G & EM: The Fundamental Forces of Everyday Life. II.

1: Galileo's Intellectual Environment

Galileo (1564 - 1642) arrives on the scene in the late renaissance, at a time when many students and scholars were examining new and heady ideas that went against the grain of long established authority. Others, fearful that the current orgy of free thinking was leading to the collapse of all reliable truths and all morality, mounted the battlements for the defense of that authority. Sounds a bit like the late 60's of the 20th century. But, in fact, the changes taking place were far deeper and long lasting than those of the late 60's.

The novelties of Galileo's mind set were: (1) to question written and scholarly authority concerning nature by *directly examining nature*, (2) to focus on *specific types of phenomena* rather than grand general schemes and aim at achieving *detailed* understanding of those phenomena, (3) to focus on the *quantitative aspects* of phenomena, rather than the merely qualitative, and aim at establishing *mathematical relationships* that hold for all instances of the phenomena in question. Before Galileo, aside from the practical purposes of farming and architecture, mathematical analysis had only been applied to the heavens, or celestial phenomena, where precision was thought to reign. The majority of the intelligentsia of the time still embraced the dogmatized Aristotelian cum Platonic view that terrestrial phenomena were the realm of corruption and instability, susceptible, at best, only to qualitative characterization along the lines of Aristotle's classification of biological species. In contrast, Galileo believed that the deeper aspects of all phenomena were the quantitative aspects and that "the book of nature is written in the language of mathematics".

Of the many things Galileo studied, we want to consider his study of falling objects and projectile motion, what we recognize as the common effects of gravity near the Earth's surface.

The official view on falling objects, was attributed to Aristotle and endlessly debated as to exactly what "the master of those who know" meant and whether or not his arguments could be improved but essentially never contemplating that Aristotle might have been simply wrong. Aristotle's view was that *heavy bodies fall faster than light bodies and at a rate proportional to their weight*. Now this view does have a certain immediate intuitive appeal. But it doesn't take *us* much reflection to see that it is flagrantly wrong! Consider dropping an open bucket of dry sand. That could be pretty heavy and, if so, should drop fast. But why wouldn't the individual sand grains, which are very light, fall slowly and, relative to the heavier and faster falling bucket, rise out of the bucket as they're left behind? We all know that doesn't happen. Since the individual grains of dry sand stay with the bucket during its fall we get the suggestion that the rate of fall doesn't depend on the weight of the falling object. That everything falls in the same way! Using a variation of the preceding argument, Galileo is then galvanized into finding out *just what that way is* in which everything falls.

But pause for a moment. Aristotle was an incredibly knowledgeable guy and, for the most part, a rather level headed and clear thinker. Famously far more down to Earth than his mentor Plato. Could he really have made such a gross blunder on something as common as falling bodies?! Some people think not and I share their view. Our view is that Aristotle, like ancient Greek thinkers in general, was not interested in the early stages of falling motion that attracted Galileo and had earlier attracted Leonardo and other renaissance minds. The Greeks were interested in the *stable* features of nature, not the changes that sometimes connected them. For the Greeks, perfection and precision lay in permanence, not in transience. Aristotle probably knew that objects dropped in resisting media such as water and oil and honey reached a fixed speed of fall quickly. That fixed speed – what we call terminal velocity – *is* proportional to the weight of the falling object if shape and size stays fixed. For objects falling in air it takes a while to reach terminal velocity and most objects don't have that far to fall. So they never reach it. But Aristotle was probably convinced that if they did they would then follow the same rule that seemed to hold in dense media. Unfortunately he didn't qualify sufficiently his statements that have come down to us to make it unequivocal that this is what he meant.

2: Galileo on Falling Objects

Back to Galileo. Does everything really fall in the same way? What about a feather? That *can* take a long time to fall but it's clear that falling feathers have an important interaction with the air and with breezes when they fall. So Galileo's quest becomes modified to finding the way everything falls when air resistance can be ignored. The important role of *approximation* in science is already present.

To guide his investigation Galileo makes some conjectures concerning the way everything falls. An early conjecture is that the speed of fall is proportional to the *distance* the object has fallen through. For some time he follows this guess. But finally he realizes that with this rule an object starting from rest would require an *infinite* amount of time to fall any distance at all (**Fig. II. 1**)! Observation and experiment seem not needed here. Logic rules this idea out.

If $v = cd$, starting from the left end, then the time for *each* portion satisfies, $T \geq (2c)^{-1}$.

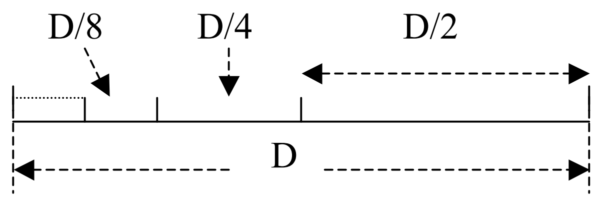

Fig. II. 1: Galileo’s argument for ruling out $v = cd$ for falling bodies.

The argument goes like this: consider any distance of fall, D, from rest. Divide that distance up in your imagination into the last half, D/2, the second quarter, D/4, the second eighth, D/8, the second sixteenth, D/16, etc. During each of these portions of the fall the speed of fall will be not greater than at the end of the portion. But since the rule is $v = cd$ where c is a constant and d the covered distance, we have $v \leq c\,D$ for the D/2 portion, $v \leq c\,D/2$ for the D/4 portion, $v \leq c\,D/4$ for D/8, etc. But since the time required to traverse these portions must satisfy, $T \geq \Delta d/v_{max}$, we must have $T \geq (D/2)/cD = 1/2c$ for D/2, $T \geq (D/4)/(cD/2) = 1/2c$ for D/4, $T \geq (D/8)/(cD/4) = 1/2c$ for D/8, etc. In other words the time required to traverse the distance D is the sum of at least the time, 1/2c, for *each* of the portions, D/2, D/4, D/8, - - - etc. an infinite total time in all!

