Math Appendix 2: The Bohr-Sommerfeld quantization rule applied to harmonic oscillators and hydrogenic atoms

Consider a particle oscillating back and forth as an oscillator and a particle orbiting a center in a circular orbit. If the first particle is under the influence of a restoring force proportional to distance from the center, it is an harmonic oscillator. If the second particle is under the influence of an attractive force proportional to the inverse square of the distance from the center, it behaves like an electron in a hydrogenic atom. Both systems exhibit periodic motion, i.e., both particles go through a cycle of positions which is then repeated again and again in exactly the same way.

The total energy of the oscillator has the form (m = mass, d = distance from center, v = velocity, k = force constant)

\[ E_{osc.} = \frac{1}{2} m v^2 + \frac{1}{2} k d^2 , \quad (1) \]

and it is constant in time. When the particle is at the center, d = 0, the squared velocity is maximized and,

\[ E_{osc} = \frac{1}{2} m v_{max}^2 . \quad (2) \]

When the particle is at one or another of the ends of its range the velocity is momentarily zero, v = 0, and d^2 is maximized at,

\[ E_{osc} = \frac{1}{2} k d_{max}^2 . \quad (3) \]

In Appendix A we saw that \( v_{max} T = 2\pi d_{max} \) where T is the time required for the particle to complete one cycle. But since,

\[ (1/2) m v_{max}^2 = (1/2) k d_{max}^2 , \quad (4) \]

we must have, \( T = 2\pi (d_{max} / v_{max} ) = 2\pi \sqrt{(m / k)} \).

Now the action of one cycle of the particle’s motion is defined as T times the average value of \( m v^2 \),

\[ A_{cycle} = T < m v^2 >_{cycle} . \quad (5) \]
For the oscillator that average value is just half the maximum value, i.e.,

\[ < m v^2 >_{\text{cycle}} = (1/2) m v_{\text{max}}^2 = E_{\text{osc}}. \]  

Hence

\[ A_{\text{cycle}} = T E_{\text{osc}}. \]  

The Bohr-Sommerfeld quantization rule was that the action must be an integer multiple of Planck’s constant, \( h \), i.e.,

\[ n h = A_{\text{cycle}} = T E_{\text{osc}}, \]  
or

\[ n h / T = E_{\text{osc}}. \]  

But \( 1 / T \) is just the frequency of the oscillations, \( f = 1 / T \). So we have,

\[ n h f = E_{\text{osc}}, \]  

Planck’s formula for the energy levels of a harmonic oscillator!

For the electron in a hydrogen atom in a circular orbit at distance, \( d \), from the center, the energy is,

\[ E_{\text{atom}} = (1/2) m v^2 - \kappa e^2 / d, \]  

Which is constant in time. Since, mass times acceleration = force, we also have,

\[ m v^2 / d = \kappa e^2 / d^2, \]  
or

\[ m v^2 = \kappa e^2 / d. \]  

Substituting this last result into the equation for \( E_{\text{atom}} \), we can express \( E_{\text{atom}} \) in terms of either \( m v^2 \) or \( \kappa e^2 / d \) separately,

\[ E_{\text{atom}} = -(1/2) m v^2 = - \kappa e^2 / 2d. \]  

For a circular orbit the time required to complete one cycle, one orbit, is just,

\[ T = 2\pi d / v. \]
Again, the action of one cycle of the particle’s motion is defined as $T$ times the average value of $m \nu^2$,

$$A_{\text{cycle}} \equiv T < m \nu^2 >_{\text{cycle}},$$  \hspace{1cm} (16)

and, as before, the quantization rule is that the action must be an integer multiple of Planck’s constant, $\hbar$, i.e.,

$$n \hbar = A_{\text{cycle}} = T < m \nu^2 >_{\text{cycle}}.$$  \hspace{1cm} (17)

For our circular orbits, with $\nu$ constant, we have,

$$A_{\text{cycle}} = (2\pi d / \nu) m \nu^2 = 2\pi d m \nu.$$  \hspace{1cm} (18)

But from above,

$$d = -\kappa e^2 / 2E_{\text{atom}}, \quad \text{and} \quad \nu = (-2E_{\text{atom}} / m)^{1/2}.$$  \hspace{1cm} (19)

Therefore,

$$A_{\text{cycle}} = 2\pi (\kappa e^2 m^{1/2}) / (-2E_{\text{atom}})^{1/2},$$  \hspace{1cm} (20)

or,

$$n \hbar = 2\pi (\kappa e^2 m^{1/2}) / (-2E_{\text{atom}})^{1/2},$$  \hspace{1cm} (21)

or, solving for $E_{\text{atom}}$,

$$E_{\text{atom}} = -2\pi^2 \kappa^2 e^4 m / n^2 \hbar^2,$$  \hspace{1cm} (22)

Bohr’s formula for the hydrogen energy levels!