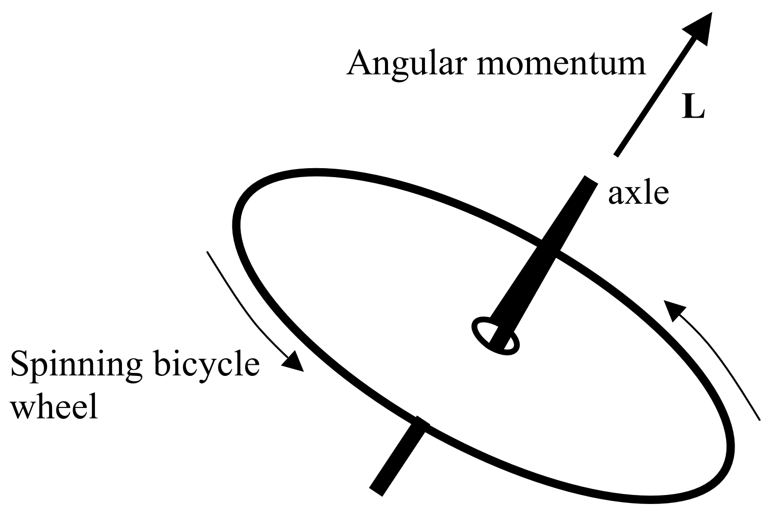
G & EM: The Fundamental Forces of Everyday Life. III

1: Angular momentum, torque and gyroscopes

If you've ever held a toy gyroscope in your hand or a disconnected bicycle wheel by its axle, you know that they both offer a noticeable resistance to twisting *when they are spinning*. The property they acquire when spinning is called **angular momentum**. The forces that have to be applied in order to twist the spinning object, i.e., to change its angular momentum, must be applied in such a way as to generate what is called a **torque**. Like forces and velocities and ordinary momentum, angular momentum and torque are vectors represented by arrows. In the cases of the spinning gyroscope or bicycle wheel, both have an angular momentum *with respect to their centers of mass* that is represented by an arrow that points along their axes of rotation in *the direction of an extended right hand thumb if the curled right hand fingers indicate the direction of rotation* (**Fig. III. 1**).



$$L \propto M R^2 \omega$$

Fig. III. 1: Angular momentum, **L**, of spinning bicycle wheel relative to the wheel's CM and in accordance with the *right hand rule*. The magnitude of **L** is proportional to the wheel's mass, M, the square of its radius, R, and its angular speed, ω.

The crucial feature *necessary, but not sufficient*, for an object to have some angular momentum with respect to some reference point, P, is that parts of the object separated from P by various displacements be moving, relative to. P, with velocities that have components perpendicular to the respective displacements. In the case of the bicycle wheel the parts on the rim are moving, relative to the center, where the CM is located, with velocities perpendicular to the radii that separate the rim parts from the center. Quite generally the angular momentum for any particle moving relative to any fixed point is defined in **Fig. III. 2a**. using the more elaborate form of the right hand rule.

The *total* angular momentum of an object or system, relative to a fixed point, is the vector sum of the corresponding angular momenta of all the particles that comprise the object or system.

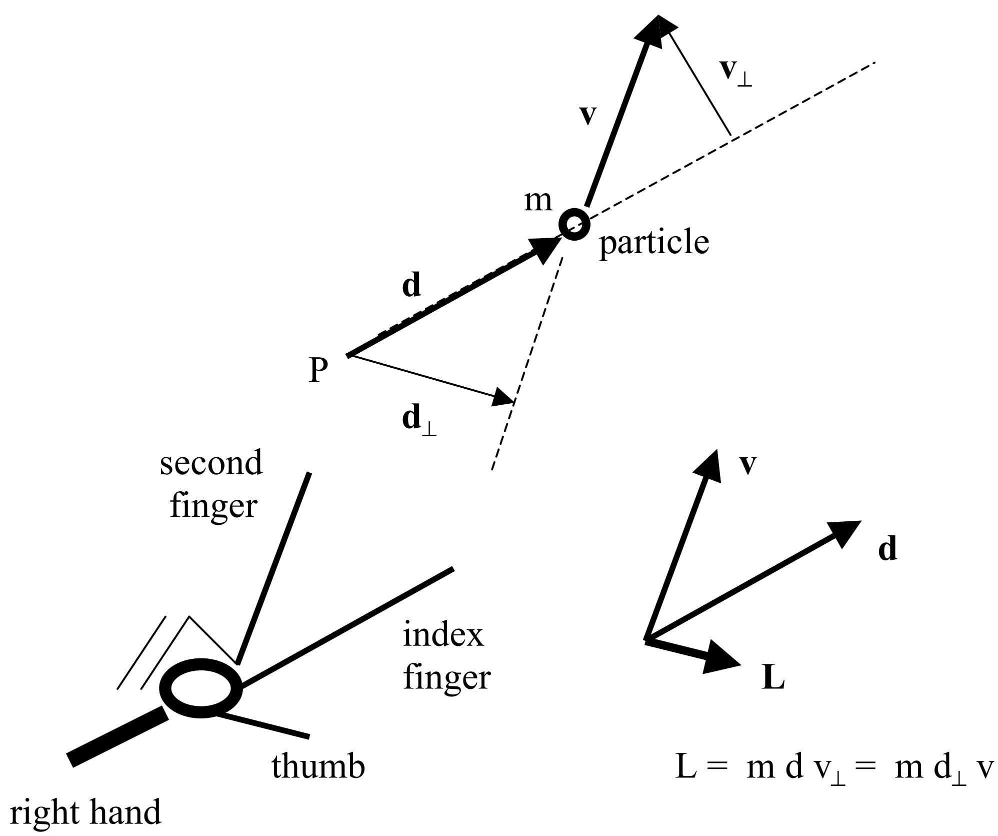


Fig. III. 2a: The notorious right hand rule for the direction of angular momentum of a moving particle relative to a fixed point.

The torque relative to a fixed point produced by a force acting on a particle is very similarly defined (**Fig. III. 2b**).

The total torque acting on an object or system relative to a fixed point, P, is the vector sum of the corresponding torques acting on all the particles of matter comprising the object or system.

