Energy I: Identifying Energies

1: The names of energy

When my hand is moving it carries energy that is just due to the motion. That energy is called **kinetic energy**, i.e., energy of motion. When my hands rub together and get warm from the rubbing, that warmth indicates the generation of **heat energy**. When you hear my hands rubbing together, or more strikingly, clapping together, that sound carries energy which we can call **sound energy**. You see my hands moving because of light that bounces off my hands and enters your eyes. That light carries **light energy**. When my outstreched arm gets tired it's because I have to keep exerting muscles to maintain the **gravitational energy** of my arm in the presence of the Earth's gravity. The mucular contraction itself imbues the muscles with **elastic energy**, as is the case for a stretched rubber band or a compressed spring. Both the gravitational energy of my arm and the elastic energy of a muscle or a spring are examples of **potential energy**, i.e., energy of relative position or configuration which has the potential to be converted to kinetic energy. If the air is dry and I rub my hand over my clothes I may generate sparks which carry **electrical energy**. Similarly, for the current in the wires when I turn on a lamp. If I strike a match, the flame is generated by a chemical reaction that involves **chemical energy**. Similarly, for the generation of heat by burning coal or oil or for the processes in my car's battery when I turn the key in the ignition. When I use a compass to check directions on the trail, the compass functions because of its interaction with the Earth's magnetism via **magnetic energy**. Similarly, for the interaction between the magnets that cause the diaphragms in your radio or TV or sound system speakers to vibrate. The radioactive decay of Uranium deposits in the Earth have been an important source of the heat of the Earth's interior. Those decay processes employ **nuclear energy**. Similarly for the fission processes that are the energy source for nuclear power plants. The light from the Sun is, ultimately, obtained from the **thermonuclear energy** liberated from the fusion of light nuclei into heavier ones deep in the Sun. Similarly for the destructive blast of a hydrogen bomb.

All of these kinds of energy have distinctive names which means there is something different about each of them. Yet they're all called 'energy' which means there's something similar about all of them. What are the differences and what are the similarities? The differences refer primarily to the contexts in which the energies manifest themselves or play an important role. The similarities go much deeper into the heart of things and are responsible for the interchangeability of the different forms into one another.

As we shall see, heat energy is primarily understood as **randomized molecular kinetic energy**. Sound energy is understood as a structured combination of **organized molecular kinetic energy** and elastic potential energy. Elastic energy, itself, can usually be understood as various combinations of chemical energy and **molecular electric potential energy**, a distinction not always easy to maintain because chemical energy is, itself, primarily molecular electrical energy governed by **quantum mechanics** combined with molecular kinetic energy. And then light energy and electric energy and magnetic energy are all understood as various manifestations of **electromagnetic energy**. Gravitational energy stands by itself in this list as does nuclear energy in both its fission and fusion garb. So, at almost the fundamental level, the major subdivisions of energy are kinetic, electromagnetic, gravitational and nuclear. Still, at the last remove, they are all just **energy**, which can take these various forms.

Now, unless you've mastered this material before coming to this course, you may already be a bit confused. Don't squelch that feeling of confusion! It's very healthy! That's your mind telling you that things must be made more simple and clear before they can be understood, let alone justifiably believed. We all tend to accept what we are told by 'authorities' far too easily, with little or no understanding!

One last point before we move on to some nitty gritty details. Like all measured quantities, energy is measured in energy units. There are many different energy units; Btu's, calories, foot pounds, joules, etc. We will encounter them all but we will focus our attention on a unit we are all familiar with because we pay for them monthly, the **kilowatt-hour**, or kwh.

2: Energy and Work

a: defining work

Energy is often *defined* as the capacity to do **work**. That does sound useful. But that would suggest that to determine how much energy was available in any situation, we would need to determine how much work could be done. This would seem to require a quantitative definition of work. There actually is one! It was proposed by James Watt around 1826 in connection with the problem of comparing the effectiveness of horses with steam engines for pumping water out of mines. It was reintroduced more formally by Clausius in 1850 in connection with the study of heat.

Usually, when there's work to be done, we want to move things around. Whether it's organizing the office, chopping logs, hauling groceries, transporting goods or lifting I-beams to the top of a construction site or pushing electrons through our computers, we, or our machines, are engaged in moving objects from one place to another. We do that by bringing **forces** to bear on the objects of interest. Forces that start them moving, forces that keep them moving against the resistance of friction and/or other opposing forces and forces that stop them from moving when we get them where we want them to be. It would seem appropriate, then, to define work in terms of the forces that are brought to bear and the duration and/or distances over which they are applied. But just exactly how should this be done?

Intuitively, one would expect more work to be done if forces are applied for longer periods of time; more work to be done if forces are applied over longer distances; more work to be done by large forces than by small. But forces and traversed distances, called **displacements**, are all **directed magnitudes** (**Fig.1**) as are **velocities** and **accelerations**, i.e., quantities that

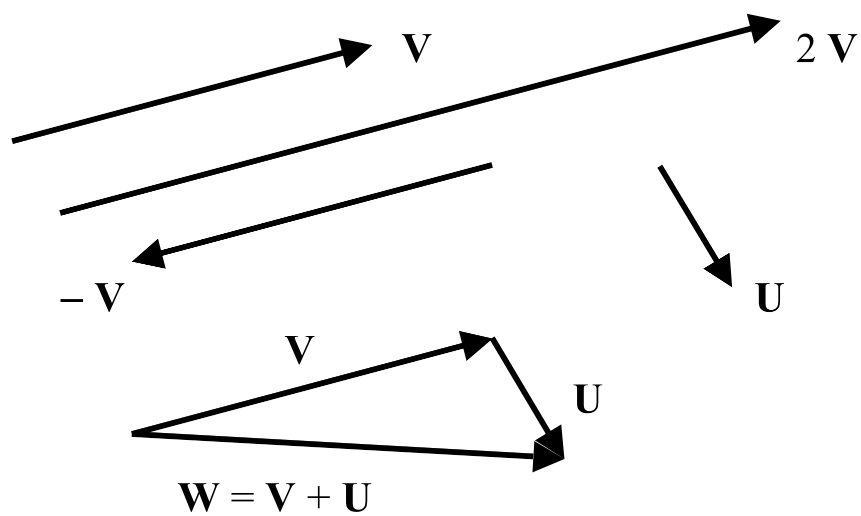


Fig. 1: Representing directed magnitudes (vectors) by arrows and the algebraic manipulation of arrows.

have not only size or magnitude, but which point in a particular direction as well. So if work depends on such quantities, it may depend on the *relationships among the directions* as well as on the magnitudes.

The decision eventually reached was to define the work done by a force of *constant* magnitude, F, acting in a *fixed* direction on an object moving *parallel* to that direction through a distance, d, as,

$$W = \pm F d . \tag{2.1}$$

Notice the absence of any reference to how fast the object is moving or how long the force is acting! But what about that $\pm$?

If the object is moving in the *same* direction that **F** is pointing, then we take $W \geq 0$, and we choose the + sign, while if the object is moving in the *opposite* direction from **F**, then we simply make W a negative quantity by choosing the minus sign. This represents the fact that in the latter case the force and the work it's doing are *opposing* the motion of the object while in the former case they are *assisting* the motion.

If the force and the displacement through which it acts point in different directions, in which case we represent the force and the displacement by arrows, **F** and **Δx**, then we define the work done as either the product of the *magnitude* of the displacement, $|\mathbf{\Delta x}|$, and the **component** (see **Fig. A1** in the **Appendices**) of the force in the direction of the displacement, $F_{\mathbf{\Delta x}}$, or the product of the magnitude of the force, $|\mathbf{F}|$, and the component of the displacement in the direction of the force, $\Delta x_{\mathbf{F}}$.

These two different products always give the same result (see **Fig. A2**).

$$W = F_{\mathbf{\Delta x}}\, |\mathbf{\Delta x}| = |\mathbf{F}|\, \Delta x_{\mathbf{F}} . \tag{2.2}$$

One way of understanding this generalization from (2.1) to (2.2) is to recognize that any force is the *sum* (**Fig. 1**) of a part that is parallel to the displacement of the object and a part that is perpendicular. The parallel part either assists or opposes the objects motion while the perpendicular part does neither. The convention, then, is to deny any work done by the perpendicular part and to treat the parallel part the same way as parallel forces were treated in the discussion of (2.1).

