Math. Appendix 4: Propagation speeds of de Broglie matter waves.

Suppose a particle has a momentum of, p, and an energy of E. According to Einstein and de Broglie, the particle should be associated with or accompanied by a harmonic wave of wavelength, λ, and frequency, f, given by,

$$\lambda = h / \mathrm{p} \qquad \text{and} \qquad \mathrm{f} = \mathrm{E} / h. \tag{1}$$

From the universal relation for the propagation speed of *any wave*, $\lambda\, \mathrm{f} = \mathrm{u}$, we find,

$$\mathrm{u} = \lambda\, \mathrm{f} = (h / \mathrm{p})(\mathrm{E} / h) = \mathrm{E} / \mathrm{p}\ . \tag{2}$$

In pre-relativistic or non-relativistic physics, the energy of a free particle is just the kinetic energy, $\mathrm{E} = (1/2)\ \mathrm{m\ v}^2 = \mathrm{p}^2 / 2\mathrm{m}$. So we get,

$$\mathrm{u} = \mathrm{p} / 2\mathrm{m} = \mathrm{v} / 2\ , \tag{3}$$

and *the de Broglie wave is moving only half as fast as the particle*!

Relativistically, we have to add in the energy due to the rest mass, $\mathrm{m\ c}^2$, and this results in the momentum and the kinetic energy having a modified form. We end up with a particle speed of,

$$\mathrm{p} = \mathrm{M\ v}, \quad \mathrm{E} = [\mathrm{m}^2\ \mathrm{c}^4 + \mathrm{p}^2\ \mathrm{c}^2]^{1/2} = \mathrm{M\ c}^2\ , \tag{4}$$

and,

$$\mathrm{v} = \mathrm{p} / \mathrm{M} = \mathrm{c}^2\ \mathrm{p} / \mathrm{E} = \mathrm{c\ p} / [(\mathrm{m\ c})^2 + \mathrm{p}^2]^{1/2}\ , \tag{5}$$

which is never faster than the speed of light.

For the de Broglie wave propagation speed we get,

$$\mathrm{u} = \mathrm{E} / \mathrm{p} = (\mathrm{c} / \mathrm{p})[(\mathrm{m\ c})^2 + \mathrm{p}^2]^{1/2}, \tag{6}$$

which is *not only faster than the particle speed, it's faster than the speed of light*!! How is this possible and what is going on?!

Suppose we represent the de Broglie waves by propagating cosine waves,

$$\cos[2\pi\ ((\mathrm{x} / \lambda) - \mathrm{f\ t})] = \cos[(2\pi / \lambda)(\mathrm{x} - \mathrm{u\ t})]\ . \tag{7}$$

To deal with such functions, it will be convenient to introduce the quantities,

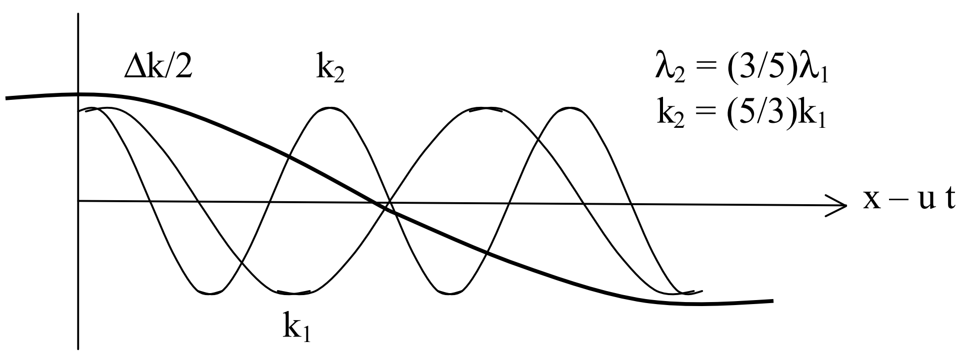
$$k \equiv 2\pi / \lambda , \qquad \text{and} \qquad \omega \equiv 2\pi f , \tag{8}$$

called the *propagation number* and the *angular frequency*, respectively. In terms of them our cosine function is,

$$\cos(k\, x - \omega\, t) = \cos[k(x - u\, t)]. \tag{9}$$

Now consider a sum of two such cosine waves with different wavelengths and frequencies as pictured below or, more elaborately, in **Fig. 2.5**.

$$\cos(k_1\, x - \omega_1\, t) + \cos(k_2\, x - \omega_2\, t). \tag{10}$$



This sum of cosines can be rewritten as

$$2 \cos(k_{av}\, x - \omega_{av}\, t) \cos[(\Delta k\, x - \Delta\omega\, t) / 2] , \tag{11}$$

where

$$k_{av} = (1/2)(k_1 + k_2), \quad \omega_{av} = (1/2)(\omega_1 + \omega_2) , \tag{12}$$

and

$$\Delta k = (k_2 - k_1), \quad \Delta\omega = (\omega_2 - \omega_1) . \tag{13}$$

The propagation speed of these cosine waves is, in the various cases, given by,

$$u_1 = \omega_1 / k_1 , \quad u_2 = \omega_2 / k_2 , \quad u_{av} = \omega_{av} / k_{av} \quad \text{and} \quad u_\Delta = \Delta\omega / \Delta k . \tag{14}$$

From our preceding discussion we know that the first three differ significantly from the speed of the associated particle. What about the last one, u_Δ?

For non-relativistic particles with $E = p^2 / 2m$, we have, $\omega = (h / 2\pi)\, k^2 / 2m$. Consequently,

$$\Delta\omega = (h / 2\pi)(k^2{}_2 - k^2{}_1) / 2m = ((h / 2\pi)\, k_{av} / m)\, \Delta k \,, \qquad (15)$$

and,

$$\Delta\omega / \Delta k = p_{av} / m \,, \qquad (16)$$

a reasonable approximation to the particle velocity. For the relativistic case we have,

$$\Delta(\omega^2) = 2\, \omega_{av}\, \Delta\omega = c^2\, \Delta(k^2) = 2\, c^2\, k_{av}\, \Delta k \,, \qquad (17)$$

or,

$$u_\Delta = \Delta\omega / \Delta k = c^2\, k_{av} / \omega_{av} = c^2\, p_{av} / E_{av} \,, \qquad (18)$$

again a good approximation to the particle velocity. Referring back to **Fig. 2.5**, u_Δ is the propagation speed of the 'beat phenomena' envelope and in a more general superposition of many cosine waves with variable amplitudes, the analogue to u_Δ is called the *group velocity* and reproduces the particle speed to the degree that that speed is well defined.