Class V: Relativity and motion (mostly)

1. Recapitulation

We have been discussing the transition from pre-relativisitc concepts of space and time, so called **Galilean space-time**, to the space and time concepts of STR, so called (for reasons that will be made clear later) **Minkowski space-time**. This transition is completely summed up in the transition from, the Galilean tranformation equations from one inertial frame to another, on one hand, to the Lorentz transformation equations, on the other. Let us re-assemble those equations here.

Consider an inertial frame, F', that is moving with velocity, **v**, relative to an inertial frame, F. Suppose a space-time interval is characterized by the displacement and time interval, ($\Delta\mathbf{x}$, Δt), in F. Separate $\Delta\mathbf{x}$ into the component perpendicular to **v**, $\Delta\mathbf{x}_\perp$, and the (algebraic) component, $\Delta x_\parallel$, parallel to **v**. Then, in the Galilean case, that same interval is characterized by ($\Delta\mathbf{x}'$, $\Delta t'$) in F' where,

$$\Delta x_\parallel' = \Delta x_\parallel - v\,\Delta t\ , \tag{1.1a}$$

$$\Delta\mathbf{x}_\perp' = \Delta\mathbf{x}_\perp\ , \tag{1.1b}$$

and

$$\Delta t' = \Delta t\ . \tag{1.1c}$$

In the relativistic, Lorentzian case, we have, instead,

$$\Delta x_\parallel' = \gamma\,[\Delta x_\parallel - v\,\Delta t]\ , \tag{1.2a}$$

$$\Delta\mathbf{x}_\perp' = \Delta\mathbf{x}_\perp\ , \tag{1.2b}$$

and

$$\Delta t' = \gamma\,[\Delta t - (v/c^2)\,\Delta x_\parallel]\ , \tag{1.2c}$$

where,

$$\gamma = (1 - (v/c)^2)^{-1/2}. \tag{1.3}$$

The relativistic, Lorentzian, set of equations becomes more and more nearly like the pre-relativistic, Galilean, set of equations as (v/c) becomes smaller and smaller. But, as we have seen, the Lorentz transformation equations contain bizarre, counterintuitive consequences of which there is not a hint in the Galilean transformation equations. In particular, since the Lorentz equations were obtained by assuming the principle of the invariance of vacuum light speed, they must be compatible with that principle. But that's not at all obvious at first glance. In fact, to show that the invariant light speed principle is actually a *consequence* of the Lorentz transformations, we have to determine what the velocity composition rule is in the relativistic case.

2. Relativistic velocity composition

We saw in **Class II** that the pre-relativistic velocity composition rule followed very simply from the Galilean transformation equations. Suppose the space-time interval, $(\Delta\mathbf{x}, \Delta t)$, consists of a spatial interval, $\Delta\mathbf{x}$, which is traversed by a moving object during the time interval, Δt. Then, from the perspective of the frame, F, the average velocity, $\mathbf{u}$, of the moving object is simply,

$$\mathbf{u} = \Delta\mathbf{x} / \Delta t \,. \tag{2.1a}$$

In F' the average velocity of the object is,

$$\mathbf{u'} = \Delta\mathbf{x}' / \Delta t', \tag{2.1b}$$

and, from (1.1), we have,

$$u_{\|}' = \Delta x_{\|}' / \Delta t' = \Delta x_{\|}' / \Delta t$$

$$= (\Delta x_{\|} - v\,\Delta t) / \Delta t = (\Delta x_{\|} / \Delta t) - v = u_{\|} - v\,, \tag{2.2a}$$

and

$$\mathbf{u}_{\perp}' = \Delta\mathbf{x}_{\perp}' / \Delta t' = \Delta\mathbf{x}_{\perp} / \Delta t = \mathbf{u}_{\perp}\,. \tag{2.2b}$$

These results combine into the single, simple equation,

$$\mathbf{u'} = \mathbf{u} - \mathbf{v}\,. \tag{2.2c}$$

For extremely short time intervals, Δt, these average velocities are, essentially, instantaneous velocities.

