving photon, tens of thousands of miles away by the time of the measurement, will become an H or a V, respectively. Or we choose to examine photon-West for R or L, guaranteeing that photon-East will become an L or R photon, respectively. This is the nature of entanglement and, as we see, it brings virulent nonlocality with it! Entangled systems, very far away from each other, can each suddenly acquire properties precipitated by acts performed locally on only one of them!

Einstein was one of the first people to realize that QM implied these possibilities and, in 1935 he and two collaborators published a paper to show how absurd QM could be! The community of physicists was initially shocked. But now we realize the world is as ‘absurd’ as QM says it is!

Some researchers think these features of the quantum world make quantum theory incompatible with Einstein's Relativity theory, with it's relativity of simultaneity and time dilation. For the question is, when does the entangled distant system acquire it's properties?! Quantum theory seems to answer, simultaneously with the local act. Simultaneously?! Relative to who?? Most don't think there's an incompatibility, but resolving the question is a subtle business!

Just to convince ourselves that this kind of weirdness is not confined to photons, we note that if we have two electrons we can have them in the entangled spin state,

$$\Sigma = |x, + > - |x, - + > = |z, - + > - |z, + - > = i ( |y, - + > - |y, + - > ).$$

Once again, neither electron has any definite spin state and a probability of 1/2 of being found in any of the two possible spin states one decides to search for, while the two electrons, regardless of how far apart they might be, have their possible spin states completely (anti)correlated.

Do states like this actually occur in nature? Yes, frequently! For instance (Fig. 5.6), there is a quanton called a 'neutral pion', denoted by $\pi^0$. It has no
**Fig. 5.6: Entanglement.** Spontaneous disintegration of a neutral pion, $\pi^0$, into two oppositely moving photons, the polarization state of which is the entangled state,

$$\Phi = (1/2)^{1/2}(RL - LR) = i(1/2)^{1/2}(HV - VH) = i(1/2)^{1/2}(DD' - D'D).$$

When intercepted by mutually perpendicular, *but otherwise arbitrary*, polarizing filters, either both photons pass through or both photons are absorbed!
electric charge (hence 'neutral' and \( \theta \)) and it does not spin. But it is unstable and spontaneously disintegrates into two oppositely moving photons in the polarization state, \( \Phi \). There are other systems that emit pairs of electrons in the spin state, \( \Sigma \).

So photons and electrons and, presumably, other quantum systems can be very counterintuitively bizarre and weird. Especially when entangled! But these are the minute quantum constituents of the physical world and we're hardly in a position to expect them to behave in the way we are familiar with from our classical macroscopic experience.

But wait! General Quantum Mechanics is still a fundamental theory. Nothing we possess goes beyond it. So, as far as we can presently tell, quantum theory applies to all physical systems, macroscopic as well as microscopic. In our best present final analysis, we are quantum systems. Maybe not just quantum systems, but quantum systems nonetheless!

Indeed there are several kinds of macroscopic systems that we understand at all only in terms of quantum theory. Liquid superfluid helium, which we can hold in pint sized vessels, and which behaves very strangely; superconductors and superconducting circuits, which conduct electric current without resistance if sufficiently cold; and the evolution of the interior of stars via thermonuclear reactions. If anything qualifies as a macroscopic object, a star surely does.

In fact there's not a shred of evidence that quantum theory does not apply to macroscopic objects. But then Schroedinger, in 1935, gives us a conundrum.

**4. Schroedinger's cat:** While quantum theory may apply to macroscopic objects, usually the quantum state of such an object would be so complex as to be hopelessly impossible to analyze or use. However, interesting and counterintuitive results follow just from assuming the *existence* of such macroscopic quantum states.