Galileo then gets the idea that maybe the speed is proportional to the *time* that has passed during the fall. He makes a graph of speed of fall versus time of fall and discovers (as we learned from the appendix of part **I**) that the area under the speed curve represents the distance that has been covered in the fall. Since the speed is proportional to the time, i.e. the speed doubles when the time doubles, triples when the time triples, etc., the 'curve' is a straight line and the area under the line is the area of a triangle. But the area of a triangle is 1/2 the product of the base and the height. In our case that's

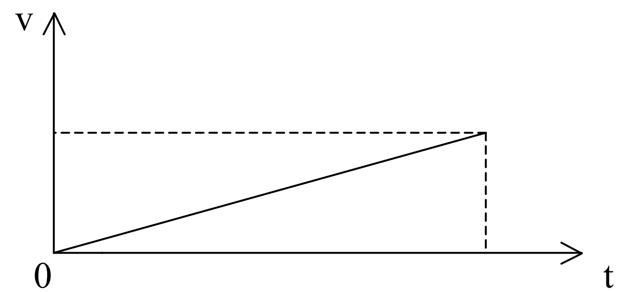

Fig. II. 2: Galileo's guess that bodies fall with a speed proportional to the time, $v = a\, t$. The constant, a, is called the acceleration of the motion.

proportional to $(1/2)\, v\, t$. So the distance covered by an object with speed proportional to the time, in the time, t, starting from rest, is $(1/2)\, v\, t$ where v is the speed at the time, t. But that speed is, itsef, proportional to the time, or, $v = a\, t$, where a is a constant. In fact, a is the **acceleration** and its constancy is reflected in the slope of the speed curve being constant since it's a straight line. So, the equations for objects moving from rest with constant acceleration, a, are,

$$v = a\, t \qquad \text{and} \qquad d = (1/2)\, v\, t = (1/2)\, a\, t^2 \ .$$

What can be done to see if falling objects actually move this way? Well, first of all, if you drop a stone from 16 ft above ground it will take very close to a second to reach the ground. So <u>if</u> these equations do describe falling objects then the value of a must satisfy,

$$16 \text{ ft} = (1/2)\, a\, (1 \text{ sec})^2,$$

or

$$(32 \text{ ft/sec})/\text{sec} = a.$$

But this then yields the prediction that a stone that falls for 2 seconds will fall through 64 ft. That can be checked pretty easily.

More generally, it follows from the equations for motion with constant acceleration that if an object moves a distance D in the time T, starting from rest, then it will move 4D in the time 2T, 9D in the time 3T, 16D in the time 4T, and so on. Consequently, in each *consecutive* time interval of duration, T, the object will move through first D, then 3D, then 5D, then 7D and so on. If the acceleration is small enough these are easy predictions to test. The acceleration of a falling object is not so small but Galileo tries the predictions out on balls rolling on inclined planes of different inclinations and finds they all satisfy these predictions. In the limit of a vertical inclined plane he has the case of a falling object.

From now on we call the acceleration induced by Earth's gravity near the surface, g. In other words, $\mathbf{g := 32\ ft/sec^2 = 9.8\ m/sec^2}$.

3: Galileo on the Motion of Projectiles

Galileo gets a start on the motion of projectiles by considering falling objects dropped on the deck of a ship moving alongside a pier. Everyone 'knows' that if you drop an object on a steadily moving ship, the object just falls straight down no differently than if you were on land. But from the perspective of an onlooker on the pier the object begins its 'fall' with the non-zero horizontal motion of the boat. So relative to the pier the object is a projectile starting with the horizontal speed of the boat. And since it falls straight down relative to the steadily moving boat, it must not lose its horizontal motion relative to the pier while it's falling (**Fig. II. 3**).

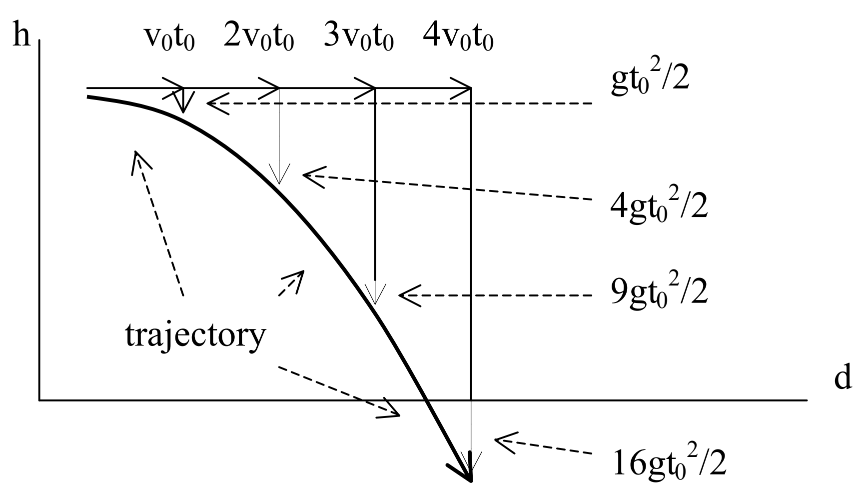

Fig. II. 3: Projectile motion with constant horizontal speed, v_0.

