## Math Appendix 5: Generalizations of numbers

**Complex numbers:** Ordinary numbers come in three varieties, **positive** numbers like 1, 3, 5.7, 3.14159, etc., **negative** numbers like – 1, – 0.25, – 5.7, – 62, etc. and, neither positive or negative, zero, 0! Sometimes we talk about the **non-negative** numbers which includes zero and all the positives,

$$\{0 \leftarrow \quad \rightarrow + \infty\}.$$

Similarly, we have the **non-positive** numbers,

$$\{- \infty \leftarrow \quad \rightarrow 0\}.$$

These two collections share the number, 0.

For *any* of these numbers we can add any two of them or subtract any one from any other or multiply any two of them. And except for zero, we can divide any number by any other number. We're not allowed to divide by zero because that either yields $\pm \infty$ , which are not quite respectable, or, in the case of $0 / 0$ it doesn't yield anything definite.

We can also multiply any number by itself an arbitrary number of times. This frequently useful operation is called *raising a number to a power* or *exponentiating*. Thus 2 to the power, 5, is

$$2^5 = 2x2x2x2x2 = 32.$$

In general, $a^n = a\ a\ a \text{ - - - } a$ (n factors).

The idea of exponentiating leads to the inverse question, for any given number, a, what is the number, b, which when raised to the power, n, equals a? Since $b^n = a$ , we write $b = a^{1/n}$ and call b the nth root of a. This notation fits nicely with the rule of exponentiation that,

$$(a^m)^n = a^{mn} \qquad \text{since} \qquad b^n = (a^{1/n})^n = a^{n/n} = a^1 = a .$$

But there's a problem! For n = even integer, like 2, 4, 6, etc. and $a < 0$, i.e., a negative, $a^{1/n}$ doesn't exist! *There are no even roots of negative numbers!*

In algebra this turns out to be a real nuisance and the question of how to eliminate the nuisance became important. Amazingly, the solution turned out to consist of just adding one new number symbol to the already existing numbers and allowing all possible additions, subtractions, multiplications and divisions of the new symbol with the standard number symbols.

The new symbol was *i* and its defining connection with ordinary numbers was the equation, $i^2 = i \times i = -1$. It was the missing square root of minus one! Now all negative numbers had (just like all positive numbers) two square roots, e.g.,

$$(-4)^{1/2} = (-1)^{1/2}(4)^{1/2} = \pm 2\,i\ .$$

By *formally* multiplying *i* by arbitrary ordinary numbers ( *i* b ) and then adding the result to arbitrary ordinary numbers ( a + *i* b ) we obtain the collection of all so-called *complex* numbers. The ordinary numbers are now called *real* numbers and the product of *i* and a real number ( *i* b ) is called an *imaginary* number. *i*, itself, is called the *imaginary unit*.

The addition and subtraction of complex numbers follows the rule,

$$(a + i\,b) \pm (c + i\,d) = (a \pm c) + i\,(b \pm d)\ .$$

Multiplication of complex numbers follows the rule,

$$(a + i\,b)(c + i\,d) = (ac - bd) + i\,(ad + bc)\ .$$

Division follows the rule

$$(a + i\,b)/(c + i\,d) = [(a + i\,b)(c - i\,d)]/[(c + i\,d)(c - i\,d)] =$$

$$[(ac + bd) + i\,(bc - ad)] / (c^2 + d^2) =$$

$$[(ac + bd)/(c^2 + d^2)] + i\,[(bc - ad)/(c^2 + d^2)]\ .$$

And finally, there is also a rule for completely general exponentiation, i.e., a rule for evaluating,