For example, there's nothing unusual anymore about a quantum state for a single unstable nucleus evolving (Fig. 5.7) from the initial state,
Fig. 5.7: The exponential decay with time of the probability of a radioactively unstable nucleus to remain intact. With \( N(t) \) as the number of undecayed nuclei left after time \( t \) when \( N(0) \) is the initial number of undecayed nuclei, the non-decay probability is, \( p(t) = N(t)/N(0) \). We then find that, \( p(nt) = p(t)^n \). In particular, \( p(2\tau_{1/2}) = p(\tau_{1/2})^2 \), and \( p(3\tau_{1/2}) = p(\tau_{1/2})^3 \), where \( \tau_{1/2} \) is the half life such that \( p(\tau_{1/2}) = 1/2 \).

| undecayed nucleus > or (for short) |Un >,

at time \( t = 0 \), to the superposition,

\[
(1/2)^{1/2} (|\text{undecayed nucleus} > + |\text{decayed nucleus} > ) \text{ or } \\
(1/2)^{1/2} (|\text{Un} > + |Dn > ),
\]
after one half life, say, at $t = 1$ hour. If we examined the nucleus after 1 hour, we would find it to be definitely undecayed or definitely decayed, each with probability $1/2$. But if we don't examine the state of decay, then the nucleus is neither decayed nor undecayed. It's in a superposition state.

Now surround the nucleus with a Geiger counter arrangement such that if the decay occurs, the Geiger counter responds and releases a hammer originally in a raised position. So now we have an initial state,

$$|\text{Un, hammer up} > \text{ or } |\text{Un, Uh} >,$$

which evolves, in an hour into

$$(1/2)^{1/2} (|\text{Un, hammer up} > + |\text{Dn, hammer down} > ) \text{ or } (1/\sqrt{2})(|\text{Un, Uh} > + |\text{Dn, Dh} > ).$$

Since the nucleus is neither decayed nor undecayed, the entangled hammer is neither up nor down, but in a superposition of the up and down states. That's rather strange!

Now place a vial of prussic acid in the way of the hammer when it falls. Then we go from,

$$|\text{Un, Uh, vial intact} > \text{ or } |\text{Un, Uh, Iv} >,$$

$$\text{to (after one hour)}$$

$$(1/2)^{1/2} (|\text{Un, Uh, vial intact} > + |\text{Dn, Dh, vial shattered} > ) \text{ or } (1/2)^{1/2} (|\text{Un, Uh, Iv} > + |\text{Dn, Dh, Sv} > )$$

(provided we haven't looked to see what has happened).

Finally, following Schroedinger, we place this "diabolical device" in an enclosed room with a live cat (Lc). And now we have (Fig. 5.8),

$$|\text{Un, Uh, Iv, Lc} > \text{ going to }$$

$$(1/2)^{1/2} (|\text{Un, Uh, Iv, Lc} > + |\text{Dn, Dh, Sv, Dc} > )$$

after one hour if we haven't looked. The rest is history.
Obviously the cat was added only for dramatic impact. The puzzle is already fully developed with a superposition of a hammer up and a hammer down. What in the world is that?! Given quantum theory, is there anything in the world that can prevent such things from happening? Well, yes, at least for all practical purposes (FAPP).

Fig. 5.8: Schroedinger’s Cat Paradox

Big things like hammers are constantly interacting in an uncontrolled and pretty nearly uncontrollable way with their environment of zillions of atoms and photons and etc. These random, thermodynamical interactions are essentially guaranteed to very quickly destroy the delicate isolation from disturbance needed to preserve the superposition. In effect, the environment is constantly ‘looking’ at the system even if we aren't.
For microscopic systems of a few photons or electrons or atoms or even molecules we can maintain isolation from the environment pretty well. For hammers and vials and mammals, forget it!

But researchers keep trying to get better at it. They've already succeeded in producing a superposition of a collection of billions of electrons moving clockwise and counterclockwise around a superconducting circuit. That's not some electrons going clockwise while the rest go counterclockwise. Each electron is in an entangled superposition state with all the rest!

So while we'll probably never see Schroedinger's cat, we may one day see Schroedinger's single virus, or Schroedinger's single biological cell, in a superposition of life and death! If we ever do, it may well change our understanding of the relationship between life and death.