Suppose the boat is a “Tall ship” moving to the right with speed v_0. Upon dropping the object (from sufficiently high up on the rigging) the object will, in the short time, t_0, move to the right with the boat a distance, v_0t_0, while falling a distance, $gt_0^2/2$. By the time $2t_0$ has passed, the object has moved to the right a distance, $2v_0t_0$, and has fallen a distance, $4gt_0^2/2$. By $3t_0$, $3v_0t_0$, and $9gt_0^2/2$, and so on. The resulting trajectory, relative to the shore, is a called a **parabola**.

To test the idea of the horizontal speed not changing, Galileo used his rolling balls on inclined paths and had them roll back up variably inclined paths after rolling down a fixed one. The results were consistent with the notion that *the more one could reduce external resistance to the rolling motion the more closely would the balls roll back up to just the same height from which they began rolling down*. And their rolling speed would change only as their height changed.

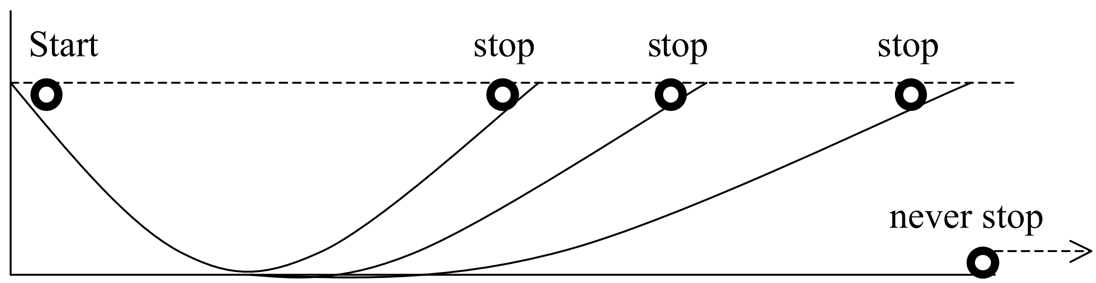

Fig. II. 4: Galileo's illustration of horizontal motion without resistance.

If this is so then a ball that rolls down the curve and then continues horizontally will never stop or slow down (in the absence of all external resistance). This is the phenomena of horizontal **inertia**. It is an instance of the very non-Aristotelian notion of a terrestrial motion that, once begun, *needs no cause to maintain it*. Causes, of whatever nature, will only *change* it. It is an instance of approximate horizontal inertia that the falling object keeps moving with the ship (neglecting air resistance) while it falls.

So much for projectiles that start out moving horizontally. What about projectiles with initial velocities inclined to the horizontal?

Galileo displays an ambivalence here which is due to his recognizing the phenomena of inertial motion only in the horizontal direction. If a projectile starts moving with an initial velocity inclined to the horizontal, Galileo considers the decomposition of that initial velocity into a horizontal and a vertical component. The horizontal component is then conserved as an instance of inertial motion while the vertical component is continuously changed by the acceleration of gravity (**Fig. II. 5**).

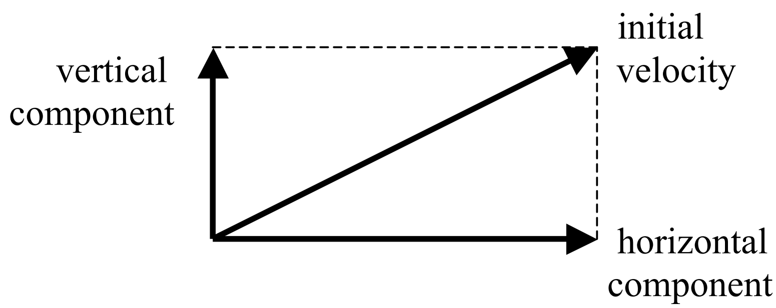

Fig. II. 5: Galileo's decomposition of initial velocity.

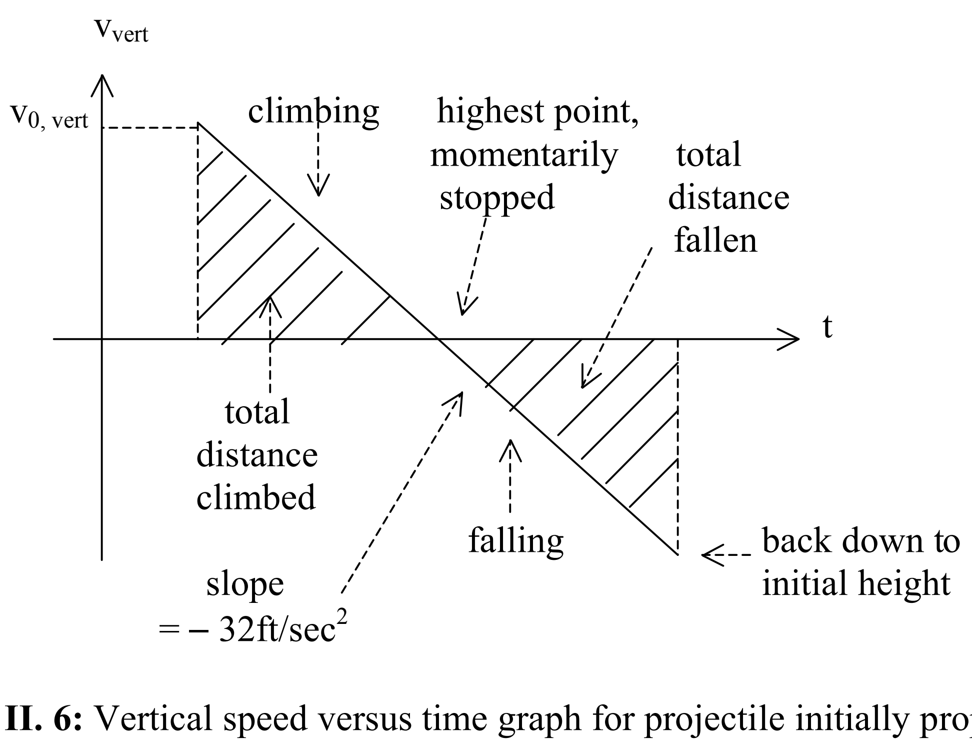

Fig. II. 6: Vertical speed versus time graph for projectile initially propelled upwards with speed, $v_{0, vert}$.