This definition has the initially puzzling feature of attributing work being done by us on a heavy object that we push along the floor but denying that work is done by us on the same object if we carry it across the floor at a fixed height above the floor (**Fig. 2**). Yet both tasks tire us!? The resolution is found at the molecular level in our muscles where molecules and atoms have work done on them as they are pushed and pulled around in chemical reactions in order to maintain the flexing of our muscles to move the heavy object.

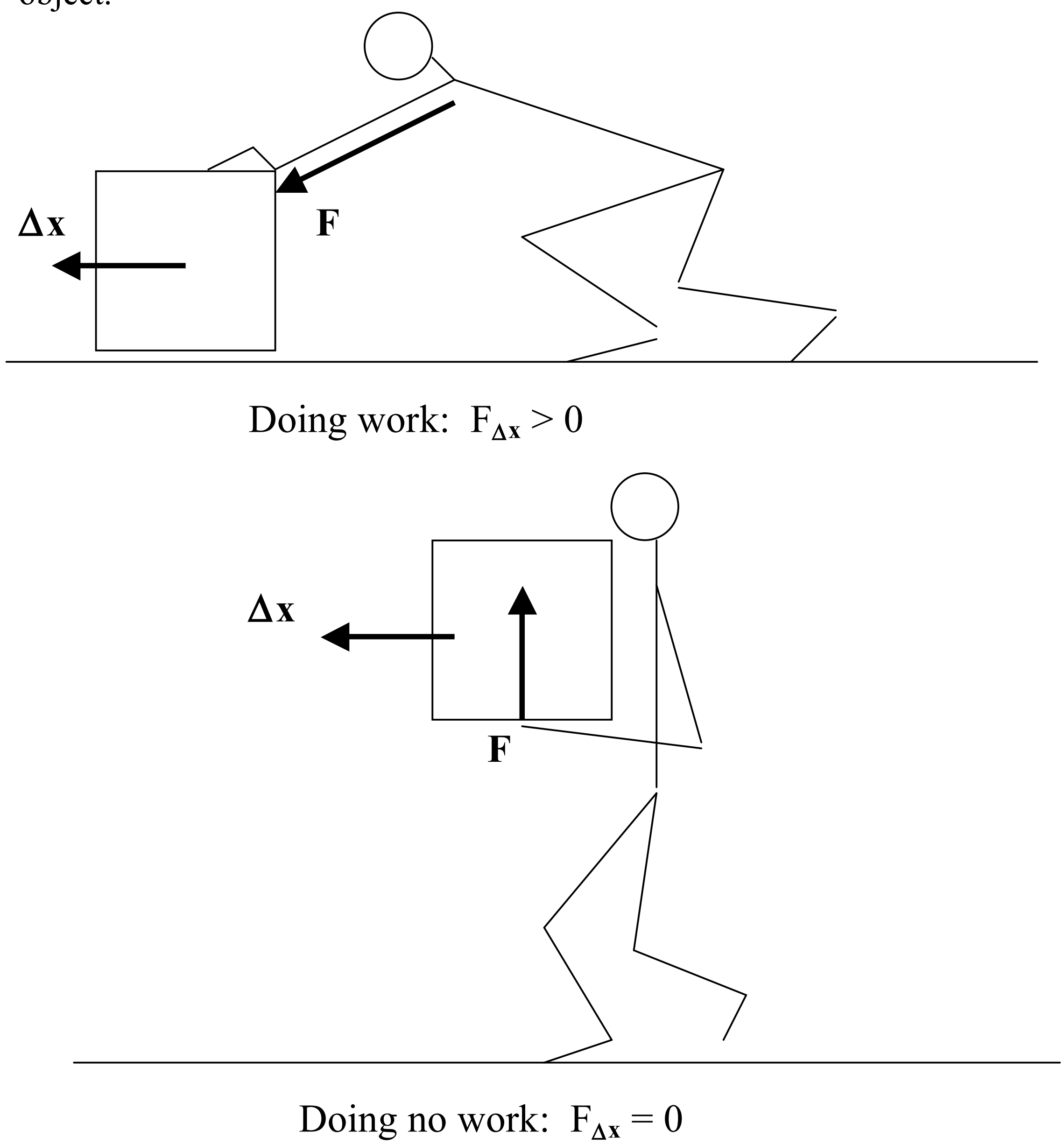


Fig. 2: Puzzling consequences of the definition of work

So much for a constant force. Suppose the force is variable, either in magnitude or direction or both. Suppose the force is applied over a curvilinear path, varying as it goes (**Fig. 3a**) . What then?

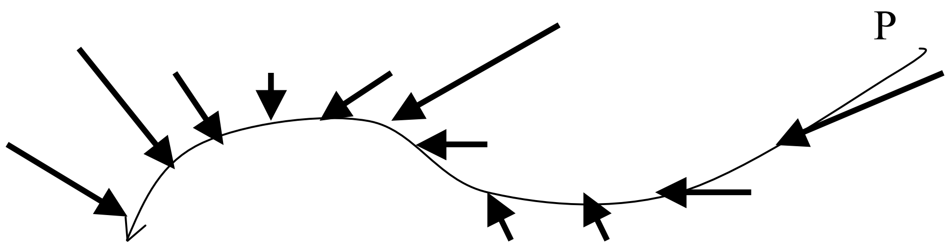


Fig. 3a: Curvilinear path, P, of object subject to varying force doing work, W.

The first part of the answer is that if the curvilinear path, P, is broken up into two parts, P_1 and P_2, (**Fig. 3b**) with the variable force doing work, W_1 and W_2 in each part, then the total work done is just the sum,

$$W = W_1 + W_2 \,. \tag{2.3}$$

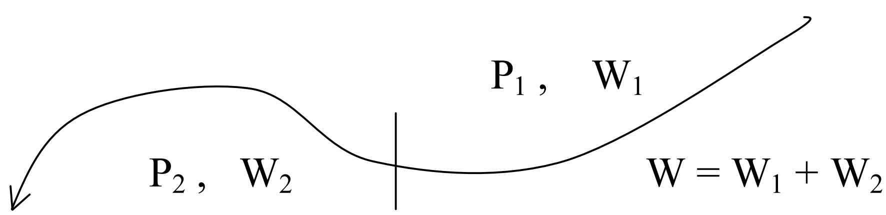


Fig. 3b: Path, P, regarded as composed of two parts, P_1 and P_2, in which the varying force does work W_1 and W_2, respectively.

The rest of the answer is that by breaking the original path up into more and more shorter and shorter parts, say N of them, (**Fig. 3c**) we eventually have, the partial paths so short as to be essentially straight, but with different lengths and directions, while the forces acting in the partial paths are essentially constant, but with different strengths and directions. Then we use the definition of work by a constant force on a straight displacement in each of the very short partial paths and add up the results to get,

$$W = W_1 + W_2 + W_3 + \text{- - -} + W_N \,, \tag{2.4}$$

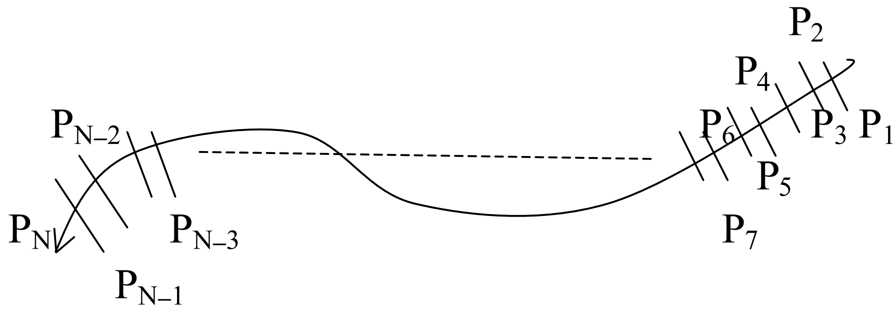


Fig. 3c: Path, P, broken up into a large number, N, of almost straight, short paths in each of which the force is almost constant. Work, W_n is done in the part, P_n.