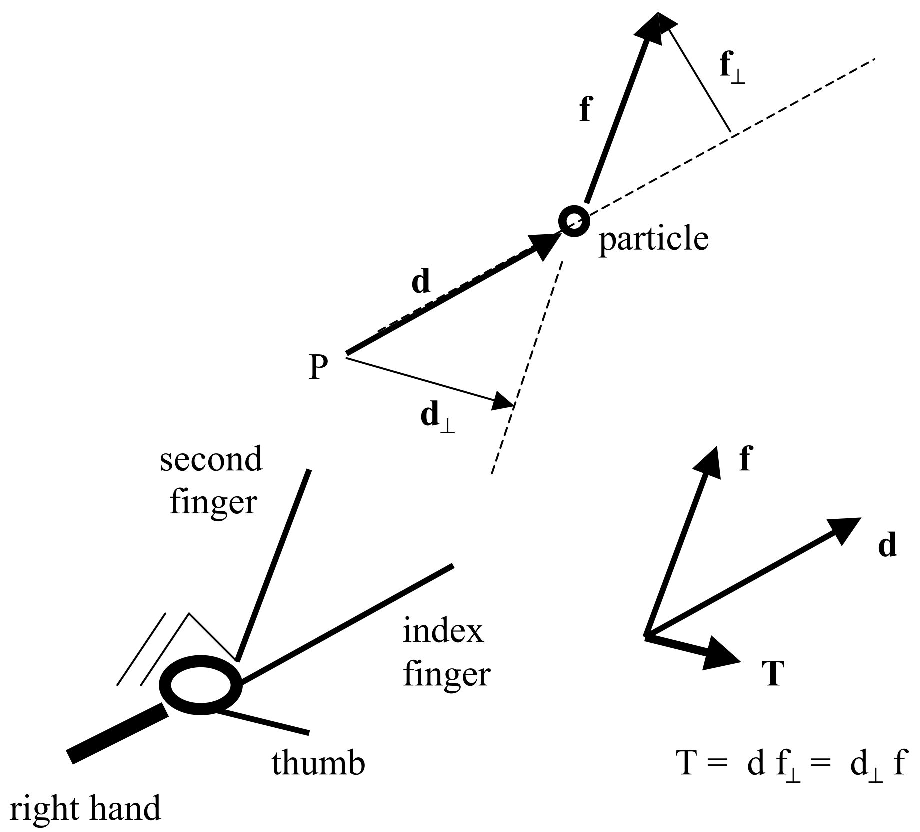


Fig. III. 2b: The notorious right hand rule for the direction of the torque of a force acting on a particle relative to a fixed point.

A general result that follows from these definitions and the basic rules we discussed in **I** is the following:

THE TOTAL TORQUE ON A SYSTEM RELATIVE TO A FIXED POINT, P, IS EQUAL TO THE RATE OF CHANGE WITH TIME OF THE TOTAL ANGULAR MOMENTUM OF THE SYSTEM RELATIVE TO THE SAME POINT, i.e.,

$\mathbf{T}_P =$ rate of change of $\mathbf{L}_P$.

We can now consider the toy gyroscope (**Fig. III. 3**).

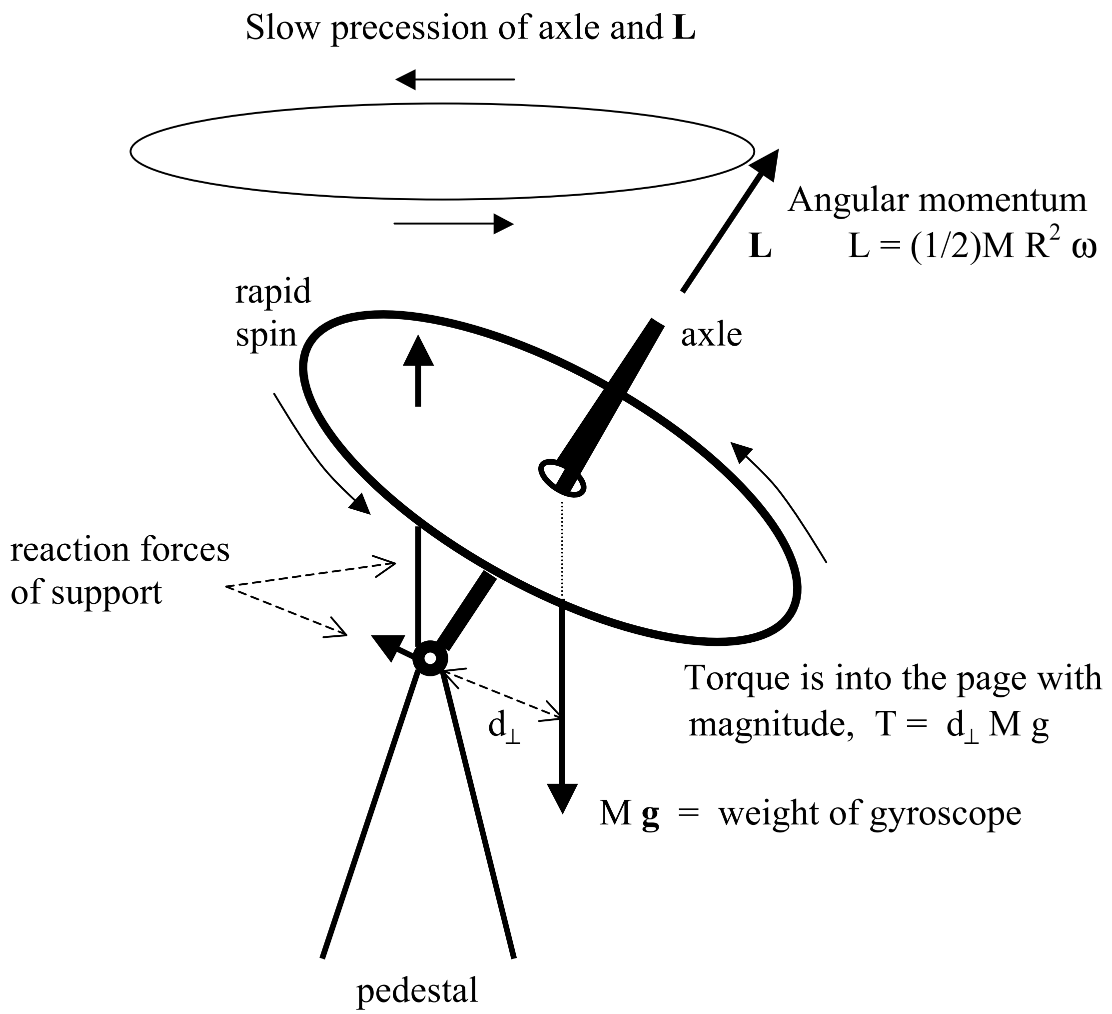


Fig. III. 3: Spinning gyroscope on pedestal induced to precess by its own weight acting as if concentrated at the CM.

Either by virtue of initial placement or inadvertently, a spinning gyroscope placed on a fixed pedestal will likely have a spin axis that is tilted from the vertical. The weight of the gyroscope, acting as if concentrated at the CM, will tend to pull the CM down tilting the spin axis further. But the angular momentum of the rapid spin produces a resistance to changing the spin axis. This causes the base of the gyroscope axle to push down on the pedestal and, therefore, the pedestal to push up on the axle base, balancing the weight and stopping the fall of the CM. But the weight of the CM also generates a torque relative to the top of the pedestal and, in the configuration of **Fig. III. 3**, that torque points *into* the page. This induces a rate of change of the angular momentum that results in the direction of the spinning axle to precess as indicated with the pedestal contact fixed. This, in turn, requires the pedestal contact to generate a small horizontal force, in direction opposite to the axle tilt, which accounts for the small acceleration of the CM in its horizontal circular motion.

Without friction this combined spinning and horizontal precessing motion would continue indefinitely. But friction at the pedestal contact and with the air will gradually slow the spinning motion, diminishing the resistance to further tilting of the axle, increasing the torque and speeding up the precessional rotation, until the gyroscope falls off the pedestal.

The torque depends on the strength of the gravitational pull on the gyroscope, indicated by Mg, where g is the acceleration of falling bodies due to Earth's gravity near the surface. If the gyroscope were on the Moon, its' mass, M, would be the same, but the gravitation induced acceleration of falling bodies, g', would be smaller than on Earth. Consequently the torque would be less and the rate of precession would be slower on the Moon.