The vertical aspect of the projectile motion is then handled by a slight extension of the graphical method for falling objects. Regarding upward displacements and velocities and accelerations as positive while downward ones are negative, the *velocity versus time graph* for a projectile with positive initial vertical velocity, $v_{0,\, vert}$, looks like **Fig. II. 6**.

By combining the inertial motion of the horizontal velocity with the accelerated vertical motion we would obtain the characteristic parabolic trajectory of the projectile. But an easier way to see that is to realize that the constant downward acceleration of gravity simply adds an accumulating downwards displacement to the displacement the initial *inclined* velocity would produce if *it* were inertial, i.e., constant, in its entirety (**Fig. II. 7**).

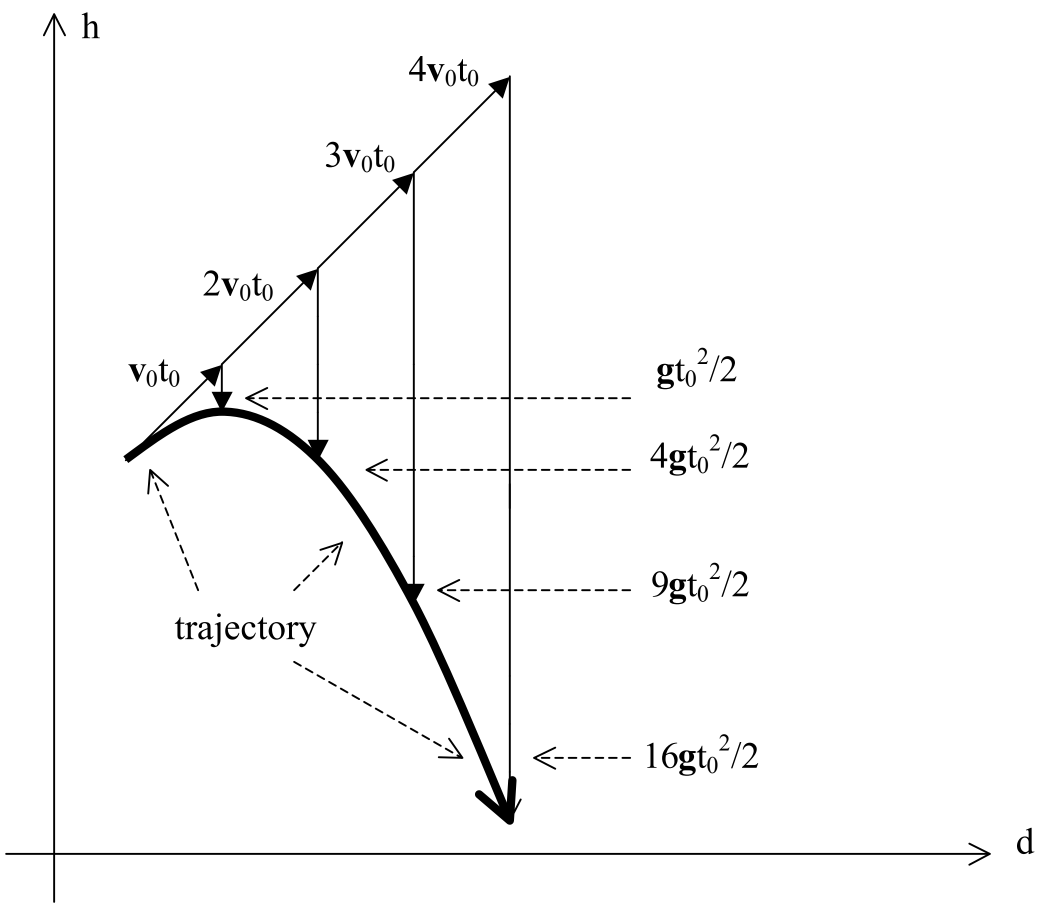

Fig. II. 7: Projectile path with inclined, inertial, initial velocity.

Galileo is aware of this and uses this kind of graphical construction. But he never seems to recognize that this means that in the absence of the acceleration of gravity the whole initial velocity would be inertial. This is probably because, unlike the external resistances to horizontal motion, which, to varying degrees can be eliminated, the vertical acceleration of gravity can not be removed. It is inherent.

4: Kepler and Gravity

Johannas Kepler (1571-1630), in possession of Tycho Brahe's (1546-1601) astronomical data of unprecedented precision and driven by the conviction that the motion of the planets must be governed by divine rules of surpassing beauty and simplicity, discovered the three laws of planetary motion which bear his name and ushered in the modern age of astronomy. Unlike Galileo, who was anomalously modern in his thinking, Kepler, although intensely mathematical, was somewhat given to what we would call mystical interpretations, a mentality that was common among intellectuals of his period.