A very important distinction now arises between forces that give rise to work that depends on the details of the path over which the force acts and forces that do work that depends only on the *location of the endpoints* of the path, but not on the shape or length of the path (**Fig. 4**). In the latter case we can always identify an **average force**, <**F**>, such that if the net displacement from the beginning to the end of the path is $\mathbf{\Delta x}$, then,

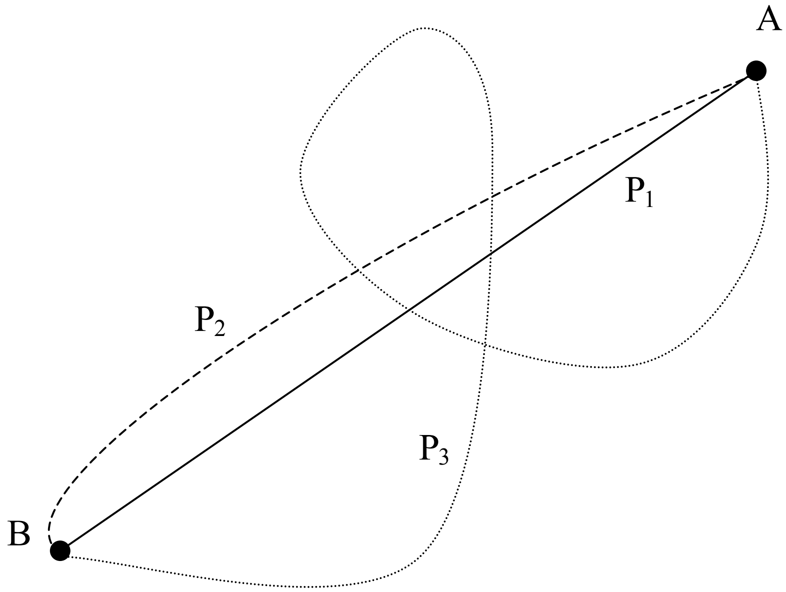
$$W = \langle F \rangle_{\mathbf{\Delta x}} \, |\mathbf{\Delta x}| \,. \tag{2.5}$$

These kinds of forces give rise to what are called **potential** energies and, for reasons that will be made clear later, are called **conservative** forces.

b: weight and gravitational potential energy

The simplest important example in nature of a working force that is *very nearly* constant is provided by the **weight** of objects near the surface of the Earth. The weight of an object is just the force with which the Earth attracts the object due to Earth's gravity. The force is (almost) always vertically downward (**Fig. 5**). If an object of weight, w, starts at a height, h, above the ground and moves to the ground, either by falling freely or by being lowered gently or in whatever way, the work done by the weight of the object is, from (2.1), just,

$$W_{P1} = W_{P2} = W_{P3} = W_{AB}$$



$$W_{AB} = \langle F_{AB} \rangle D_{AB} \qquad\qquad W_{BA} = - W_{AB}$$

Fig. 4: The work done by a conservative force acting on an object moving from one point, A, to another point, B, does not depend on the path taken by the object, but just on the endpoints, A and B.

$$W_{grav} = w\ h. \tag{2.6a}$$

If the object is *lifted* from the ground to the height, h, by some *other* force, Earth's gravity is still acting. But now the weight and the displacement are in opposite directions, so the work done by gravity is now, again from (2.1),

$$W_{grav} = -\ w\ h. \tag{2.6b}$$

If an object moves in a curvilinear way such that it started out at a height, h_1, above the ground and ended up at a height, h_2, above the ground, then the work done by the Earth's gravity, i.e., by the weight, is

$$W_{grav} = w\ (h_1 - h_2). \tag{2.7}$$

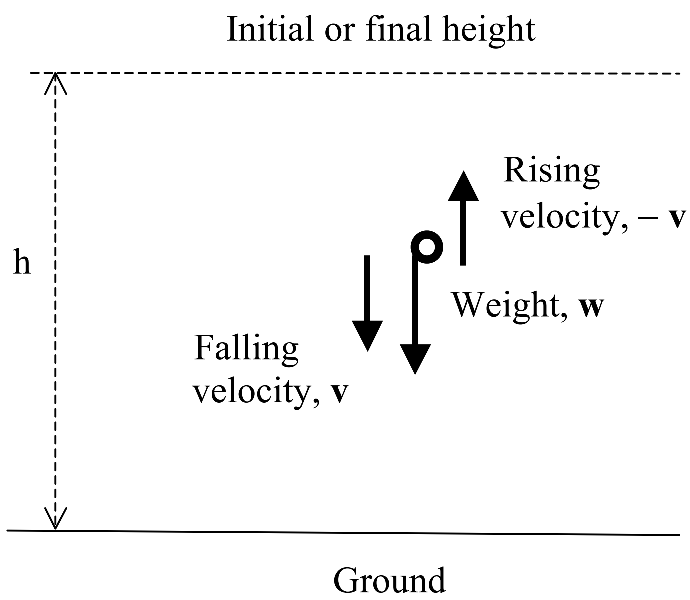


Fig. 5: Object falling or rising in Earth's gravity

The reason is that the only parts of the curvilinear path that count are the parts with a displacement component in the direction of the force and the force is always vertically downwards. The work done depends on the path only through the height of the endpoints!

So, relative to the ground, Earth's gravity has the capacity to do an amount of work, wh, on any object of weight, w, situated at a height, h, above the ground. Therefore we say such an object has a **gravitational potential energy** *relative to the ground* of

$$E_{grav} = wh. \tag{2.8}$$

In the course of moving from height, h_1, to height, h_2, the work done by gravity is just the negative of the change in the gravitational potential energies,

$$W_{grav;1,2} = wh_1 - wh_2 = E_{grav,1} - E_{grav,2} = -\Delta E_{grav}\ . \tag{2.9}$$

If we dug a hole in the ground under the object and it moved so as to end up somewhere below the ground level in the hole at a depth, d, say, the work done in moving the object from the height, h, to the depth, d, would be,

$$W_{grav} = w(h + d). \tag{2.10}$$

Accordingly, we extend the concept of potential energy to levels below the surface by saying that at a depth, d, the potential energy is,

$$E_{grav} = -\,wd\,. \qquad (2.11)$$

Then the rule that the work done is just the negative change in potential energy holds for levels both above and below the surface. We can unify all our expressions for the gravitational potential energy by introducing a vertical coordinate, y, which above ground level will have the value of the height, h, while below ground level will have the value of the negative of the depth, − d. Then we can always write (**Fig. 6**),

$$E_{grav} = wy. \qquad (2.12)$$

At this point some of you may be wondering when I'm going to make reference to the promised familiar item of the kilowatt hour. As things are getting a little abstract, this might be a good time to do that. By all means!

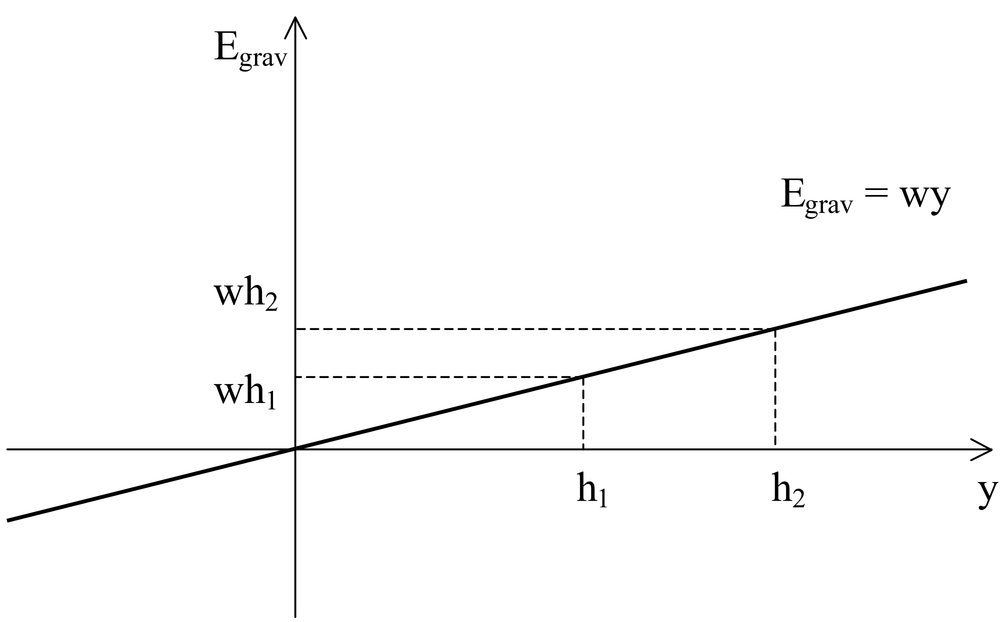


Fig. 6: Gravitational potential energy curve for object of weight, w, near the Earth's surface.