Now, to derive the velocity composition rules in the relativistic case we do the same thing again using the Lorentz transformation equations. This time, for $u_{\|}'$, we get (details in **Appendix**),

$$u_{\|}' = \Delta x_{\|}' / \Delta t' = \gamma\,[\Delta x_{\|} - v\,\Delta t] / \gamma\,[\Delta t - (v/c^2)\,\Delta x_{\|}]$$

$$= (u_{\|} - v) / [1 - (v\,u_{\|} / c^2)]\,, \qquad (2.3a)$$

The dependence of $u_{\|}'$ on $u_{\|}$, for fixed v, that this equation asserts, is schematically represented in **Figs. 2.1**.

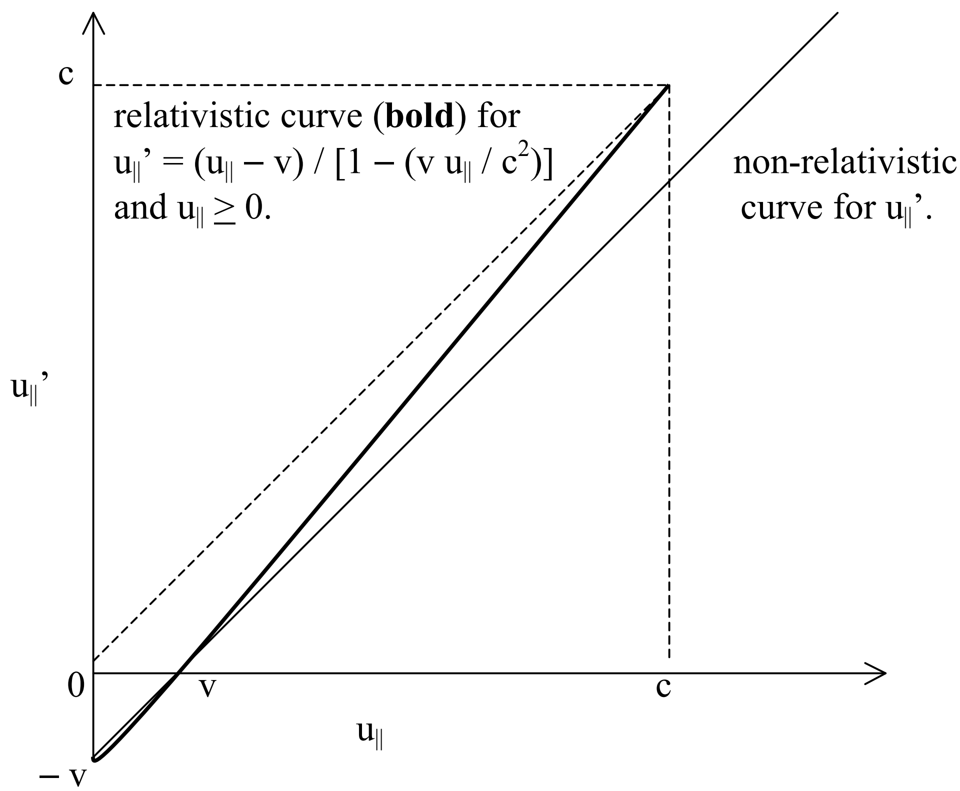


Fig. 2.1a: Schematic dependence of $u_{\|}'$ on $u_{\|}$ for a fixed, positive, value of $v = v_{F'F}$ and non-negative values of $u_{\|}$. The relativistic $u_{\|}'$is slightly more negative than the non-relativistic $u_{\|}'$ for $u_{\|}$ lying between zero and v. For $u_{\|} > v$, the relativistic $u_{\|}'$ is more and more positive than the non-relativistic $u_{\|}$, approaching the light speed value, c, as $u_{\|}$ approaches c.