As with everything else in science, Kepler's three laws are approximations. He didn't think they were only approximations. I dare say that had he thought that all he could achieve was improved approximations, he never would have lifted a finger on behalf of astronomy. So important are our fantasies in leading us to our real achievements!

Kepler's three laws of planetary motion are (**Fig. II. 8**):

(1) Each planet moves in an elliptical orbit around the Sun with the Sun located at a focus of the ellipse.

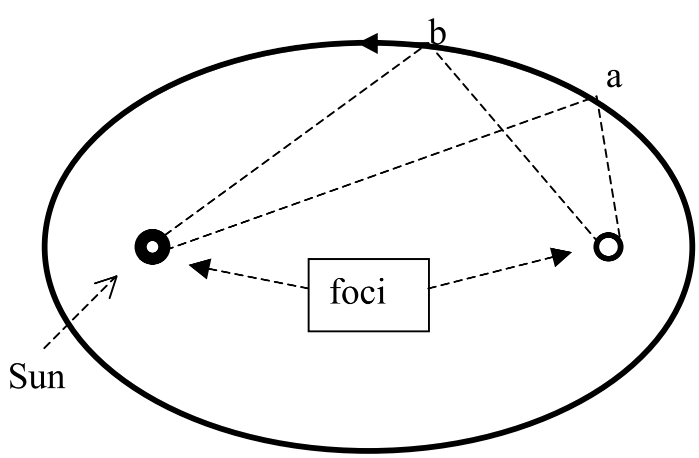

Fig. II. 8: Elliptical planetary orbit around the Sun at one focus.

The two foci of an ellipse are points inside the ellipse such that *any* point on the ellipse has the same *sum* of distances from the foci.

With this law the long (2000 yr) tortuous history, from the ancient Greeks through Ptolemy and Copernicus, of trying to account for planetary motion via the use of only "perfect" circles, and circles moving on circles, is broken with the insight that a different kind of closed curve, an ellipse, provides an acceptable account and a "perfect" fit to the data – as far as Kepler can tell.

> *(2) Each planet moves on its elliptical orbit so that the line from the Sun to the planet sweeps out area at a constant rate.*

So planets move faster when they're closer to the Sun than when they're farther from the Sun. We now know that this constant rate of sweeping out area is equivalent to saying that the angular momentum of the planet relative to the Sun is constant.

> *(3) For all of the planets the ratio of the cube of a planets mean distance to the Sun divided by the square of that planets orbital period is the same, i.e.,*
>
> $R^3 / T^2 = C$ *, a constant, the same for all planets.*

It is this third law that will give Newton a big assist towards *formulating* his theory of gravity. Kepler's first and second laws will then be *tests* for Newton's theory. Tests the theory will pass with flying colors!

But before we can see how Kepler can assist Newton, we have to understand acceleration more generally than just the constant vertical acceleration that Galileo studied. In particular, what about the acceleration of an object moving in a closed curve, like an ellipse, or even a circle?!

5: The General Nature of Acceleration*

The general connection of acceleration to velocity is *exactly* the same as the general connection of velocity to relative position. All of these quantities are *directed magnitudes*, i.e. they point in some direction and they have a magnitude. In other words, they, like forces, are **vectors**. So relative position (the displacement from a reference point to a position), velocity and acceleration are all vectors. And to repeat; the relationship of the vector, velocity, to the vector, relative position, is *exactly* the same as the relationship of the vector, acceleration, to the vector, velocity. What is that common relation?

Suppose that $\mathbf{x}_1$ and $\mathbf{x}_2$ are the position vector of some object relative to a reference point, P, at the instants of time, t_1 and t_2, respectively, where $t_1 \leq t_2$.

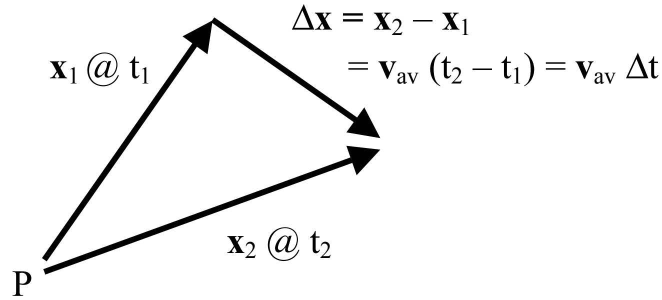

Fig. II. 9: Relation of average velocity to change of relative position

Then the **average velocity**, $\mathbf{v}_{av}$, of the object between t_1 and t_2 is just the difference of the relative positions divided by the time interval, i.e. (**Fig. II. 9**),

$$\mathbf{v}_{av} := (\mathbf{x}_2 - \mathbf{x}_1) / (t_2 - t_1) = \Delta\mathbf{x} / \Delta t .$$

What we call the **instantaneous velocity**, or simply *the* velocity at a moment of time is just the *limit* of the average velocity for smaller and smaller time intervals including the moment in question.

