Math Appendix 2: The Bohr-Sommerfeld quantization rule applied to harmonic oscillators and hydrogenic atoms

Consider a particle oscillating back and forth as an oscillator and a particle orbiting a center in a circular orbit. If the first particle is under the influence of a restoring force proportional to distance from the center, it is an *harmonic* oscillator. If the second particle is under the influence of an attractive force proportional to the inverse square of the distance from the center, it behaves like an electron in a hydrogenic atom. Both systems exhibit periodic motion, i.e., both particles go through a cycle of positions which is then repeated again and again in exactly the same way.

The total energy of the oscillator has the form (m = mass, d = distance from center, v = velocity, k = force constant)

$$E_{osc.} = (1/2)\, m\, v^2 + (1/2)\, k\, d^2 , \tag{1}$$

and it is constant in time. When the particle is at the center, d = 0, the squared velocity is maximized and,

$$E_{osc} = (1/2)\, m\, v_{max}^2. \tag{2}$$

When the particle is at one or another of the ends of its range the velocity is momentarily zero, v = 0, and d^2 is maximized at,

$$E_{osc} = (1/2)\, k\, d_{max}^2. \tag{3}$$

In **Appendix A** we saw that $v_{max}\, T = 2\pi\, d_{max}$ where T is the time required for the particle to complete one cycle. But since,

$$(1/2)\, m\, v_{max}^2 = (1/2)\, k\, d_{max}^2, \tag{4}$$

we must have, $T = 2\pi\, (d_{max} / v_{max}) = 2\pi \sqrt{(m / k)}$.

Now the **action** of one cycle of the particle's motion is defined as T times the *average* value of $m\, v^2$,

$$A_{cycle} \equiv T < m\, v^2 >_{cycle} . \tag{5}$$

For the oscillator that average value is just half the maximum value, i.e.,

$$< m v^2 >_{cycle} = (1/2) m v_{max}^2 = E_{osc} . \qquad (6)$$

Hence

$$A_{cycle} = T E_{osc} . \qquad (7)$$

The Bohr-Sommerfeld quantization rule was that the action must be an integer multiple of Planck's constant, *h*, i.e.,

$$n h = A_{cycle} = T E_{osc} , \qquad (8)$$

or

$$n h / T = E_{osc} . \qquad (9)$$

But 1 / T is just the frequency of the oscillations, f = 1 / T. So we have,

$$n h f = E_{osc} , \qquad (10)$$

Planck's formula for the energy levels of a harmonic oscillator!

For the electron in a hydrogen atom in a circular orbit at distance, d, from the center, the energy is,

$$E_{atom} = (1/2) m v^2 - \kappa e^2/ d , \qquad (11)$$

Which is constant in time. Since, *mass times acceleration = force*, we also have,

$$m v^2 / d = \kappa e^2 / d^2, \qquad (12)$$

or

$$m v^2 = \kappa e^2/ d . \qquad (13)$$

Substituting this last result into the equation for E_{atom} , we can express E_{atom} in terms of *either* $m v^2$ *or* $\kappa e^2/ d$ separately,

$$E_{atom} = - (1/2) m v^2 = - \kappa e^2/ 2d . \qquad (14)$$

For a circular orbit the time required to complete one cycle, one orbit, is just,

$$T = 2\pi d / v . \qquad (15)$$

Again, the *action* of one cycle of the particle's motion is defined as T times the average value of $m v^2$,

$$A_{cycle} \equiv T < m v^2 >_{cycle} , \qquad (16)$$

and, as before, the quantization rule is that the action must be an integer multiple of Planck's constant, *h*, i.e.,

$$n h = A_{cycle} = T < m v^2 >_{cycle} . \qquad (17)$$

For our circular orbits, with d and v constant, we have,

$$A_{cycle} = (2\pi d / v) m v^2 = 2\pi d m v . \qquad (18)$$

But from above,

$$d = -\kappa e^2 / 2E_{atom} , \quad \text{and} \quad v = (- 2E_{atom} / m)^{1/2} . \qquad (19)$$

Therefore,

$$A_{cycle} = 2\pi (\kappa e^2 m^{1/2}) / (- 2E_{atom})^{1/2}, \qquad (20)$$

or,

$$n h = 2\pi (\kappa e^2 m^{1/2}) / (- 2E_{atom})^{1/2}, \qquad (21)$$

or, solving for E_{atom} ,

$$E_{atom} = - 2\pi^2 \kappa^2 e^4 m / n^2 h^2 , \qquad (22)$$

Bohr's formula for the hydrogen energy levels!