2: Variation of Atmospheric Pressure with Altitude

Air is *compressible*. If the temperature is the same everywhere (and not too cold), a fixed mass, m, of air satisfies, to a good approximation, **Boyle's Law**,

$$P\,V = (\text{constant})\; m\;, \qquad (1)$$

where P is the pressure (not too high) and V the volume of the gas. Therefore, on a quiet day (not even a breeze!), with no temperature variation with altitude, the weight of air within a thin *horizontal* layer of given thickness and area (and therefore given volume) is determined by the pressure. But the pressure must drop as we rise so that the weight of the air in the layer can be *supported* by the *pressure difference* from the bottom to the top of the layer over the area of the layer. In the previous course of this series, **The Physical Forces of Everyday Life**, we considered the implications of these ideas for the atmosphere near the surface of the Earth where we could neglect the curvature of the Earth's surface and the decrease in the strength of Earth's gravity with increasing distance from its' center. This lead to the conclusion that the atmospheric pressure would decrease with altitude in accordance with a so-called *exponential curve* which is graphically represented in **Fig. III. 4**.

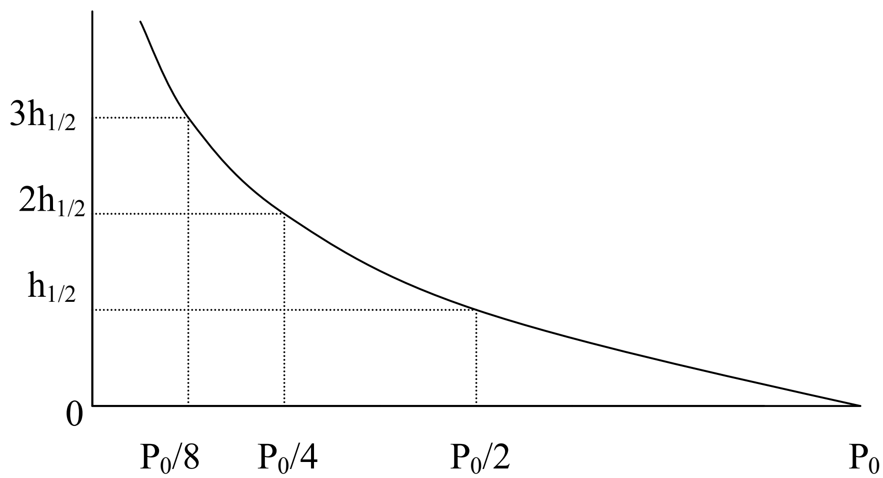


Fig. III. 4: Exponential drop in atmospheric pressure with altitude

In particular we see that if the atmospheric pressure drops from P_0 at ground level to $P_0 / 2$ in the climb to altitude $h_{1/2}$, then at altitude, $2h_{1/2}$, the pressure would be $P_0 / 4 = P_0 / (2^2)$ and at altitude $3h_{1/2}$ the pressure would be $P_0 / 8 = P_0 / (2^3)$. More generally, our considerations lead to the result that the ratio of atmospheric pressure to ground pressure at an altitude of Nh, $P(Nh) / P_0$, was related to the corresponding ratio at altitude, h, $P(h) / P_0$, by,

$$[P(Nh) / P_0] = [P(h) / P_0]^N . \qquad (2)$$

The actual altitude at which the pressure does drop by half is between 5 and 6 km (~ 3 and 4 mi). Our exponential dependence result would then predict a drop in pressure at 55 km altitude to about $(1/1000)P_0$.

$$[P(55 \text{ km}) / P_0] = [P(5.5 \text{ km}) / P_0]^{10} = (1/2)^{10} \sim 10^{-3} = 1/1000$$

But 55 km is getting up there! Can we trust the approximations of (1) neglecting the drop in the strength of Earth's gravity and (2) neglecting the curvature of the Earth's surface that far above the Earth's surface? As it turns out, our complete neglect of the variation in *temperature* with altitude is a much more questionable approximation! So we have to regard our analysis as pertaining to a very idealized atmosphere in which the temperature doesn't vary! But a variable temperature would make the problem much more complex, and seeing how the pressure would vary with a fixed temperature is at least a beginning to understanding. But let's see if we can improve our understanding by taking into account the spherical nature of the Earth and the variation of gravity with distance, albeit still assuming a non-varying temperature.

We consider a small vertical cylinder of base area, A, and extending from a radial distance from the Earth's center of r to a radial distance of $r + \Delta r$, where $r > r_0$, the Earth's radius, and $\Delta r \ll r_0$ and $A \sim (\Delta r)^2$ (**Fig. III. 5**). The atmospheric pressure pushing up on the bottom of the cylinder is

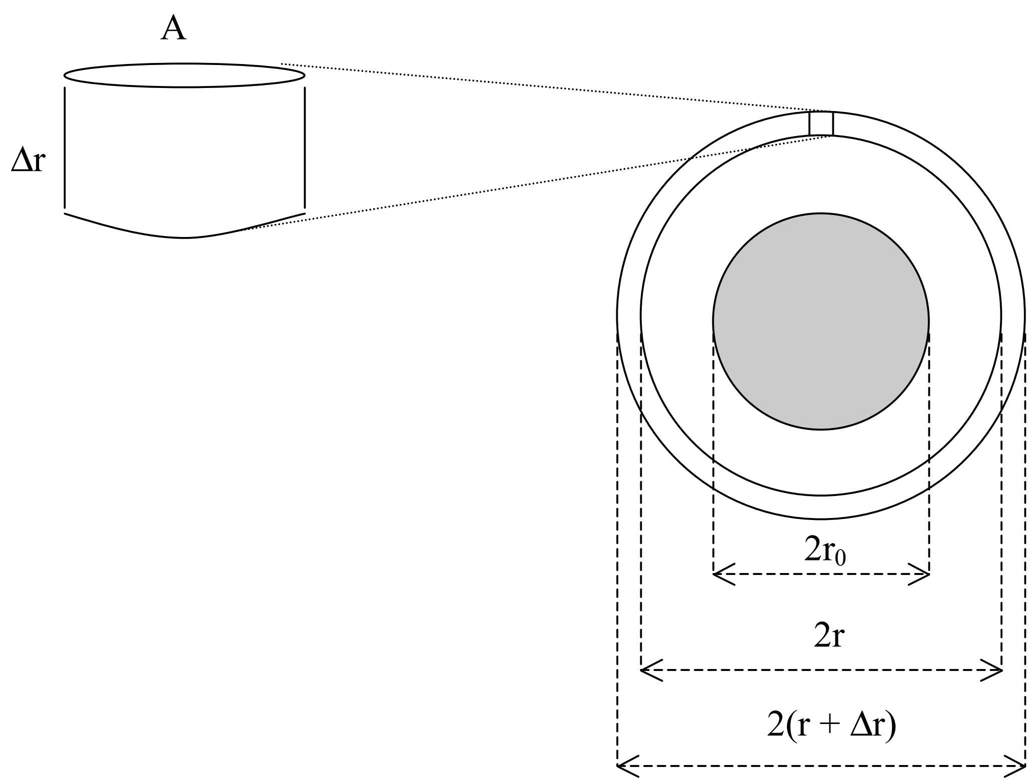


Fig. III. 5: A small vertical cylinder of air of base area, A, and height, Δr, located at a radial distance, r, from the center of the Earth (shaded).