If you lift a 25 lb sack of bird seed from the ground to a 4 ft high table top or shelf, you will have done

$$\text{w h} = 25 \text{ lb x } 4 \text{ ft} = 100 \text{ lb ft} = 100 \text{ ft lb} \qquad (2.13)$$

of work on the sack. As a consequence the sack then has 100 ft lb of gravitational potential energy in it (relative to the ground) which you put there by your work. So a **foot-pound** or ft lb is a natural unit of energy. If you were an employee at a Walmart Super Store and you shelved 100 of those sacks you would have done 10,000 ft lb of work and you'd feel it!

How does that amount of work (and the equal amount of gravitational potential energy deposited in the shelved sacks) compare with a kilowatt hour? What we find when we turn to the reference manuals are statements like,

$$550 \text{ ft lb/sec} = 1 \text{ hp} = 746 \text{ w.} \qquad (2.14a)$$

These are units of **power**, i.e., energy per unit time, and if we shelved our 100 sacks at the rate of 11 sacks every 2 seconds (!) we'd be working at a rate of one horsepower (1 hp) or 746 watts (w). More human (not to mention *humane*) would be 1 sack every 4 sec., or about 1/22 hp. In any case from the preceding equations we see that,

$$550 \text{ ft lb} = 746 \text{ w sec,} \qquad (2.14b)$$

or about,

$$0.73 \text{ ft lb} \simeq 1.0 \text{ w sec.} \qquad (2.14c)$$

But,

$$1 \text{ kwh} = (1000 \text{ w}) \text{ x } (3600 \text{ sec}) = 3.6 \text{ x } 10^6 \text{ w sec } ! \qquad (2.15a)$$

So

$$1 \text{ kwh} \simeq 3.6 \text{ x } 10^6 \text{ x } 0.73 \text{ ft lb} = 26{,}300 \text{ x } 100 \text{ ft lb.} \qquad (2.15b)$$

Shelving 26,300 sacks would take about 29 hours at ~ \$6.00 / hr. Via electric power we get that kwh (after taxes and fees) for about 4.6 cents! Still, the quantity of useful energy involved is the same in both cases.

c: the elastic potential energy of stressed springs

Suppose we have a coiled spring of length, L, when it's neither stretched nor compressed. If we stretch it through a distance, d, where d << L, it will pull back against our effort and require an external force that is proportional to d,

$$F = kd . \tag{2.16}$$

The restoring force of the spring will, accordingly, be represented by - kd since it has the same magnitude as the external force and the opposite direction (**Fig. 7**). This restoring force provides the stressed spring's capacity to do work. The stiffness of the spring is measured by the constant, k.

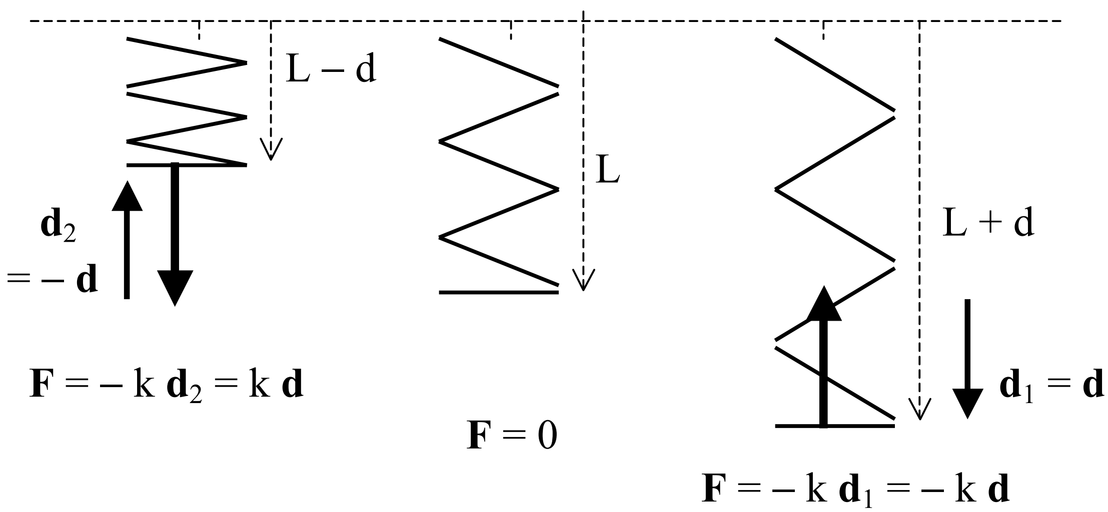


Fig. 7: Forces exerted *by* a stressed spring

Similarly, if we compress the spring by a displacement of its end by, – **d**, the spring will push back with a force that grows as the compression progresses like kd.

Since the stressed spring has the capacity to do work, the external force which stressed the spring must have deposited potential energy in the spring (or in the mass attached to the spring) by the work it did. But the task of calculating just how much work is done is more complicated now since, unlike near Earth gravity, the force isn't constant over the displacement. What is the average force?

Since the external force grows steadily from zero to kd or – kd as we stretch or compress, respectively, the spring by an amount, d, a reasonable *guess* for the average force would be

$$< F >_{[0, \pm d]} = \pm (1/2)\, kd, \qquad (2.17a)$$

respectively. It turns out that this guess is exactly correct (see **A3** for a

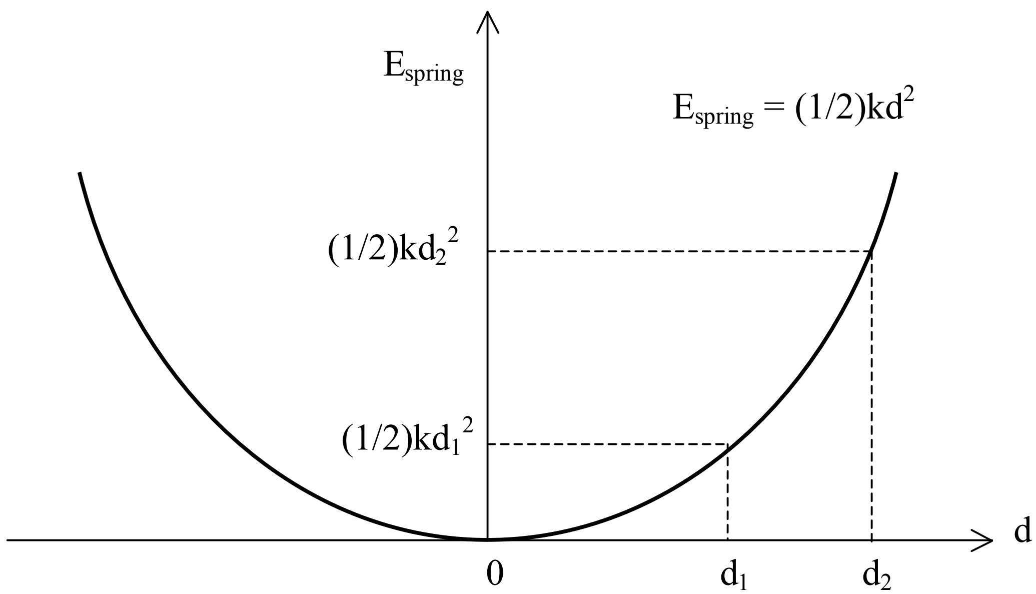


Fig. 8: Elastic potential energy curve, relative to the relaxed state, for a spring of stiffness, k.

derivation) and, consequently the work done *on* the spring when being stressed through a displacement of $\pm$ d is just

$$W = < F >_{[0, \pm d]} (\pm d) = [\pm (1/2)\, kd\,] (\pm d) = (1/2)\, kd^2. \qquad (2.17b)$$