Similarly, suppose that $\mathbf{v}_1$ and $\mathbf{v}_2$ are the velocities of some object at the times, t_1 and t_2, respectively, where $t_1 \leq t_2$. Then the **average acceleration**, $\mathbf{a}_{av}$, of the object between t_1 and t_2 is just the difference of the velocities divided by the time interval, i.e. (**Fig. II. 10**),

$$\mathbf{a}_{av} := (\mathbf{v}_2 - \mathbf{v}_1) / (t_2 - t_1) = \Delta\mathbf{v} / \Delta t .$$

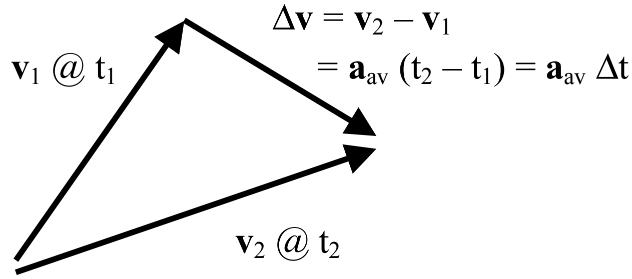

Fig. II. 10: Relation of average acceleration to change of velocity

What we call **instantaneous acceleration**, or simply *the* acceleration at a moment of time is just the *limit* of the average acceleration for smaller and smaller time intervals including the moment in question.

For the simple case of projectiles near the surface of the Earth, the average acceleration is constant, always pointing to the ground and with magnitude, g. So we denote the near-Earth gravitational acceleration vector by **g**.

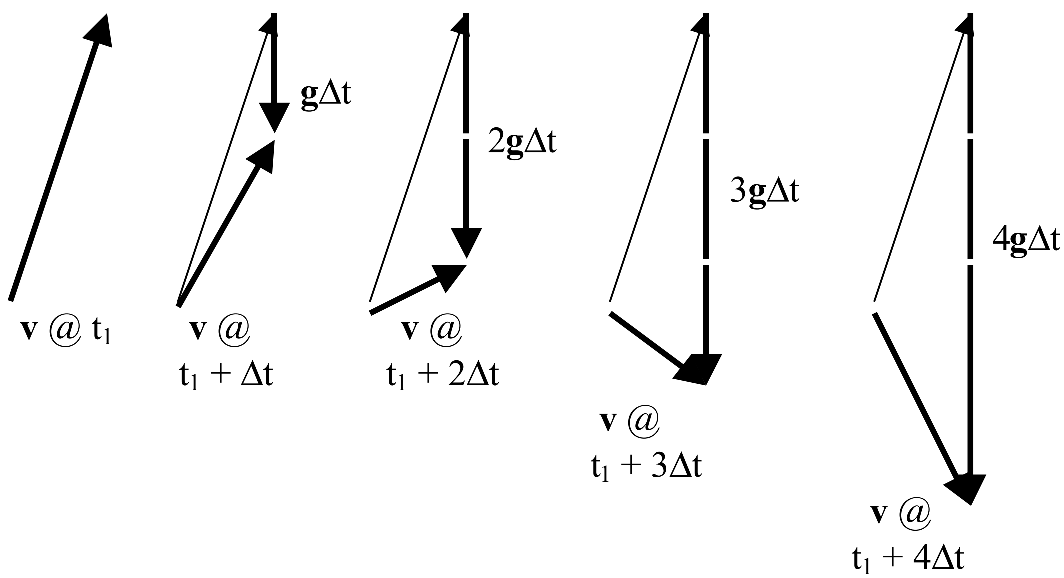

Fig. II. 11: Changes in velocity due to constant acceleration acting through consecutive Δt time intervals.

The resulting temporal change of the velocity due to constant acceleration (**Fig. II. 11**) gives rise to the parabolic trajectories of projectiles that was discussed previously.

A second important example of acceleration is provided by uniform motion around the rim of a circle. The circular trajectory need not be enforced by some circular object to which the moving object is attached. Some Earth satellites move in very nearly circular orbits, but there is no circular track on

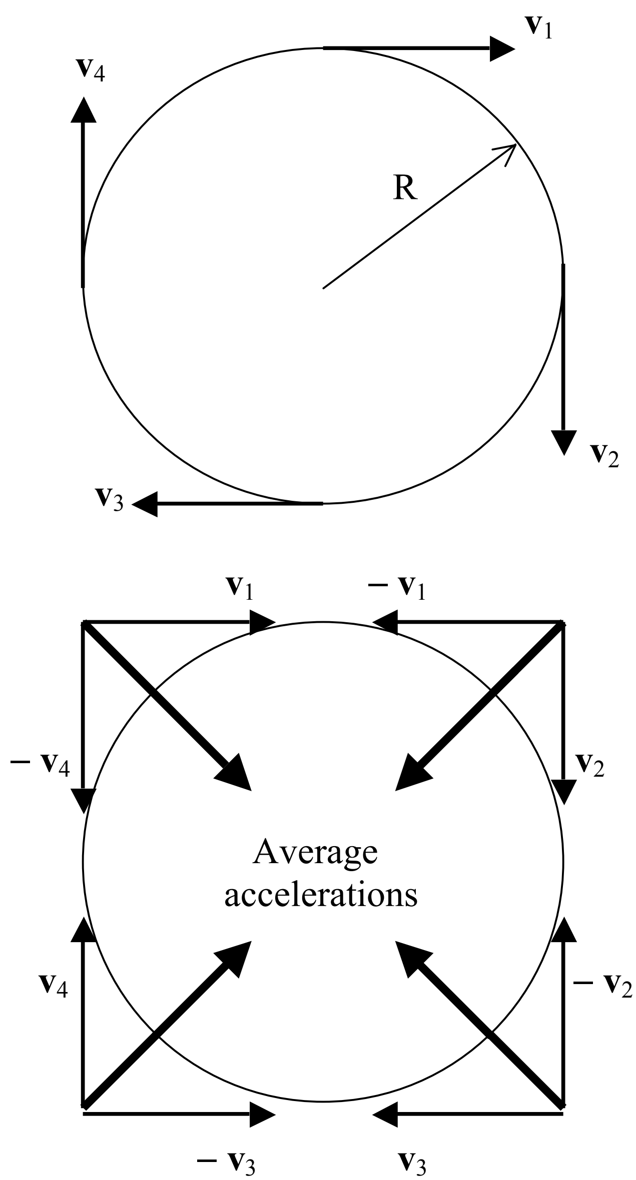

Fig. II. 12: Some instantaneous velocities and average accelerations in uniform circular motion.

which they are running. They are simply held in their orbits by Earth's gravity. We usually call this *uniform motion in a circle* (**Fig. II. 12**).