P(r) and the pressure pushing down on the top of the cylinder is $P(r + \Delta r)$. Consequently, the upward force on the air in the cylinder due to the pressure drop with altitude is just, $[P(r) - P(r + \Delta r)]\ A$.

If the air in the cylinder is not accelerating, this *buoyant* force, due to the pressure difference, must be balanced by the downward force of gravity on the air. If the mass of air in the cylinder is m , then that force of gravity is just,

$$G\ M_E\, m / r^2.$$

Equating the two forces, we have,

$$[P(r) - P(r + \Delta r)]\ A = G\ M_E\ m / r^2\ .$$

We now invoke Boyle's Law, (1), which here takes the form,

$$P(r)\ A\ \Delta r = (\text{const.})\ m\ .$$

If we divide this Boyle's Law equation into the balanced forces equation, left side into left side and right side into right side, we get,

$$\{\ [P(r) - P(r + \Delta r)]\ A\ /\ P(r)\ A\ \Delta r\ \} = \{\ G\ M_E\ m\ /\ (\text{const.})\ m\ r^2\ \}\ .$$

Noting that A cancels out on the left side and m cancels out on the right side, we then multiply both sides by Δr to obtain,

$$1 - (P(r + \Delta r) / P(r)) = (G\ M_E / (\text{const.}))\ \Delta r / r^2\ .$$

This result can be rewritten as,

$$P(r + \Delta r) / P(r) = 1 - (G\ M_E / (\text{const.}))\ \Delta r / r^2\ ,$$

and the small ratio, $\Delta r / r^2$, on the right hand side is just the negative change in the reciprocal of the radial distance, i.e.,

$$-\Delta(1 / r)\ =\ -[(1 / (r + \Delta r)) - (1 / r)]\ =\ \Delta r / [r\ (r + \Delta r)]\ \simeq\ \Delta r / r^2\ ,$$

the last approximation getting better and better as Δr gets smaller and smaller.

So if we consider a sequence of increasing radii, r_0, r_1, r_2, r_3, ---, r_n, --- such that the differences,

$$(1 / r_{n+1}) - (1 / r_n) = -\varepsilon ,$$

are all the same and very small, then we know that the ratios of the corresponding pressures at those radii are also all the same, i.e.,

$$P_{n+1} / P_n = 1 - (G M_E / (\text{const.}))\, \varepsilon .$$

But this also means that for any large N,

$$P_N / P_0 = (P_N / P_{N-1})(P_{N-1} / P_{N-2}) \text{ - - - } (P_2 / P_1)(P_1 / P_0) = (P_1 / P_0)^N,$$

an exponential dependence on N, just like the case with constant gravity! Nor, with this result in hand, are we confined to using basic ratios, P_1 / P_0 , corresponding to very small ε . Thus if $N = K L$, we also have,

$$P_{KL} / P_0 = (P_K / P_0)^L \; !$$

The previous equation from constant gravity, (2), and the graph of **Fig. III. 4**, both still hold exactly if we now just change h to mean,

$$h = r_0^2 ((1 / r_0) - (1 / r)) .$$

The *big* difference from the constant gravity situation is that now, for a given h , the multiple, Nh , is *limited* by the requirement that, $r < \infty$, or

$$Nh \leq r_0^2 ((1 / r_0) - (1 / \infty)) = r_0 .$$

Thus if $h = 4.0$ mi, then, since $r_0 \sim 4000$ mi, $N_{max} \sim 1000$. But since $[P(h = 4.0 \text{ mi}) / P_0] \sim 1 / 2$, this yields,

$$[P(N_{max}h \sim 1000 \text{ x } 4.0 \text{ mi}) / P_0] \sim [P(r = \infty) / P_0] \sim (1 / 2)^{1000} \sim 10^{-300}.$$

In other words, *for a fixed temperature atmosphere extending to infinity* (!!), the pressure does *not* drop to zero at infinite distance, but only to the *very* small value near $P_0 / 10^{300}$!

But, of course, the Earth's atmosphere does not extend to infinity and is already being seriously dissipated by ultraviolet radiation and solar X-rays by an altitude of less than 400 mi. At that considerable altitude (the space station orbits at 220 mi) the acceleration due to Earth's gravity is essentially,

$$g_{400\text{ mi}} \sim (32\text{ ft / sec}^2)\ (4000\text{ mi / }4400\text{ mi})^2 \sim (32\text{ ft / sec}^2)(1 - 0.2)$$

$$\sim 26\text{ ft / sec}^2.$$

Noticeably diminished!

So while our analysis has been applied to an idealized (fixed temperature), fairy world, it's still informative of how gravity works by telling us how gravity *would work* in that fairy world. Because the real world is often very difficult to analyze successfully, physicists (and scientists in general) often learn a lot by *first* analyzing idealized fairy worlds!

3: Comparing the masses of the Earth and the Sun

A novel astronomical insight allowed by Newton's Law of Gravitation is the comparison of the mass of the Earth with that of the Sun. Establishing the comparison requires that we talk about the Moon as well as the Earth and the Sun.

To keep things reasonably simple we will treat the Sun's gravitational pull on the Earth *as if* it dominates over all other gravitational pulls on the Earth, such as the Moon's or that of Mars or Venus. This is a good approximation for our purposes, but, if we were willing to do more calculations, we could improve upon it by including the Moon's gravitational pull. Similarly we will treat the Earth's gravitational pull on the Moon *as if* it dominates over all other gravitational pulls on the Moon, such as the Sun's. *This* approximation may seem more doubtful to you, as it should, and we will eventually test it for consistency with our procedure for comparing the masses of the Earth and the Sun.

First notice that if we combine Newton's 2nd Law of motion, our **Rule 3**, with Newton's Law of Gravitation and apply them to the Sun's pull on the Earth we have,

$$M_E\, A_E = F_{ES} = GM_E\, M_S / (d_{ES})^2 ,$$

Where M_E is the Earth's mass, A_E is the Earth's acceleration in it's orbit around the Sun, M_S is the Sun's mass and d_{ES} is the distance from the Earth to the Sun.

The first thing to notice is that the Earth's mass, M_E , being a common factor on both sides, cancels out of the equation. This leaves,

$$A_E = GM_S / (d_{ES})^2 . \qquad (3)$$

In other words, the *acceleration* produced by the Sun's gravity is independent of the mass of the Earth. The Sun would accelerate *anything* that was in the position of the Earth by the same amount. More massive objects are harder to accelerate, but gravity always compensates by pulling harder by just the right amount to produce the same acceleration. The acceleration depends only on *where* the object that is being accelerated is. This is not something peculiar to the Sun. *All* objects, according to Newton's Law of Gravity, produce a gravitational attraction that, because of its' proportionality to the mass of the object being attracted, generates an acceleration that is independent of the mass of the accelerated object and dependent only on its position.

We're really just repeating here some of the ideas that we used last time to understand how Newton got his Law of Gravity in the first place.

So OK, just what is the acceleration of the Earth around the Sun? Since we're only after a rough estimate we'll treat the Earth's orbit as if it were a perfect circle (**Fig. III. 6**). Kepler's discovery that it was an ellipse rather than a circle was very important in the development of Newton's theory. But now that we have the theory we can ignore the ellipse for a circle if we don't want too much accuracy. This will be what scientists call a "back of the envelope" calculation. Anyhow, the Earth's orbit is *very* close to being a circle!