On the other hand a word of caution is in order. There’s a lot of loose talk these days about the weird entangled quantum states enabling objects (the hammer for example) to “be in two places at the same time” or enabling Schroedinger’s cat to be “both alive and dead at the same time”. Now while entanglement is bizarre it is not self contradictory. The entangled superpositions yield comparable probabilities of finding the object or the cat to be in one or the other of two incompatible states, but they never require or allow the object or cat to be simultaneously in both states. Instead the hammer can be in a state in which its neither up nor down and the cat in a state in which its neither dead nor alive! That’s very strange because it asserts third, alternative, possibilities to what we’re accustomed to regarding as exhaustive dichotomies. But it’s not self contradictory.

5. Hidden variables and Bell inequalities

From the early days of QM many people have wanted to be able to show that the probabilities of QM were just like those of classical physics and simply reflected our ignorance of the details of what was really going on, our ignorance of so-called hidden variables. Many efforts to build such hidden variable models of QM have been undertaken and the most successful of them was the model created by David Bohm in 1951. But even then the probabilities continued to have bizarre, counterintuitive properties which troubled people. Then, in the 70s, John Bell showed that all possible attempts at hidden variable reinterpretations of QM would necessarily fail if
they involved only intuitively satisfying probabilities, so-called LOCAL probabilities, and could succeed in being compatible with QM only if the model retained much of the counterintuitive features of QM which the modeling was intended to explain. Bell did this by deriving inequalities which all intuitively appealing, i.e., all LOCAL models, must satisfy, but which QM violates.

To see an example of a Bell inequality, consider a system of two electrons. Since each electron has spin 1/2, the total spin of the 2-electron system can be either \( 1 = 1/2 + 1/2 \), or \( 0 = 1/2 - 1/2 \). Suppose the total spin is 0. This requires that for any given direction, the values of each electron spin component in that direction are opposite, i.e., \( (1/2, -1/2) \) or \( (-1/2, 1/2) \) denoted below by \((+, -)\) or \((- ,+)\) respectively. In this way the electrons are entangled.

Now consider an apparatus that can measure simultaneously one spin component for each electron. The directions of these measured spin components can be independently chosen to be any of three directions labeled, 1, 2, 3. The probabilities for obtaining the results, a, b for any pair, d, d’ of the three directions will be denoted by, \( P_{dd'}(a, b) \), where a, b can take the values, +, +; +, -; -, + or -, -. In other words, \( P_{dd'}(a, b) \) is the probability that one electron’s spin component in the direction, d, registers the value, a, and the other electron’s spin component in the direction, d’, registers the value, b.

From the previous comments about total spin zero we have,

\[
P_{11}(+,+) = P_{22}(+,+) = P_{33}(+,+) = P_{33}(-,-) = P_{22}(-,-) = P_{11}(-,-) = 0
\]

and

\[
P_{11}(+,-) = P_{22}(+,-) = P_{33}(+,-) = P_{33}(-,+) = P_{22}(-,+)= P_{11}(-,+)= 1/2
\]

For distinct directions, d and d’, quantum mechanics tells us that,

\[
P_{dd'}(+,+) = 1/2 \sin^2 \left( \theta_{dd'}/2 \right) = P_{dd'}(-,-)
\]

and

\[
P_{dd'}(+,-) = 1/2 \cos^2 \left( \theta_{dd'}/2 \right) = P_{dd'}(-,+)
\]
where $\theta_{dd'}$ is the angle between the directions $d$ and $d'$. If $d = d'$, then $\theta_{dd'} = 0$ and since, $\sin (0) = 0$ and $\cos (0) = 1$, the formulae agree with the previous lines. So far we’ve just been discussing the standard QM for the electrons.

Now suppose there are hidden variables that determine the spin components in all directions at every moment. Since we don’t know the values of these hidden variables, we can only assume their values are distributed according to certain probabilities. The probability distributions for these hidden variables would then also determine the probabilities for the electron spin components to have anyone of their possible values during a measurement.