In such motion the time required for the object to complete one entire transit of the circle is called the **period** of the motion and is denoted by T. If the radius of the circular trajectory is R, then the **circumference** is 2πR and the constant speed of the motion is,

$$v = 2\pi R / T .$$

But while the speed is constant, the **velocity** is not. For the velocity includes the direction of the motion and as the object moves around the circle its' direction of motion is changing continuously. Since the velocity is changing, the motion is accelerated. What is the direction and magnitude of the acceleration?

We first note that the velocity is everywhere **tangent** to the circular trajectory. Next, let us look at the **differences of these consecutive velocities,** i.e. $\mathbf{v}_2 - \mathbf{v}_1$, $\mathbf{v}_3 - \mathbf{v}_2$, $\mathbf{v}_4 - \mathbf{v}_3$, and $\mathbf{v}_1 - \mathbf{v}_4$, to determine the directions of the average accelerations during each quarter of an orbit.

Pretty clearly, all the average accelerations point toward the center of the circular trajectory. To keep an object moving uniformly in a circle it appears that you have to accelerate the object towards the center of the circle. To assess the magnitude of the acceleration we consider the average acceleration for a very small time interval, Δt.

In the **Fig. II. 13**, below, $\mathbf{r}_1$ and $\mathbf{r}_2$ are displacements from the center of the circular trajectory to two positions of the moving object on the circle at Similarly, the instantaneous velocities when the object is at $\mathbf{r}_1$ and $\mathbf{r}_2$ are $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively, and these velocities, being perpendicular to their corresponding **r**'s (because they are tangent to the circular trajectory) have the same angle between them as the **r**'s do. Since the difference of the velocities is $\mathbf{a}_{av}\,\Delta t$, where $\mathbf{a}_{av}$ is the average acceleration during Δt, and since both $\mathbf{v}_1$ and $\mathbf{v}_2$ have the same magnitude, v, namely $v = 2\pi R / T$, it follows that the triangle composed of the **r**'s and $\mathbf{v}_{av}\,\Delta t$ has the same shape as

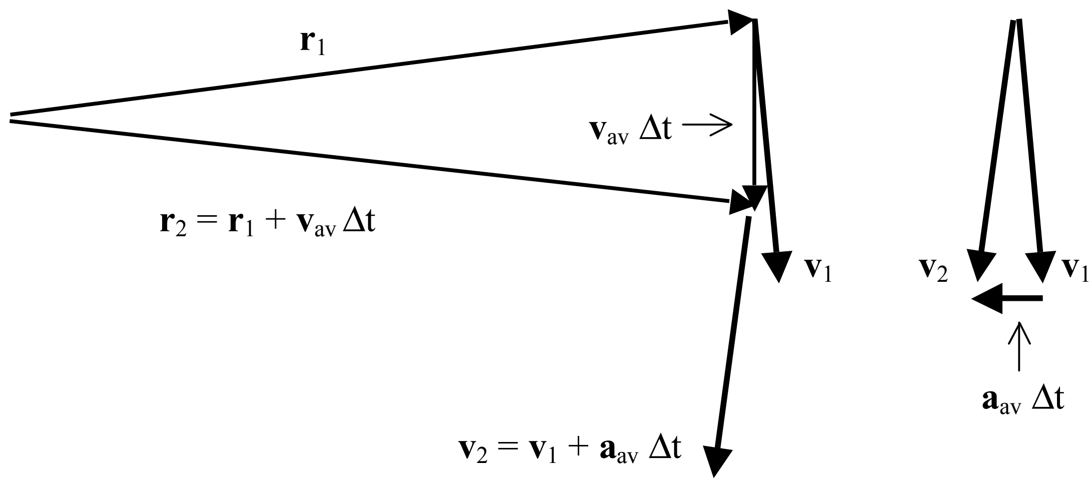

Fig. II. 13: Positions relative to the center and corresponding instantaneous velocities separated by a time interval, Δt, and average velocity and acceleration for that time interval in the case of uniform circular motion.

the triangle composed of the **v**'s and $\mathbf{a}_{av}\ \Delta t$. Consequently,

$$a_{av}\ \Delta t / v = v_{av}\ \Delta t / R\ ,$$

or, canceling the common factor, Δt, and multiplying both sides by v, we get,

$$a_{av} = v\ v_{av} / R\ .$$

But as Δt gets smaller and smaller, v_{av} (magnitude of the average velocity) becomes the same as v (magnitude of the instantaneous velocity) and a_{av} (magnitude of the average acceleration) becomes the magnitude of the instantaneous acceleration, a. Therefore we get,

$$a = v^2 / R = 4\pi^2 R / T^2\ .$$

as the relation between the magnitudes of the instantaneous acceleration and the instantaneous velocity or speed of the object. So for uniform motion in a circle the acceleration always points toward the center of the circle and has a constant magnitude given by the previous equation.