Last time we saw that the acceleration for uniform motion around a circle

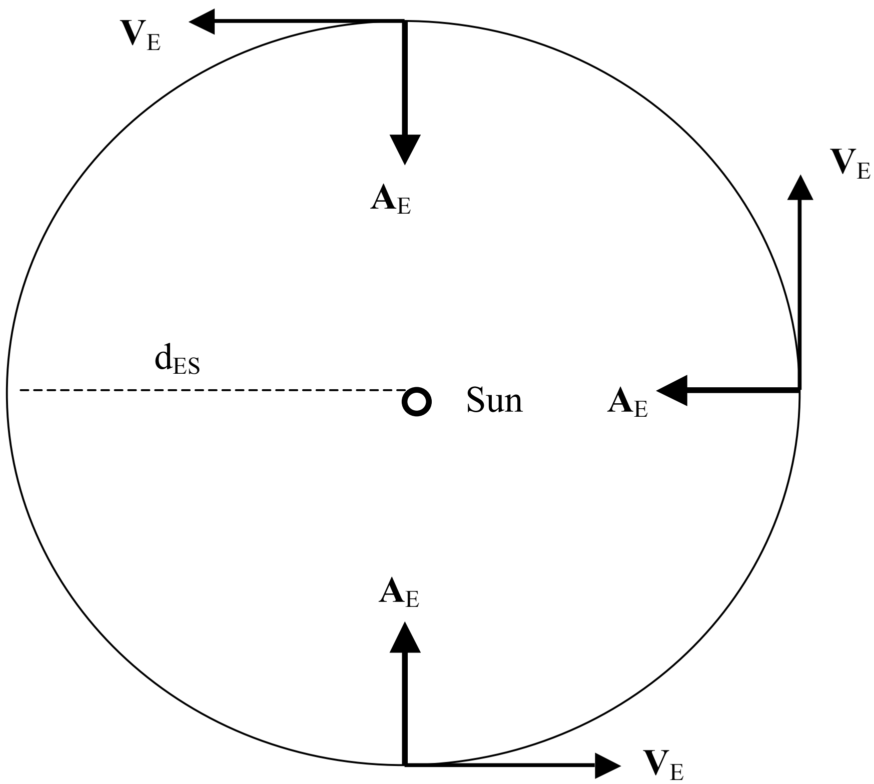


Fig. III. 6: Earth orbiting the Sun with acceleration magnitude, $A_E = GM_S/(d_{ES})^2$ and velocity magnitude, $V_E = (A_E d_{ES})^{1/2} = (GM_S/d_{ES})^{1/2}$.

was given by,

$$A = V^2 / R = (2\pi R / T)^2 / R = 4\pi^2 R / T^2 .$$

In our case, $R = d_{ES} \simeq 93 \times 10^6$ mi and $T = 1$ yr $\simeq 365$ d $= 8760$ hr . So

$$V = V_E \simeq 2\pi \times 93 \times 10^6 \text{ mi} / 8760 \text{ hr} \simeq 6.67 \times 10^4 \text{ mi / hr} .$$

The Earth is clipping right along! But the radius of the orbit is BIG! So the acceleration turns out to be,

$$A_E = (6.67 \times 10^4 \text{ mi/hr})^2 / 93 \times 10^6 \text{ mi} \sim 48 \text{ mi / hr}^2 \sim 0.02 \text{ ft / sec}^2.$$

This is miniscule compared to the acceleration of falling objects near the Earth's surface, $g = 32\ ft / sec^2$.

If we now plug this value for A_E back into equation (3) and multiply both sides by $(d_{ES})^2$ we get,

$$G M_S \sim 48\ mi / hr^2 \times (93 \times 10^6\ mi)^2 \simeq 4.2 \times 10^{17}\ mi^3 / hr^2. \qquad (4)$$

This, of course, is an odd-ball quantity, and since we don't know what G is we don't know what M_S is. Nevertheless, as you will see, this result will be useful.

Let's now turn to the Moon (**Fig. III. 7**), again adopting a circular orbit. Assuming the Earth's gravity dominates the Moon, we must have,

$$M_M A_M = F_{ME} = G M_M M_E / (d_{ME})^2 ,$$

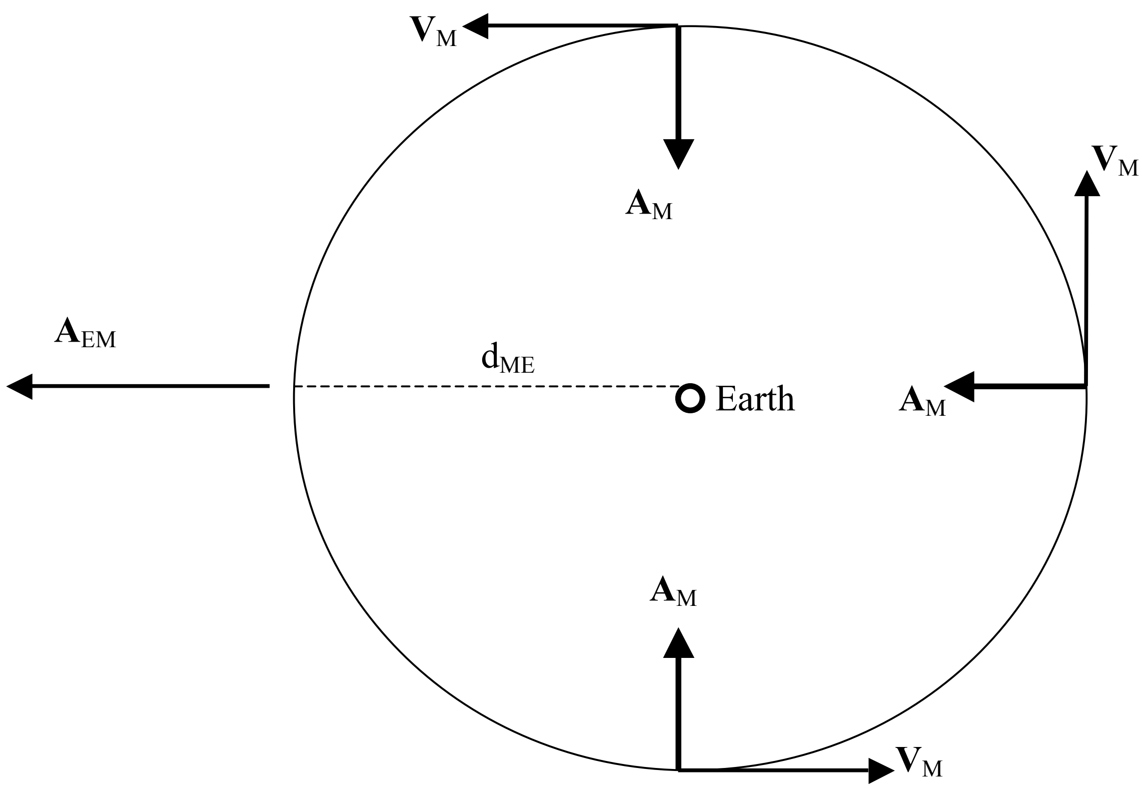


Fig. III. 7: Moon orbiting the Earth with $A_M = GM_E/(d_{ME})^2$ and $V_M = (A_M d_{ME})^{1/2} = (GM_E/d_{ME})^{1/2}$.The entire EM system accelerates towards the Sun with average acceleration, $A_{EM} \sim 48\ mi / hr^2$.

where M_M and A_M and d_{ME} is the Moon's mass, acceleration and distance from Earth, respectively.