By relaxing the spring we can get that amount of work back *from* the spring and so the potential energy of the stressed spring is just (**Fig. 8**),

$$E_{spring} = (1/2)\, kd^2. \qquad (2.18a)$$

In changing the stressing of the spring from d_1 to d_2 (both either positive or negative), the spring force changes from $-\, kd_1$ to $-\, kd_2$, with an average force of $-(1/2)(kd_1 + kd_2)$ and a displacement of $(d_2 - d_1)$, yielding a work done *by the spring* of,

$$
\begin{aligned}
W_{spring} &= -(1/2)(kd_1 + kd_2)(d_2 - d_1) \\
&= (1/2)kd_1^{\,2} - (1/2)kd_2^{\,2} = -\Delta E_{spring}\ . \qquad (2.18b)
\end{aligned}
$$

d: electrostatic potential energy

Two small charged objects, carrying amounts of charge, q and q', which can be positive or negative, and separated by a distance, r, exert forces on each other with a magnitude of

$$| \mathbf{F}_{elec.}| = \kappa\, | qq'| / r^{\,2}, \qquad (2.19)$$

where κ is a universal constant of proportionality. The letter, r, is usually used in this case because the distance is thought of as a radius or radial distance since the force due to each charge is spherically symmetric with respect to that charge as a center.

If the charges are both positive or both negative so that qq' is positive, the forces are repulsive; the charges 'try' to push each other apart. If the charges have opposite sign so that their product is negative, the forces are attractive; the charges 'try' to pull each other together.

If the radial distance between the charges changes from r_1 to r_2 via some curvilinear path of one or both charges, the radial direction of the forces guarantee that only the radial component of the displacements comprising the short segments of the path(s) will contribute to the work done by the forces. Consequently the work done will only depend on the radial distances of the end points of the path(s) and so an average force exists. For the case in which the end points lie on the same radius the average force has a magnitude which is the *geometric mean* of the force magnitudes at r_1 and r_2, i.e., it is the square root of the product of the end point magnitudes (see **A4** for a derivation),

$$|\langle \mathbf{F}_{elec} \rangle_{[r1,\, r2]}| = \kappa |qq'| / r_1 r_2\ . \qquad (2.20)$$

If $(r_2 - r_1) > 0$, then the work done *by* the electrostatic force will be positive if the forces are repulsive and negative if the forces are attractive. The opposite is the case if $(r_2 - r_1) < 0$. Therefore, the work done is,

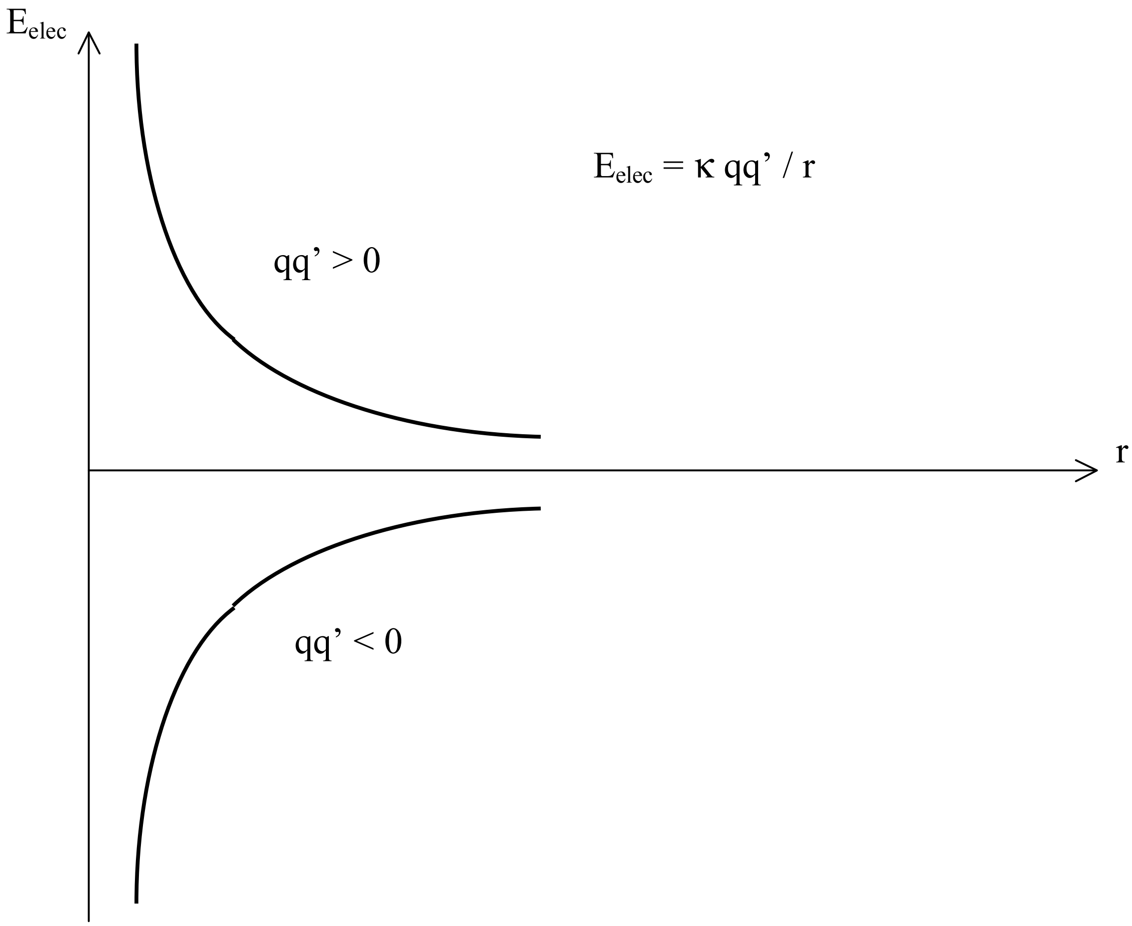


Fig. 9: Electrostatic potential energy curves for qq' positive or negative.

$$W_{elec} = (\kappa qq' / r_1 r_2)(r_2 - r_1) = (\kappa qq' / r_1) - (\kappa qq' / r_2). \qquad (2.21)$$

This is equal to the negative change in the potential energy, $-\Delta E_{elec}$, if we define (**Fig. 9**),

$$E_{elec} = \kappa qq' / r . \qquad (2.22)$$

3: Work and kinetic energy

a: resultant force and acceleration

Potential energies exist for a variety of individual forces (but not for all forces, as we mentioned at the end of **1**) and we have examined three such. Kinetic energy, the capacity to do work that resides in the motion of an object, is not related to any particular kind of force but, rather, to the total or **resultant force** that acts on an object and may be composed of many forces acting simultaneously. The reason is that, according to **Newton's laws of motion**, [the basic laws governing the behaviour of objects that are not moving too fast (much less than light speed) and are neither too small (atoms) nor too large (galaxies)], it is the resultant force on an object that determines how its motion changes and, therefore, how its kinetic energy changes. What do Newton's laws of motion say about this?

For a *constant* resultant force, $\mathbf{F}_{res,}$, acting on a particle of **mass**, m (mass is proportional to weight but they are not the same thing, as we will see later), for a time interval, Δt, Newton's first and second laws are expressed by the equation,

$$\mathbf{F}_{res}\, \Delta t = m\, \mathbf{\Delta v}\,, \tag{3.1}$$

where $\mathbf{\Delta v}$ is the change in the velocity of the particle during the time interval, Δt. So the *change* in the velocity points in the same direction as the resultant force and has a magnitude given by,

$$|\mathbf{\Delta v}| = |\mathbf{F}_{res}|\, \Delta t / m\,. \tag{3.2}$$

For objects that are too large to be regarded as particles, there is a point 'inside' them, called their **center of mass**, and the Newton law equation refers to the change in velocity of that point.