6: Back to Kepler and Newton's Theory

Let's now go back and recall Kepler's third law that the ratio of the cube of the average radius of a planetary orbit to the square of the orbital period for the planet was the same for all the planets, i.e.,

$$R^3 / T^2 = C, \text{ a constant, the same for all the planets}$$

Using the approximations that the planetary orbits are circles and a planet's speed is constant in its orbit, it follows that the acceleration is

$$a = 4\pi^2 R / T^2 = 4\pi^2 (R^3 / T^2)(1 / R^2) = 4\pi^2 C / R^2 .$$

The planetary accelerations decrease as the inverse square of their orbital radii!

And what about the Moon and falling objects near the Earth? The acceleration of the Moon in its orbit is roughly,

$$a_{Moon} = 4\pi^2 (240{,}000 \text{ mi.}) / (28 \text{ days})^2 \sim 21 \text{ mi} / \text{hr}^2 \sim 86 \times 10^{-4} \text{ ft} / \text{sec}^2$$

$$\sim (1 / 60)^2\, 32 \text{ ft} / \text{sec}^2 \sim (\text{radius of Earth} / \text{distance to Moon})^2\; a_{falling}.$$

The inverse dependence of gravitational acceleration on distance squared seems to be holding. Finally there is Galileo's report that the moons of Jupiter he discovered telescopically satisfy Kepler's third law in their Jovian orbits.

So Newton is in a good position to assert that the gravitational force from a particle and from the Sun and from the Earth and from Jupiter varies with distance by decreasing as the inverse square of the distance,

$$F_{grav} \sim 1 / (\text{distance})^2 \quad .$$

One might wonder why Newton and others would have been motivated to work through the steps we've just followed to obtain the dependence of the planetary accelerations on their orbital distances from the Sun. Why, for instance, would they expect there to be any definite dependence of acceleration on distance at all? Certainly Galileo never investigated such a possibility. But Galileo, so far as we know, never contemplated the accelerating gravity of the Earth to extend to other astronomical bodies. His confinement to the motion of projectiles *near* the Earth's surface guaranteed that the distance dependence Earth's gravity does have is sufficiently small, in the cases Galileo considered, to be undetectable by him. This was historically fortunate since analyzing the problem of motion under the influence of a spatially varying acceleration would have been a much harder problem for Galileo to solve!

But the later understanding of the accelerated nature of uniform motion in a circle, due to Huyghens, combined with Kepler's laws of planetary motion in ellipses makes it clear to Newton that Kepler's planets are undergoing *variable* acceleration rather than the *constant* acceleration studied by Galileo. The acceleration varies from planet to planet and, for any given planet, the acceleration depends on where the planet is in it's orbit.

Beyond this there had arisen, ever since Copernicus and greatly enhanced by Kepler, the notion that, somehow, the Sun must be the organizing center of the motions in the solar system. The question of how the Sun could extend an influence through the vastness of (empty?) space to all the planets was an intensely debated issue in Newton's time. Kepler had written that the rotating Sun swept the planets along in their orbits with a force tangential to their orbits. But the later, improved understanding of acceleration led Newton and others to suspect that the planets were accelerating *towards the Sun* at all times. The dominant guess as to how the Sun could cause such an acceleration was based on the analogy with how the brilliance of an unfocused source of light diminishes with distance. As the distance increases the light is spread over an increasingly larger spherical surface, the area of which is proportional to the square of the radius of the surface. If the totality of the light is, in some sense, conserved, then the intensity of the light at any point of the surface must decrease as the inverse square of the radius, i.e., like

$$1 / (\text{distance})^2 \; !$$

This was the basis for expecting the acceleration to depend on distance.

Now, what about the way in which gravity depends on mass? Galileo showed that, near the Earth's surface, all falling objects, regardless of their mass (and neglecting air resistance), fall with the *same* acceleration. Since Newton's second law of force (our Third Rule (?)) asserts the resultant force on an object to equal the product of the mass and the acceleration of the center of mass, it must be that *Earth's gravitational force is proportional to the mass of the object it acts upon* !

Furthermore, Kepler's laws make no reference to the masses of the planets. They only refer to how the motion of a planet depends on its position. The simplest assumption is, again, that the Sun's gravity is proportional to the mass of any object it acts upon so that the acceleration the Sun's gravity will produce depends only on where the object is relative to the Sun.

And apparently this assumption also works for the Earth's gravity on the Moon and for Jupiter's gravity on her satellites. So now Newton has,

$$F_{grav} \sim \text{(accelerated mass)} / \text{(distance)}^2 \quad .$$

But now consider Newton's third law of forces (our First Rule). The gravitational force of the Earth on a falling object is reciprocated by an equal and opposite gravitational force of the object on the Earth! Presumably *that* gravitational force must be proportional to the mass of the Earth! The gravitational forces of the Sun on the planets are each reciprocated by equal and opposite gravitational forces of the separate planets on the Sun. Presumably *those* gravitational forces are proportional to the mass of the Sun. The gravitational reciprocation of the Moon back on the Earth is, presumably, proportional to the mass of the Earth. And that of the Jovian satellites on Jupiter proportional to the mass of Jupiter.

It appears, then, that of any two mutually gravitating objects, they will each exert an equal and opposite force on the other that will be proportional to each of their masses and inversely proportional to the square of the distance between them. This will be reliably accurate only if the distance between them is much larger than their sizes. For other cases one must analyse each case carefully. The *basic* law, however, for *tiny* particles of matter, is just,

$$F_{grav,\,12} = F_{grav,\,21} = G\, m_1 m_2 / (d_{12})^2 \quad ,$$

Where G is the constant of proportionality, Newton's universal gravitational constant.

Next we will derive consequences and explanations from all this.