This time the *Moon's* mass cancels out on both sides and we have,

$$A_M = G\, M_E / (d_{EM})^2 \quad . \tag{5}$$

The value of d_{EM} is $\simeq 2.4 \times 10^5$ mi and the Moon's speed in its orbit is roughly

$$V_M = 2\pi\, d_{EM} / T_M \simeq 6.28 \times 2.4 \times 10^5 \text{ mi} / 28 \times 24 \text{ hr}$$

$$= (6.28 \times 10^4 / 28) \text{ mi / hr} \simeq 2.2 \times 10^3 \text{ mi / hr.}$$

This, then, gives an acceleration of

$$A_M = V_M^{\,2} / d_{EM} \simeq (2.2 \times 10^3 \text{ mi / hr})^2 / (2.4 \times 10^5 \text{ mi})$$

$$\simeq 20 \text{ mi / hr}^2.$$

Oh oh! Our assumption that the Earth dominated the gravitational influence on the Moon was **wrong!** Since the Moon is less than 1/3 % as far from the Earth as it is from the Sun, the Sun will accelerate the Moon roughly the same as it does the Earth. But, we've seen that that's 48 mi / hr ^{2}. More than twice the Earth's acceleration of the Moon!

Fortunately we don't have to scrap our calculations. While the Sun's accelerating effect is strong, it doesn't *vary* much from the Earth to the Moon. The variation from when the Moon is on the far side of the Earth from the Sun to when it's on the near side of the Earth is just ~ 0.5 mi / hr ^{2}. So the Sun just pulls the whole Earth-Moon system along without changing the Moon's acceleration *relative to the Earth* by much. We're still OK.

So now, if we substitute the value for A_M and the value of d_{EM} into (5) and then multiply both sides of (5) by $(d_{EM})^2$ we get,

$$G\, M_E \sim (20 \text{ mi / hr}^2) \times (2.4 \times 10^5 \text{ mi})^2 \sim 1.2 \times 10^{12} \text{ mi}^3 / \text{hr}^2 \,, \tag{6}$$

another one of those odd-ball quantities! But if we now divide (6) into (4), left side into left side and right side into right side, the unknown quantity, G, cancels out and we get,

$$M_S / M_E \sim (4.2 / 1.2) \times 10^{5} \sim 3.5 \times 10^{5} .$$

The Sun seems to be about 350,000 times more massive than the Earth! This estimate turns out to be about 6% too high. It should be 330,000.

Now if the Moon had a satellite and we knew its orbital period and distance from the Moon, we could do the same kind of calculation to find GM_M. Then we could divide out G and get M_E / M_M . Of course, we've put satellites in orbit around the Moon temporarily and the data from them would enable us to do the calculation. And astronauts have measured the acceleration of falling objects dropped on the Moon and that data would suffice. But we could never have put satellites and astronauts there in the first place without knowing M_M / M_E beforehand. So how could we determine M_M / M_E prior to artificial satellites? First by comparing the tidal effects of the Sun and Moon (see **4**). Second, by taking into account the fact that *both the Moon and the Earth orbit their common center of mass* (**Fig. III. 8**) so that it is the *sum* of both their orbital radii that equals the Earth-Moon distance, 240,000 mi. And by observing planetary parallax due to Earth's motion. *It turns out that*,

$$M_E / M_M \simeq 81$$

Taking into consideration the Moon's 1100 mi radius, this would give an acceleration of falling objects near the surface of the Moon of about 5.0 ft/sec ^{2}. It would take more than two seconds for an object to fall 10 ft!

4: The Tides

We all know that the Moon produces the tides by its gravitational attraction. Water being fluid, that would make intuitive good sense if it weren't for the fact that the tides cycle roughly every 12 hrs rather than every 24 hrs. This means that, roughly speaking, we have high tide underneath the Moon on the near side of the Earth AND on the far side of the Earth. How does the Moon's gravitational attraction *elevate* the water level on the *far* side of the Earth?! And given what we just saw about the gravitational effectiveness of the Sun compared to the Earth on the Moon, what effect does the Sun have on Earth tides compared to the Moon?

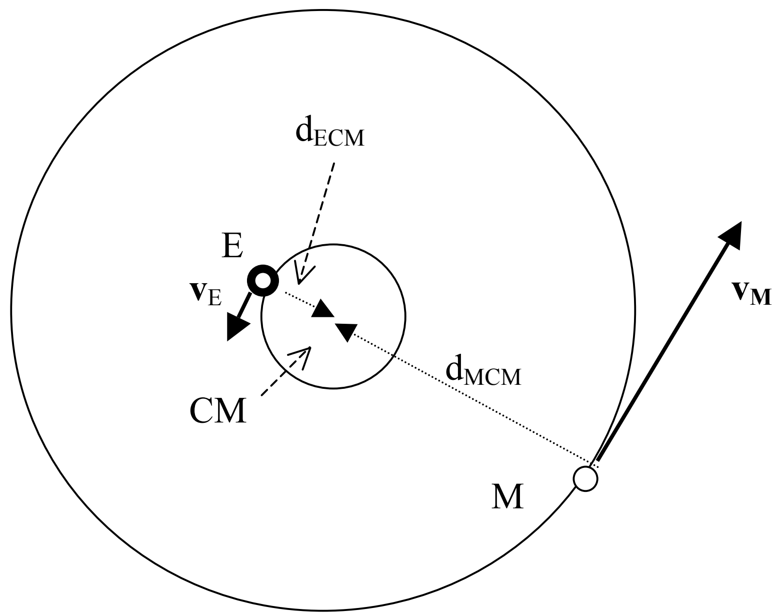


Fig. III. 8: Earth and Moon orbiting common center of mass (CM) with $d_{ME} = d_{MCM} + d_{ECM}$ and $(v_E / d_{ECM}) = (v_M / d_{MCM})$. **Not to scale!**

The short answer is that the tides are not really due to the mere fact that the Moon exerts a gravitational attraction on the Earth but, rather, to the fact that the Moon contributes a *different* gravitational force per unit mass (**pull**) to different parts of the Earth (**Fig. III. 9a**). The part of the Earth's surface facing the Moon is pulled a little bit more than the center of the Earth and the center is pulled a little bit more than the part of the surface facing away from the Moon. The parts of the Earth's surface that face perpendicular to the Earth-Moon direction are pulled in a slightly different direction than the previous parts. These differences produce the tides (**Fig. III. 9b**).

The Earth is not perfectly rigid and, consequently, besides being accelerated as a whole by the Moon, it is stretched a little bit in the direction of the Moon and squeezed a little bit in directions perpendicular to the Earth-Moon direction. Still, the solid portions of the Earth are much more rigid than the water on its surface. So, to a first approximation, the Earth as a whole accelerates towards the Moon as its center does, while the water on the surface is pulled *by* the Moon *relative* to the solid Earth by the tiny differences in gravitational force per unit mass (pull) (**Fig. III. 9b**).