Denoting a possible value for the spins of the two electrons in direction 1 by $a$ and $a'$, in direction 2 by $b$ and $b'$ and in direction 3 by $c$ and $c'$, the hidden variable probability for two electrons and all three directions can be denoted by, $P_{HV}(a, b, c; a', b', c')$. For our two electrons in the total spin 0 state, the only non-zero probabilities are for the cases of the form, $P_{HV}(a, b, c; -a, -b, -c)$. That is, there are eight different, non-zero probabilities,

$$P_{HV}(+,+,+; -,-,-), \ P_{HV}(+,+,-; -,-,+), \ P_{HV}(+,+-; -,+,-), \ P_{HV}(-,+; +,-,-),$$

$$P_{HV}(-,-; +,+,-), \ P_{HV}(-,+; +,-,+)$$

These hidden variable probabilities, whatever they are, determine the QM probabilities, $P_{12}(+,+), P_{23}(+,+)$ and $P_{13}(+,+)$ to be,

$$P_{12}(+,+) = P_{HV}(+,+,+; -,-,-) + P_{HV}(+,+-; -,+,-),$$

$$P_{23}(+,+) = P_{HV}(+,+,-; -,-,+) + P_{HV}(-,+; +,-,-),$$

$$P_{13}(+,+) = P_{HV}(+,+; -,-,+) + P_{HV}(+,+-; -,+,-).$$

But notice what this tells us!

$$P_{12}(+,+) + P_{23}(+,+) = [P_{HV}(+,+,+; -,-,-) + P_{HV}(+,+-; -,+,-)] +$$

$$[P_{HV}(+,+,-; -,-,+) + P_{HV}(-,+; +,-,-)] \geq$$

$$P_{HV}(+,+; -,-,+) + P_{HV}(+,-; -,+,-) = P_{13}(+,+).$$

In other words,
\[ P_{12}(+,+) + P_{23}(+,+) \geq P_{13}(+,+) , \]

an example of what is called a \textit{Bell inequality}. Using the QM values for \( P_{dd'}(+,+) \), this is,

\[ \frac{1}{2} \sin^2(\theta_{12}/2) + \frac{1}{2} \sin^2(\theta_{23}/2) \geq \frac{1}{2} \sin^2(\theta_{13}/2) . \]

But this is violated by many angle choices! For example, suppose all three directions lie in a plane so that,

\[ \theta_{13} = \theta_{12} + \theta_{23} . \]

Now choose \( \theta_{13} = 120^0, \theta_{12} = \theta_{23} = 60^0 \). Since \( \sin^2 60^0 = 3/4 \) and \( \sin^2 30^0 = 1/4 \), the inequality requires,

\[ (1/2)(1/4) + (1/2)(1/4) \geq (1/2)(3/4) , \]

and is clearly violated. More generally, since

\[ \sin^2[(\theta_{12} + \theta_{23})/2] = \sin^2(\theta_{12}/2) + \sin^2(\theta_{23}/2) + 2\sin(\theta_{12}/2) \sin(\theta_{23}/2) \cos[(\theta_{12} + \theta_{23})/2] , \]

the inequality is violated for any angles satisfying,

\[ 0 < \theta_{12} , \theta_{23} , \theta_{12} + \theta_{23} < 180^0 . \]

So LOCAL deterministic hidden variables can NOT reproduce all of QM! Why do I say LOCAL? Because the hidden variable probabilities do not depend on which of the spin directions we choose to measure, they only depend on the spin values, themselves, whether we measure them or not. In other words the hidden variable probabilities refer “locally” only to the hidden variables and not to decisions or choices made by external observers.
Notice that if the hidden variable probabilities *did* depend on which of the spin directions we choose to measure, then we could not derive the Bell inequality. The expression of the QM probabilities in terms of the hidden variable probabilities would then be,

\[
P_{12}(+,-) = P_{HV,12}(+,-,+; -,-,+) + P_{HV,12}(+,-,-; -,+,+),
\]

\[
P_{23}(+,-) = P_{HV,23}(+,-,+; -,-,+) + P_{HV,23}(-,+,-; +,-,+),
\]

\[
P_{13}(+,-) = P_{HV,13}(+,-,+; -,-,+) + P_{HV,13}(+,-,-; -,+,+),
\]

and our Bell inequality need not hold. But now we’ve lost much of the motivation for a hidden variable explanation because the way in which QM probabilities depend on our choice of what to measure is a major part of the strangeness of QM. In fact this is just how David Bohm’s hidden variable account of QM works; by retaining the dependence of the hidden variable probabilities on the choices of what we measure.