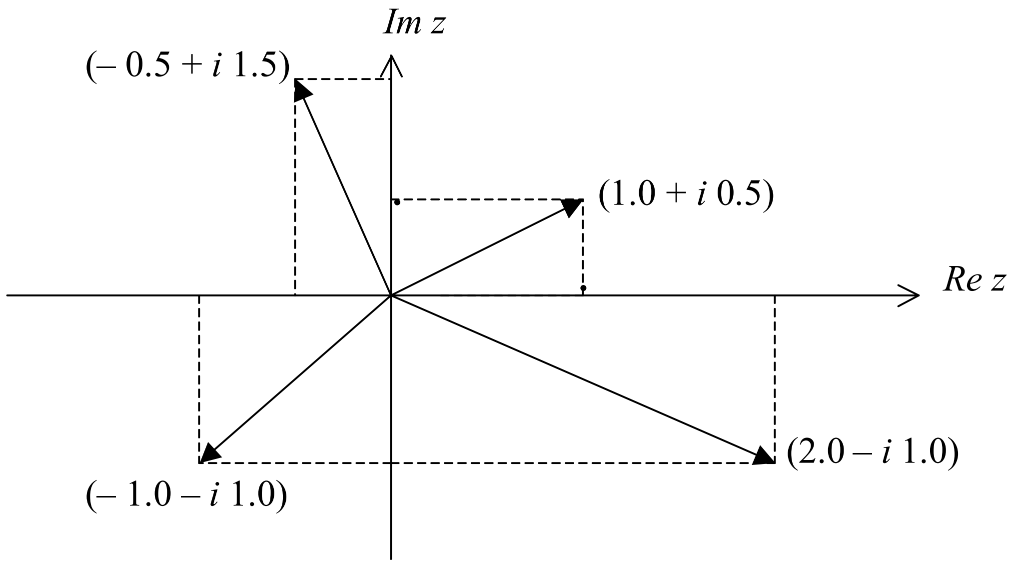
$$(a + i\,b)^{(c + i\,d)}\quad ,$$

which is why *i* was introduced in the first place. Unfortunately, *that* rule is a bit complicated for this appendix. Nevertheless, here it is,

$$(a + i\, b)^{(c + i\, d)} = [(a^2 + b^2)^{1/2} \exp(i \tan^{-1}(a/b))]^{(c + i\, d)} =$$

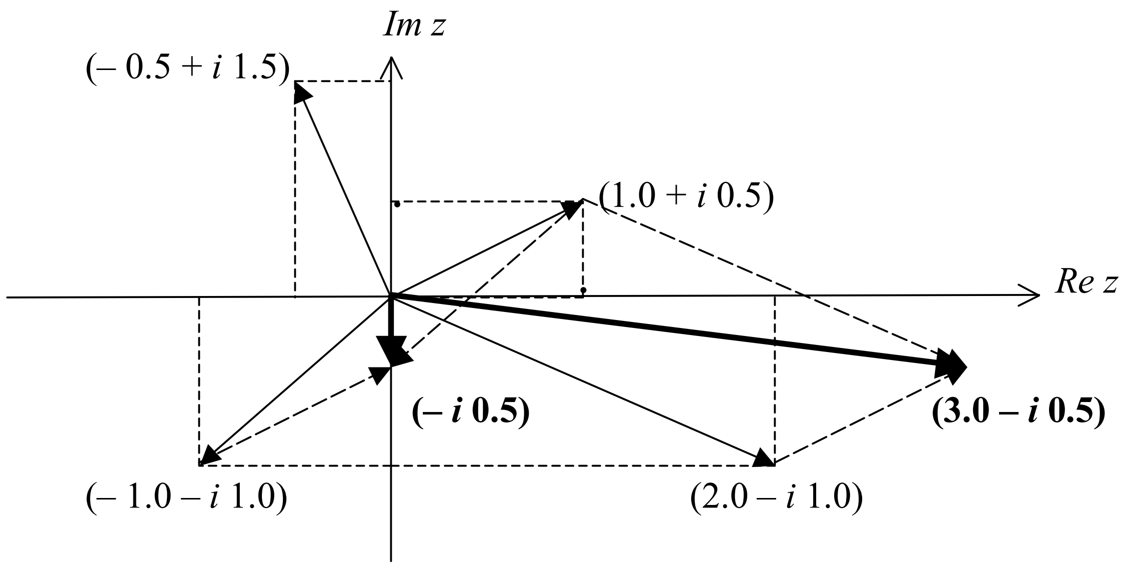
$$(a^2 + b^2)^{c/2} \exp[-d \tan^{-1}(a/b)] \exp\{i\, [(d/2) \ln(a^2 + b^2) + c \tan^{-1}(a/b)]\}.$$

There is a convenient way to represent complex numbers geometrically. Since complex numbers have two parts, their real part and their imaginary part, they can be represented in a two dimensional plane with the real part plotted along the horizontal axis and the imaginary part along the vertical axis.