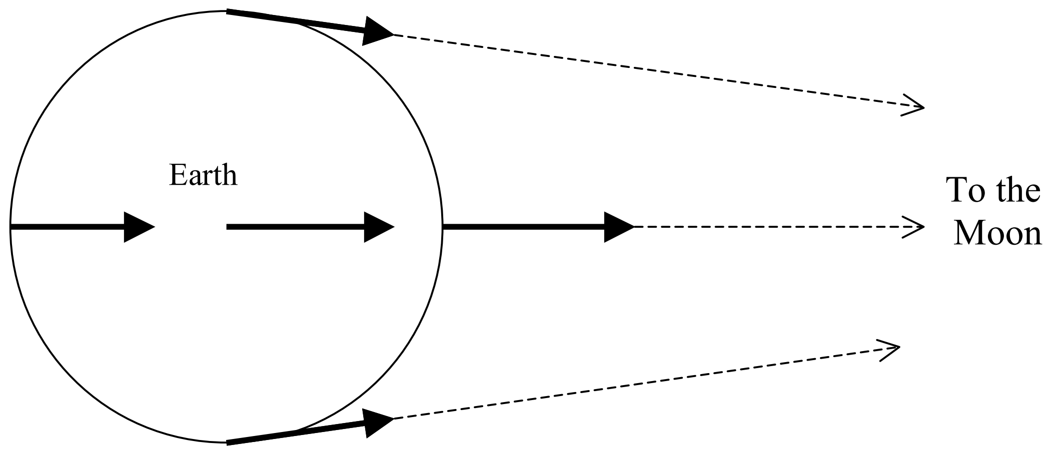


Fig. III. 9a: Gravitational force per unit mass (**pull**) contributed by the Moon (convergence and differences exaggerated).

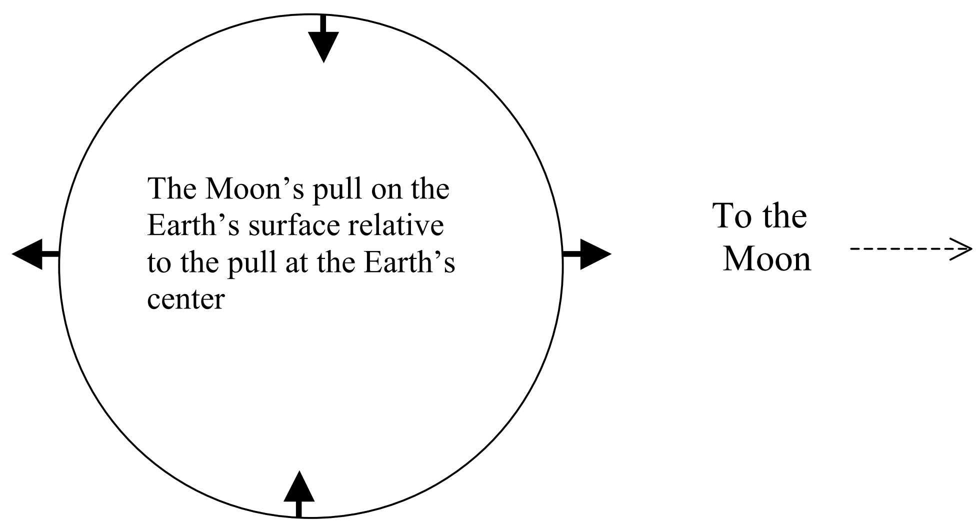


Fig. III. 9b: The residual lunar pull at the Earth's surface relative to the accelerating pull at the Earth's center. The residual pull generates the tides.

Just how tiny are these differences? Well, remember that the total gravitational forces exerted by the Earth and Moon on each other are equal and opposite. That means that the products of their masses times their accelerations relative to one another are (assuming the Sun's contribution to their accelerations is the same for both of them) equal, or

$$M_E\, A_{EM} = M_M\, A_{ME}\ .$$

We found earlier that $A_{ME} \sim 20$ mi / hr ^{2}. So, dividing through by the mass of the Earth,

$$A_{EM} \sim (M_M / M_E)(20 \text{ mi/hr}^2).$$

Substituting in the value, $M_M / M_E \sim\ 1 / 81$, we obtain,.

$$A_{EM} \sim 0.25 \text{ mi/hr}^2\ \sim 1.0 \text{ x } 10^{-4} \text{ ft/sec}^2\ .$$

That's the Moon's acceleration of the Earth's center and the solid Earth as a whole. The tiny deviation from that which tends to *raise* the tide on the near and far side of the Earth from the Moon *turns out to be*,

$$A_{EM} \text{ x } 2 \text{ x } (4000 / 240{,}000) = A_{EM} \text{ x } (1 / 30)$$

The factor of 4000 / 240,000 comes from comparing the Moon's gravitational effect at locations 4000 mi closer to or farther from the Moon than the Earth's center, which is 240,000 mi from the Moon. The factor of 2 comes from the inverse *square* dependence of that gravitational effect on distance.

The tiny deviation which tends to *lower* the tide on the rim of the Earth between the near and far side is

$$A_{EM} \text{ x } (4000 / 240{,}000) = A_{EM} \text{ x } (1 / 60)$$

Here the distance from the Moon along the rim is essentially the same as from the Earth's center so there is no factor of 2 and the factor of 4000 / 240,000 comes from the small *angle* between the Moon's gravitational pull on the Earth's rim and that on the Earth's center.

The water is not actually accelerating relative to the Earth at these rates. These are the differential pulls with which the Moon is acting on the water relative to the Earth. Pulls being forces per unit mass, they have the physical dimension of accelerations. The Moon is "trying" to accelerate the surface water by this amount relative to the Earth. It doesn't succeed because the Earth is holding the water to it with an overwhelming pull of 32 lb / sl.

The net effect of the Earth's pull and the Moon's differential pull on the water is that the water on the near and far sides of the Earth is very slightly lighter than the normal 64 lbs / ft ^{3} and the water on the rim between the near and far sides is very slightly heavier. Consequently, the water pressure increase with depth on the rim is very slightly higher than normal and on the near and far sides of the Earth it's very slightly lower than normal. This produces a pressure gradient from the rim towards the near and far sides and that pressure gradient pushes water from the rim towards the near and far sides.

The detailed change in water level at any given location is heavily dependent on the local topography and weather. In particular, as a consequence of inertia, friction and solid obstacles, the tides are delayed by roughly six hours from an instantaneous response to the gravitational agencies producing them. High tide *follows* the Moon around the rotating Earth but it does not sit directly under the Moon. Accounting for this delay in detail is very complicated. But one interesting regularity of tides is easily accounted for by our explanation. If the Moon did not orbit the Earth as the Earth rotated under it, then one days' local high tides would occur essentially at the same time as the preceding and succeeding days'. But in the course of a 24 hr rotation the Moon has gone through 1/28 th of its orbit. It takes the Earth about 24 hr / 28 $\simeq$ 52 min to catch up, and, sure enough, each days' tides are delayed about 50 min from the previous days' tides at any locale.

The Sun contributes to the tides in the same *way* as the Moon, but to a smaller degree and in a varying direction relative to the Moon. While the central acceleration of the Earth due to the Sun is about 192 times larger than the central acceleration of the Earth due to the Moon, i.e., 48 vs 0.25, the differential pulls due to the Sun require the factor

$$(4000 / 93,000,000) = (1 / 23,250)$$

rather than the Moon's factor of (4000 / 240,000) = (1 / 60). So the Sun's differential pulls are (192 / 23,250) x 60 = (11,520 / 23,250) ~ 0.5 as strong as the Moon's. When the Sun and Moon are on the same or opposite sides of the Earth they produce the largest (*Spring*) tides and when they are at right angles relative to the Earth they produce the smallest (*Neap*) tides (**Fig. III. 10**).