If we divide both sides of the Newton law equation, (3.1), by Δt then the ratio,

$$\mathbf{\Delta v} / \Delta t = \langle \mathbf{a} \rangle, \tag{3.3}$$

that will occur on the right hand side, is called the **average acceleration** for the time interval, Δt. If the resultant force is not constant during Δt, the equation still holds if we put in place of $\mathbf{F}_{res}$ the *time average* resultant force, $\langle \mathbf{F}_{res} \rangle$, i.e.,

$$\langle \mathbf{F}_{res} \rangle = m \langle \mathbf{a} \rangle, \tag{3.4}$$

holds generally. If the time interval, Δt, is so short as to be, for all practical purposes (FAPP), instantaneous, then $\langle \mathbf{F}_{res} \rangle$ and $\langle \mathbf{a} \rangle$ are called the **instantaneous resultant force** and **instantaneous acceleration**, respectively, and we usually drop the angle brackets in writing the equation,

$$\mathbf{F}_{res} = m\, \mathbf{a}\ . \tag{3.5}$$

For example, suppose we hold an object above the ground and then let go of it (**Fig. 10**). To a good approximation the resultant force on the object will be just that of Earth's gravity pointing vertically downwards and with a magnitude we call the weight, w, of the object. This constant resultant force will produce a constant acceleration downwards while the object falls and

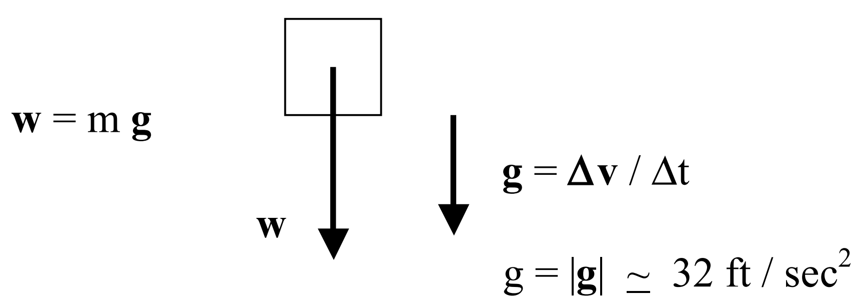


Fig. 10: The relationship between weight and mass near Earth's surface. If the weight, **w**, is the resultant force, then the object falls with the universal acceleration, $g \simeq 32$ ft / sec^2.

Galileo's observation and subsequent careful measurement indicates that *all objects fall with the same acceleration*,

$$g \simeq (32 \text{ ft /sec})/\text{sec} \simeq (9.8 \text{ m/sec})/\text{sec}. \tag{3.6}$$

Consequently, Newton's law equation for just the magnitudes in this case is,

$$w = m\,g\,. \tag{3.7}$$

This is the relationship between mass and weight. Mass is (for now!) an intrinsic property of an object. Weight is the strength of the Earth's pull on the mass. At very high altitudes the mass of an object is the same but the Earth pulls less strongly and the weight is less. On the Moon the mass is the same but the Moon-weight is less. For a given object, it's just as hard to produce a given acceleration on the Moon as on the Earth and so the acceleration of a falling body near the surface of the Moon, produced by Moon-weight, is less than that near the surface of the Earth, produced by Earth-weight.

b: resultant work and kinetic energy

Now, back to kinetic energy. Suppose a constant resultant force, $\mathbf{F}_{res}$, acts on a particle of mass, m, from a time when the particle has a velocity, $\mathbf{v}$, to the time, Δt, later, when it has a velocity, $\mathbf{v} + \mathbf{\Delta v}$. Then we have,

$$\mathbf{F}_{res}\,\Delta t = m\,\mathbf{\Delta v}. \tag{3.8}$$

If the particle moves through the displacement, $\mathbf{\Delta x}$, during Δt, then the work done by $\mathbf{F}_{res}$ is (according to (2.2)),

$$W_{res} = |\mathbf{F}_{res}|\,\Delta x_{\mathbf{F}res} = m\,|\mathbf{\Delta v}|\,\Delta x_{\mathbf{F}res} / \Delta t\,. \tag{3.9}$$

Now since $\mathbf{F}_{res}$ and $\mathbf{\Delta v}$ point in the same direction, the magnitude of $\mathbf{\Delta v}$ is the same as the component, $\Delta v_{\mathbf{F}res}$. Also the ratio on the right hand side is just the component of the average velocity in that same direction, $\langle v_{\mathbf{F}res} \rangle$. Therefore (see **A5** for a derivation),

$$W_{res} = m\,\Delta v_{\mathbf{F}res}\,\langle v_{\mathbf{F}res} \rangle = (m/2)\,\Delta(|\mathbf{v}|^2). \tag{3.10}$$

So for a constant resultant force the work done is just the change in the quantity (where $v = |\mathbf{v}|$),

$$K = (1/2)\,m\,v^2, \tag{3.11}$$

called the kinetic energy (**Fig. 11**).

For a general variable resultant force the same result holds since we can divide the time interval up into many short parts in each of which the resultant force is effectively constant and then we get,

$$W_{res} = W_1 + W_2 + - - - + W_N = \Delta K_1 + \Delta K_2 + - - - + \Delta K_N = \Delta K \,. \qquad (3.12)$$

This result, that the total work done by the resultant force equals the change in kinetic energy, is called the **work-energy theorem**.

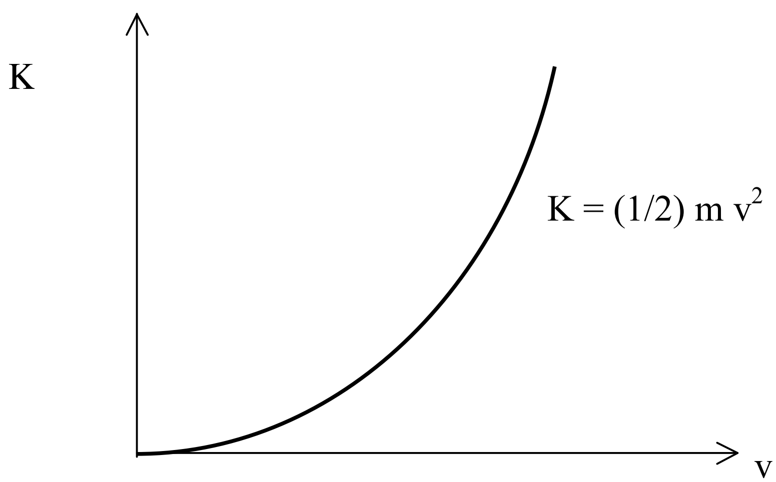


Fig. 11: The dependence of kinetic energy on speed

Appendices:

A1: Definition of components of arrows (vectors)

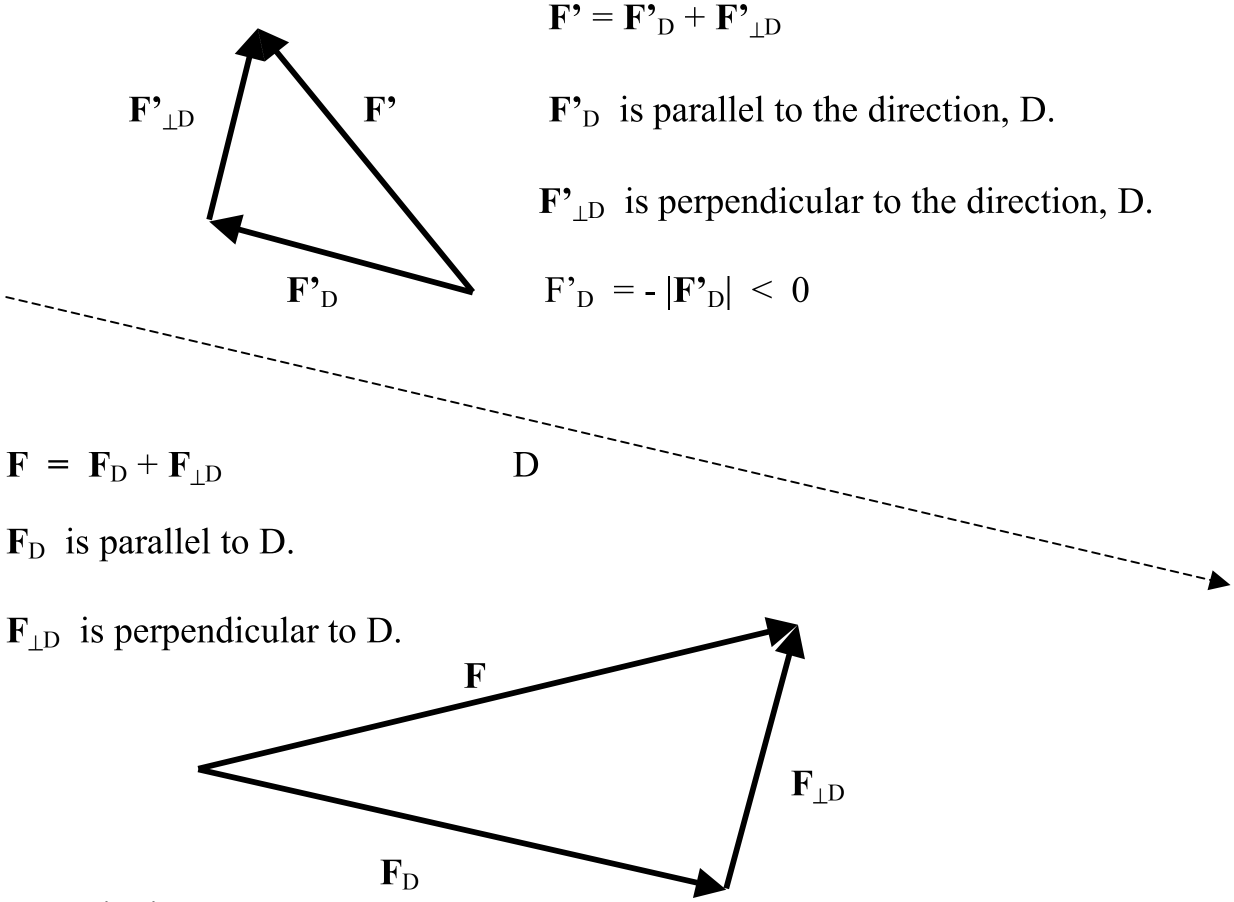
A2: Argument for the second equality in (2.2)