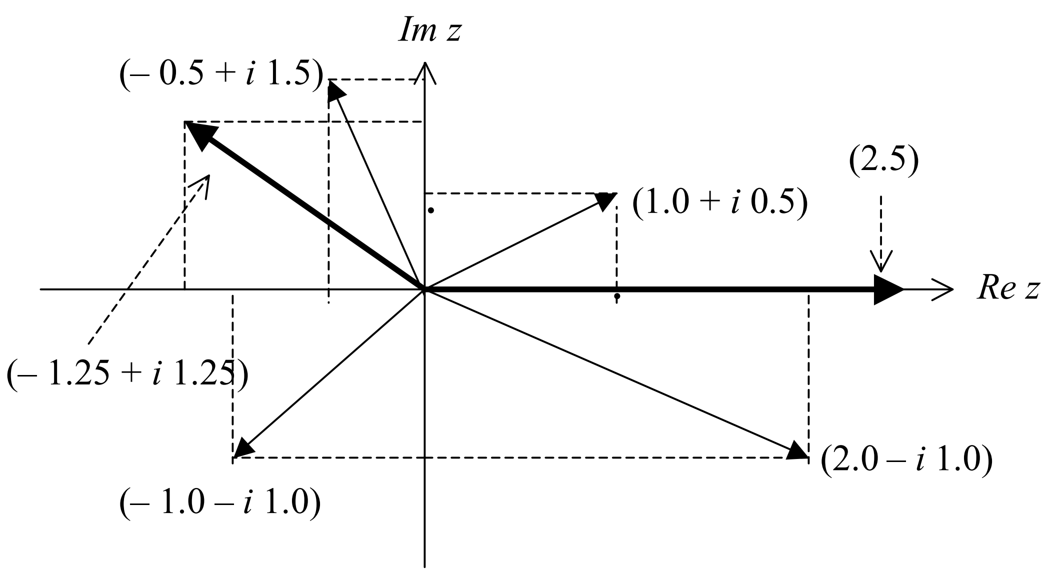



**Representing complex numbers in a plane:** When a plane is used to represent complex numbers it is called an Argand plane or a complex plane or *the* complex plane.

The addition and subtraction of complex numbers is represented in the complex plane in the same way the addition and subtraction of *arrows* or *directed quantities* or *vectors* would be represented. The real and imaginary parts of the numbers play the role of mutually perpendicular components of the vectors. For example, see the next figure. The geometrical representation of the *multiplication* of complex numbers is a bit more subtle. If the two factors in the product are represented by arrows that make angles of $\theta_1$ and $\theta_2$ with the positive real axis, then the product arrow makes an angle of $\theta_1 + \theta_2$ with the positive real axis. The magnitude of the product arrow is the product of the magnitudes of the factor arrows. For examples see two figures down.




**Examples of addition and subtraction in the complex plane:**
$(1.0 + i\ 0.5) + (2.0 - i\ 1.0) =$ $\mathbf{(3.0 - i\ 0.5)}$,
$(1.0 + i\ 0.5) - (1.0 + i\ 1.0) = (1.0 + i\ 0.5) + (-\ 1.0 - i\ 1.0) =$ $\mathbf{(-\ i\ 0.5)}$




**Examples of multiplication in the complex plane:**
$(1.0 + i\ 0.5)\text{x}(2.0 - i\ 1.0) = (2.5)$; angles: $\tan^{-1}(0.5) - \tan^{-1}(0.5) = 0$.
$(1.0 + i\ 0.5)\text{x}(-\ 0.5 + i\ 1.5) = (-\ 1.25 + i\ 1.25)$; angles:
$\tan^{-1}(0.5) + \tan^{-1}(-\ 3.0) = \tan^{-1}(-\ 1.0) = 3\pi/4$.