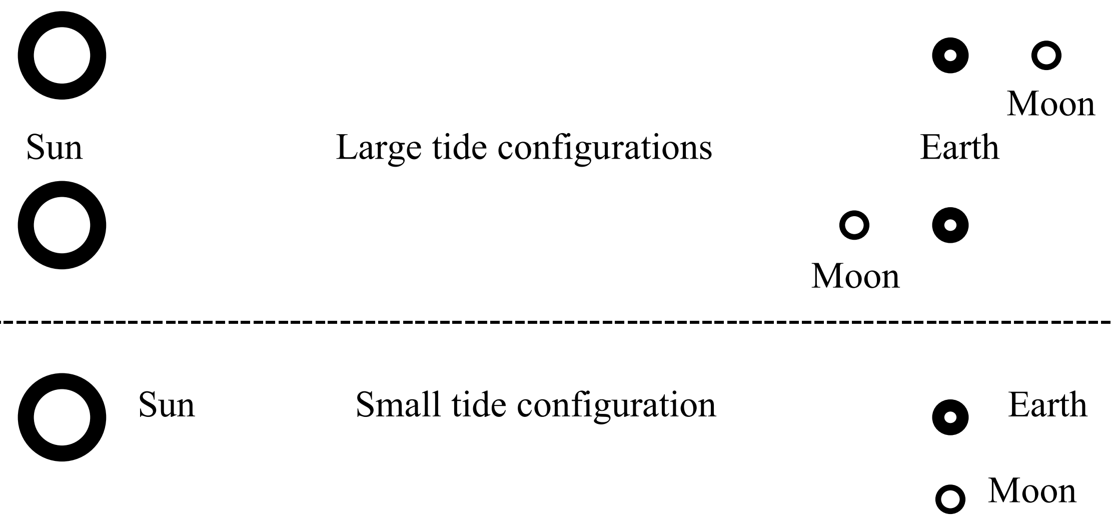


Fig. III. 10: Sun-Earth-Moon configurations for generating large (*Spring*) and small (*Neap*) tides.

5: Determining the Gravitational Constant, G:

To determine the individual masses of the Earth, Sun, Moon and other astronomical objects we need to know the value of Newton's universal gravitational constant, G. To obtain that requires measuring the gravitational force of attraction between objects the masses of which are already known. That means terrestrial objects. But if the entire Earth exerts a gravitational attraction of only 32 lbs on a mass of 1 slug, or an attraction of 9.8 N on a mass of 1 kg, then the gravitational attractive force between two "ordinary" terrestrial objects is going to be very tiny indeed – and hard to measure! The first efforts tried to detect the attraction a mountain exerted upon a suspended lead ball. Some pretty accurate values were obtained this way.

The first reasonably precise measurement was done by Henry Cavendish in 1798 (**Fig. III. 11**). He placed small lead balls at the ends of a balanced bar suspended from a thin fiber and then put large lead balls equidistant from the small ones and on opposite sides of them. The attraction turned the bar which twisted the fiber until the resisting torque of the twisted fiber stopped the bar. He then moved the large balls to the other sides of the small balls and the turning-twisting went in the opposite direction. The small amount of turning was amplified by attaching a small mirror to the fiber and having it reflect a light beam onto a far wall. The whole apparatus was so delicate that it had to be protected from air currents by being enclosed in a large glass cylinder which, in turn, was in a closed room. Moving the large balls and watching the reflected light beam was done from outside the room.

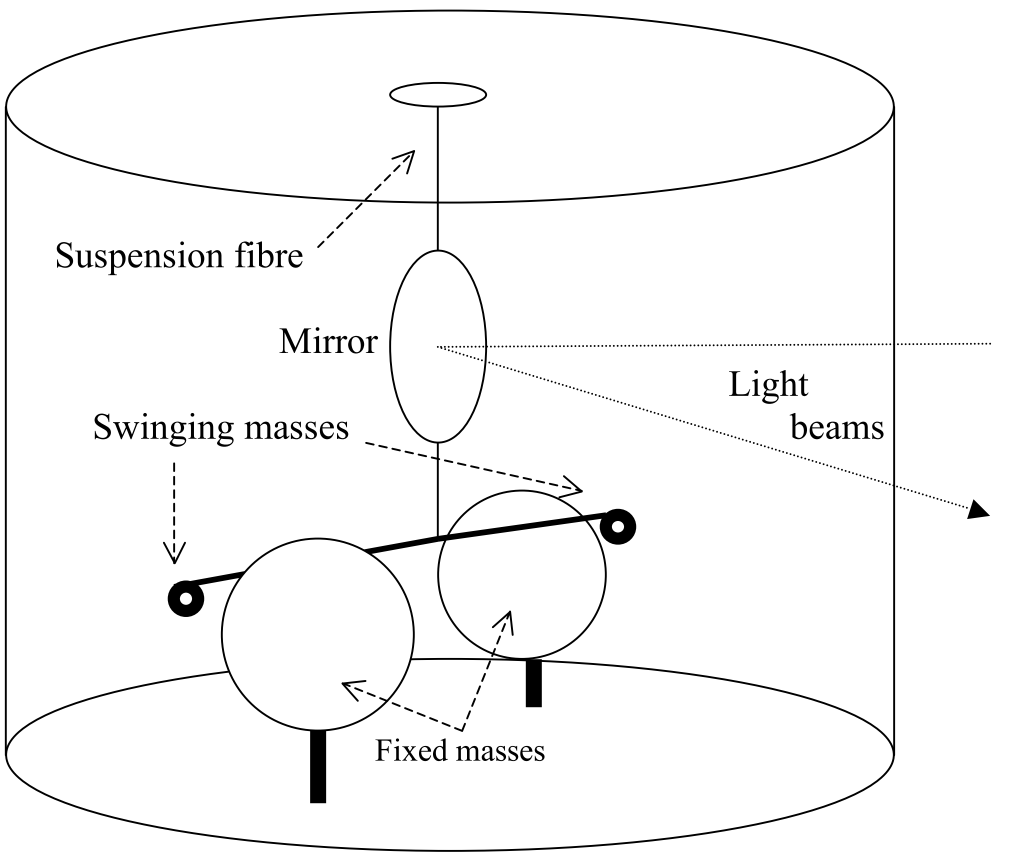


Fig. III. 11: Cavendish's arrangement for measuring Newton's gravitational constant, G.

The value obtained for G was,

$$G = 6.75 \times 10^{-11}\ \mathrm{m}^3/\mathrm{kg\ sec}^2 \sim 3.5 \times 10^{-8}\ \mathrm{ft}^3/\mathrm{sl\ sec}^2$$

$$\sim 3.0 \times 10^{-12}\ \mathrm{mi}^3/\mathrm{sl\ hr}^2 .$$

Combining this with equation (6) we find

$$M_E \sim 4.0 \text{ x } 10^{23} \text{ sl} \simeq 6.0 \text{ x } 10^{24} \text{ kg} ,$$

i.e., six trillion trillion kilograms.