Of course we might still hope to perform experiments that test whether the Bohm hidden variable model is correct. But because the Bohm model (and the infinite number of variations of it that work equally well) reproduces QM exactly, there is no possibility of designing an experiment that tells us which variant, if any, is correct. They’re all correct as far as reproducing QM is concerned, but their different additional details are, as a matter of principle, inaccessible to experimental test.

What some still hope for is a hidden variable theory that reproduces all the features of QM that have THUS FAR been tested and confirmed but which differs from QM in some as yet untested features and which, when those tests are finally made, will pass the test as QM fails it. But I wouldn’t call that a hidden variable theory. I’d just call it an improvement on QM.

### Appendix A: Doped Semiconductors and the n-p junction Diode

Silicon is a semiconductor. The Silicon atom has 4 electrons in the outermost, *n* = 3 state. At ordinary temperatures a Silicon crystal has some
of these electrons lifted into the conduction band with corresponding holes left in the valence band. These electrons and holes can yield weak currents under the influence of a voltage source across the crystal. However, if the crystal is ‘contaminated’ with a small percentage of impurity atoms, say Phosphorous with 5 electrons in the n = 3 state, many of the extra electrons go into the conduction band and much more current can flow. Similarly, if the Silicon crystal is ‘contaminated’ with a small percentage of Boron with only 3 electrons in the outermost n = 3 state, many of the ‘missing’ electrons act like mobile holes in the valence band and again yield stronger currents when voltage is applied. Semiconductors like Silicon with these kinds of impurities are called doped semiconductors. The doped semiconductor that

With positive bias (V > 0) current flows easily to the right

- - - - - - extra electrons

p-type

extra holes + + + + + +

n-type

Very little current can flow to the left with negative bias (V < 0)

\[ V \]

Exponentially sensitive positive current

Negative minimum current

\[ V \]

\[ I \]

\[ V \]

Fig. A1: n-p junction diode and current – voltage response curve
has extra electrons in the conduction band is called an n (for negative) type semiconductor while the doped semiconductor with missing electrons or extra positive holes in the valence band is a p (for positive) type semiconductor.

If an n-type semiconductor is placed in contact with a p-type semiconductor (Fig. A1) a voltage difference across the pair that would pull electrons and holes away from the junction can only generate very small currents from the n-type to the p-type semiconductor because this would further separate the extra electrons in the n-type from the extra holes in the p-type. But the junction yields strong and very sensitive currents to a voltage that pulls the extra n- electrons and p-holes towards each other. Thus the n-p junction acts like a one way diode for current. The great advantage of replacing the old vacuum tube diodes with semiconductor n-p junction diodes is that the latter can operate much faster and more sensitively to small voltages and they can be made much smaller! The same is true for the semiconductor transistor which replaces the old vacuum tube triode.

Appendix B: Light emitting diodes

In some forward biased n-p junctions the conduction band electrons emit visible light photons as they combine with valence band holes. This is the basis for light emitting diodes or LEDs. The emission process is very efficient because it is accompanied by almost no heat loss.
Until 1927 no form of quantum mechanics was recognized. Physicists, like Heisenberg, treated the problem of the wave nature of light. Bohr asserted, "No, but I was deeply disturbed by the thought that this new form of quantum mechanics might be necessary to explain the behavior of matter and energy."

Heisenberg, in his work on the uncertainty principle, suggested that the path of a particle was not determined until it was observed. This idea was revolutionary and led to the development of the modern theory of quantum mechanics.