A3: Argument for the potential energy of a stretched spring

A4: Argument for the potential energy of electric charges

A5: Argument for the work – energy theorem

A1: Definition of components of arrows (vectors)



$\mathbf{F'} = \mathbf{F'}_D + \mathbf{F'}_{\perp D}$

$\mathbf{F'}_D$ is parallel to the direction, D.

$\mathbf{F'}_{\perp D}$ is perpendicular to the direction, D.

$F'_D = -|\mathbf{F'}_D| < 0$

$\mathbf{F} = \mathbf{F}_D + \mathbf{F}_{\perp D}$

$\mathbf{F}_D$ is parallel to D.

$\mathbf{F}_{\perp D}$ is perpendicular to D.

$F_D = |\mathbf{F}_D| > 0$

Fig. A1: Two arrows (*vectors*), **F** and **F'**, expressed as sums of their (*vector*) components parallel to the direction, D, ($\mathbf{F}_D$ and $\mathbf{F'}_D$, respectively) and perpendicular to the direction, D ($\mathbf{F}_{\perp D}$ and $\mathbf{F'}_{\perp D}$, respectively). Also the *algebraic* components parallel to D (F_D and F'_D , respectively) are related to the magnitudes of their corresponding vector components. This can always be done for any arrows (*vectors*) and any direction.

A2: Argument for the second equality in (2.2)

Consider two vectors, **V** and **W**. Join their origins together as drawn.

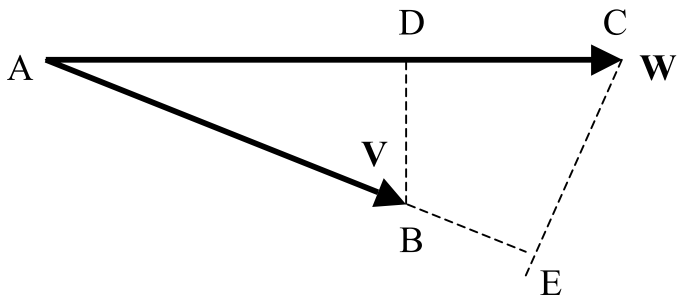


Fig. A2: Two vectors and lines from their ends extending or perpendicular to one or other of them.

From B, the end of **V**, draw a perpendicular line to **W** (extended if needed) ending at D.

From C, the end of **W**, draw a perpendicular line to **V** (extended if needed) ending at E.

The triangles ABD and ACE have the same shape. Therefore the lengths (AB), (AD), (AC) and (AE) satisfy,

$$(AB) / (AD) = (AC) / (AE). \qquad \text{(A2.1)}$$

But $(AB) / (AD) = |\mathbf{V}| / V_\mathbf{W}$ and $(AC) / (AE) = |\mathbf{W}| / W_\mathbf{V}$. Consequently,

$$|\mathbf{V}| / V_\mathbf{W} = |\mathbf{W}| / W_\mathbf{V} \quad \text{or} \quad W_\mathbf{V}\, |\mathbf{V}| = V_\mathbf{W}\, |\mathbf{W}| \qquad \text{(A2.2)}$$

This equality holds regardless of the sizes and directions of the vectors, even if $V_\mathbf{W}$ and $W_\mathbf{V}$ are (both) negative. If the vectors are a force, **F**, and a displacement, $\Delta\mathbf{x}$, we have (2.2).

A3: Argument for the potential energy of a stretched spring

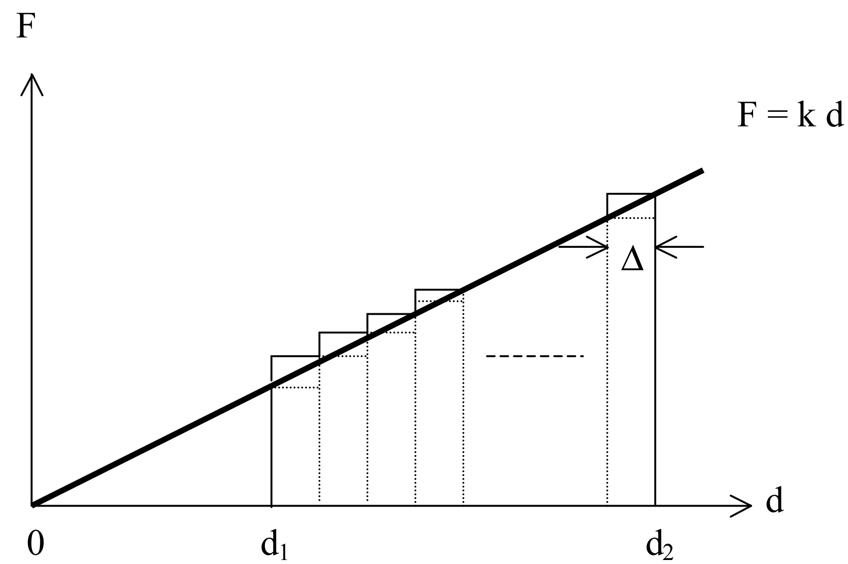


Fig. A3: Graph of the force, F, (vertical axis) required to stretch a spring against the length of the stretch, d, (horizontal axis). The vertical rectangles are referred to in the discussion below.

In the graph above, d measures the stretch of a spring along the horizontal axis due to an applied, external force, F, measured along the vertical axis. With a spring stiffness constant, k, a stretch of d requires the applied force, $F = kd$.

Now suppose the spring is stretched from d_1 to d_2. During the stretch the magnitude of the applied force is getting larger and larger. If we divide the interval, (d_1, d_2), into N equal sub-intervals of length $\Delta = (d_2 - d_1) / N$, then in the nth sub-interval, from $(d_1 + (n - 1)\,\Delta)$ to $(d_1 + n\,\Delta)$, F is never greater than $k(d_1 + n\,\Delta)$ and is never less than $k(d_1 + (n - 1)\,\Delta)$. Consequently *the work done on the spring* in stretching it is less than if the force changed in accordance with the solid steps in the graph and is more than if the force changed in accordance with the dotted steps in the graph.

But the work corresponding to the solid step force graph is proportional to the area of all the taller rectangles between d_1 and d_2, while the work corresponding to the dotted step graph is proportional to the area of all the shorter rectangles between d_1 and d_2. As we increase the number of sub-intervals, N, the difference in area between these two sets of rectangles gets closer and closer to zero while the area of each set of rectangles gets closer and closer to the area between the straight line curve for the actual force, $F = kd$, and the horizontal axis and between the vertical lines at d_1 and d_2. But this enclosed area is just the difference in area between the large triangle with vertices at $(0, d_2, F = kd_2)$ and the small triangle with vertices at $(0, d_1, F = kd_1)$. Since the area of *any* triangle is given by 1/2 the product of its base and height we find the work done *on* the spring to be,

$$W_{12} = (1/2)\, d_2\, (kd_2) - (1/2)\, d_1\, (kd_1) \qquad \text{(A3.1)}$$

or

$$W_{12} = (1/2)\, k(d_2^2 - d_1^2). \qquad \text{(A3.2)}$$

This is the work we can get back *out* of the spring if it relaxes *from* the d_2 stretch *back to* the d_1 stretch. But that work, being just the negative of the change in the potential energy, $-(E_{spring,\,1} - E_{spring,\,2})$, tells us that the potential energy is

$$E_{spring} = (1/2)\, kd^2, \qquad \text{(A3.3)}$$

as asserted in equation (2.18a). We didn't use the *average force* to get the work done and the potential energy in this case. It was easier to use the *triangle area* method. But our result determines $< F >_{[d1,\, d2]}$, the average of kd in the interval, $[d_1, d_2]$, through the equation, (A3.2) and,

$$W_{12} = (d_2 - d_1) < F >_{[d1,\, d2]} . \qquad \text{(A3.4)}$$

Combining (A3.2) and (A3.4), we find that

$$< F >_{[d1,\, d2]} = (1/2)(kd_1 + kd_2)\, , \qquad \text{(A3.5)}$$

A4: Argument for the potential energy of electric charges

The same procedure used in **A3** could be used again here, i.e., referring to the geometry of the graph of the force and its dependence on distance. But in the case of a force that varies with the radial distance, r, between two charges, as $1/r^2$, as electrostatic attraction or repulsion between charges does, the geometric method is more difficult. Instead we will use a *method of averages*. This method could have been used in **A3** and would have led to the result we obtained there.

Suppose we consider two positive charges, q and q', that move further apart, from an initial distance apart of r_1 to a final, greater distance apart, r_2. The work done during this change of distance by the repulsive force between them is equal to the distance change, $r_2 - r_1$, multiplied by the *average* magnitude, $< \kappa qq'/r^2 >_{[r1,\, r2]}$, of the force in the interval, $[r_1, r_2]$, i.e.,

$$W(r_1, r_2) = (r_2 - r_1) < \kappa qq'/r^2 >_{[r1,\, r2]} . \qquad \text{(A4.1)}$$

This average force plays the same role in calculating the work done that the actual force plays when it is constant. When the force is not constant, as is the case here, the average force for an interval must not be more than the strongest value of the actual force in the interval nor less than the weakest value for the actual force, i.e., since $r_1 \leq r_2$,

$$\kappa qq' / r_2^2 \;\leq\; < \kappa qq'/r^2 >_{[r1,\, r2]} \;\leq\; \kappa qq' / r_1^2 \;. \qquad \text{(A4.2)}$$

Furthermore, since, if the charges continued to separate from r_2 to an even greater r_3, then,

$$W(r_1, r_3) = W(r_1, r_2) + W(r_2, r_3), \qquad \text{(A4.3)}$$

and so we must also have,

$$(r_3 - r_1) < \kappa qq'/r^2 >_{[r1,\, r3]} =$$

$$(r_2 - r_1) < \kappa qq'/r^2 >_{[r1,\, r2]} + (r_3 - r_2) < \kappa qq'/r^2 >_{[r2,\, r3]} . \qquad \text{(A4.4)}$$

Finally, for an interval of zero distance, $[r_n, r_n]$, the average force must equal the actual force at r_n, i.e.,

$$< \kappa qq'/r^2 >_{[r\,n,\, r\,n]} \;=\; \kappa qq' / r_n^2. \qquad \text{(A4.5)}$$

The conditions (A4.2, 4, 5) uniquely determine the value of the average force for any interval. So if you can find *a* solution, you have found *the* solution! The solution is,

$$< \kappa qq'/r^2 >_{[r\,m,\, r\,n]} = \kappa qq' / r_m r_n \qquad \text{(A4.6)}$$

Can you check that (A4.6) actually satisfies each of the requirements, (A4.2), (A4.4) and (A4.5)?

If we now substitute (A4.6) into (A4.1) we find,

$$W(r_1, r_2) \;=\; (r_2 - r_1)\, \kappa qq' / r_1 r_2 \;=\; (\kappa qq' / r_1) \;-\; (\kappa qq' / r_2)\,. \qquad \text{(A4.7)}$$

This is the negative of the change in the electrostatic potential energy,

$$- \Delta E_{elec} \;=\; - (E_{elec.\,2} - E_{elec.\,1}) \qquad \text{(A4.8)}$$

if and only if we identify,

$$E_{elec.} \;=\; (\kappa\, qq' / r) + \text{const.} \qquad \text{(A4.9)}$$

This method would yield the same result in the case of two negative charges or the attractive case of one positive and one negative charge.

Since the constant makes no difference in calculations of work or *changes* in energy, it is by convention set equal to zero! This yields (2.22) with the interpretation that it is the potential energy relative to infinite separation of the charges.

A5: Argument for the work-energy theorem

The crux of the derivation of equation (3.10) from (3.9) is the recognition that for a constant resultant force we have a constant acceleration. That means that $v_{\mathbf{F}res}$ changes, proportionally to the elapsed time, from its initial value, $v_{\mathbf{F}res,\,1}$, to its final value, $v_{\mathbf{F}res,\,1} + \Delta v_{\mathbf{F}res}$, at the end of the time interval, Δt. The dependence on time is, mathematically, of the same form as the dependence of the force of a stretched spring on the length of the stretch. Consequently, as in that case, the average value, $<v_{\mathbf{F}res}>$, is half the sum of the initial and final value, i.e.,

$$<v_{\mathbf{F}res}> = (1/2)((v_{\mathbf{F}res,\,1} + \Delta v_{\mathbf{F}res}) + v_{\mathbf{F}res,\,1}) = v_{\mathbf{F}res,\,1} + (1/2)\,\Delta v_{\mathbf{F}res}\,. \qquad \text{(A5.1)}$$

Therefore, for a constant resultant force, we have,

$$W_{res} = m\,\Delta v_{\mathbf{F}res} <v_{\mathbf{F}res}> = m\,\Delta v_{\mathbf{F}res}\,(v_{\mathbf{F}res,\,1} + (1/2)\,\Delta v_{\mathbf{F}res})$$

$$= (m / 2)[2v_{\mathbf{F}res,\,1}\,\Delta v_{\mathbf{F}res} + \Delta v_{\mathbf{F}res}{}^2]\,. \qquad \text{(A5.2)}$$

But

$$[2v_{\mathbf{F}res,\,1}\,\Delta v_{\mathbf{F}res} + \Delta v_{\mathbf{F}res}{}^2] = (v_{\mathbf{F}res,\,1} + \Delta v_{\mathbf{F}res})^2 - v_{\mathbf{F}res,\,1}{}^2$$

$$= \Delta(v_{\mathbf{F}res}{}^2)\,. \qquad \text{(A5.3)}$$

So

$$W_{res} = (m / 2)\,\Delta(v_{\mathbf{F}res}{}^2). \qquad \text{(A5.4)}$$

Now, using Pythagoras' theorem, we have,

$$\Delta(|\mathbf{v}|^2) = \Delta(v_{\mathbf{F}res}{}^2 + v_{\mathbf{F}res,\,\perp}{}^2) = \Delta(v_{\mathbf{F}res}{}^2) + \Delta(v_{\mathbf{F}res,\,\perp}{}^2)$$

$$= \Delta(v_{\mathbf{F}res}{}^2)\,, \qquad \text{(A5.5)}$$

where the last equality holds because the constant force can't change the component of the velocity perpendicular to the direction of the force, i.e.,

$$\Delta(v_{\mathbf{F}res,\,\perp}{}^2) = 0\,. \qquad \text{(A5.6)}$$

Therefore, we finally have,

$$W_{res} = (m / 2)\,\Delta(|\mathbf{v}|^2)\,, \qquad \text{(A5.7)}$$

as (3.10) claims.