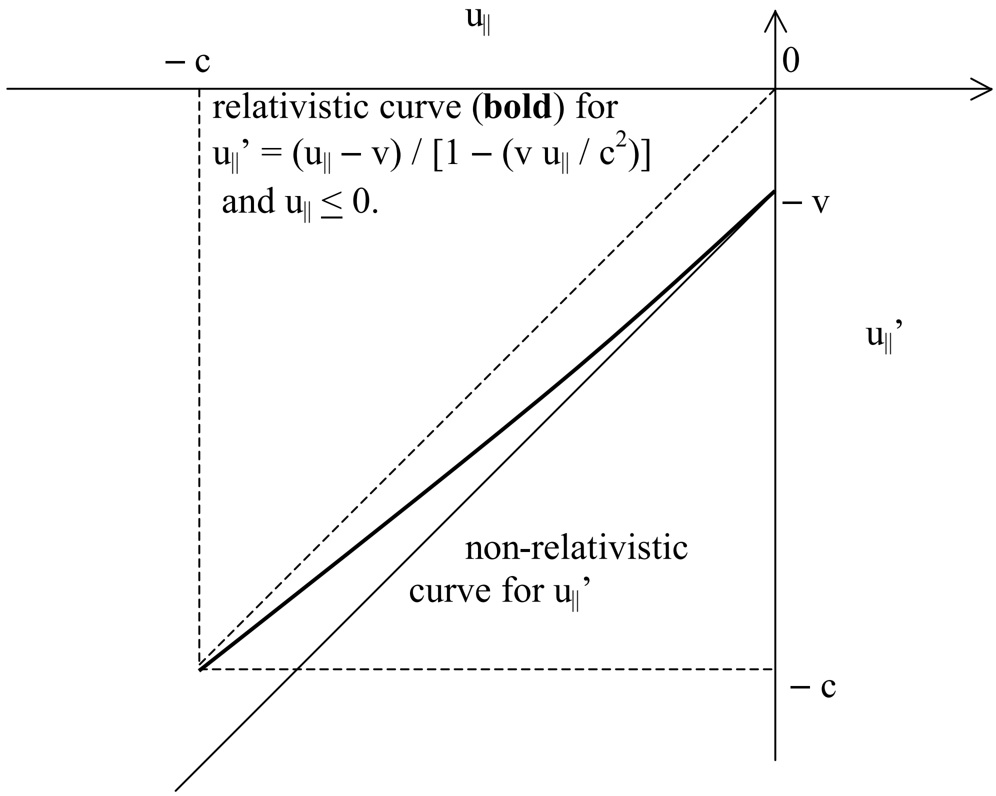


Fig. 2.1b: Dependence of $u_{||}'$ on $u_{||}$ for a fixed, positive, value of $v = v_{F'F}$ and negative values of $u_{||}$. The relativistic $u_{||}'$is more and more positive than the non-relativistic $u_{||}'$ for $u_{||}$ lying between zero and − c, again approaching the light speed value, − c, as $u_{||}$ approaches − c. Clearly, c is a critical value for the speed of *any* object and not *just* an invariant value for the vacuum speed of light.

The graph for the dependence of $u_{||}'$ on v for fixed values of $u_{||}$ would look very similar. In particular, just as in **Figs. 2.1** where $u_{||}'$ approaches $\pm$ c as $u_{||}$ approaches $\pm$ c, respectively, regardless of the value of v, so $u_{||}'$ also approaches $\pm$ c as v approaches $\pm$ c, respectively, regardless of the value of $u_{||}$. The speed, c, clearly has a special significance for the motion of any object and not just for the motion of light in vacuum!

Next we determine the relativistic composition rule for the components of velocity perpendicular to the relative motion between the reference frames. Here the modification from the non-relativistic result, (2.2b), is even more complicated. We find (details in **Appendix**),

$$\mathbf{u}_\perp' = \Delta\mathbf{x}_\perp'/\,\Delta t' = \Delta\mathbf{x}_\perp \,/\, \gamma\,[\Delta t - (v/c^2)\,\Delta x_{||}]$$

$$= \mathbf{u}_\perp\,(1 - (v/c)^2)^{1/2} \,/\, [1 - (v\,u_{||}\,/\,c^2)]\ . \qquad (2.3b)$$

In this case a graphical representation is not very informative, but a lot can be learned by using both (2.3a, b) to calculate the square of the magnitude of **u**', i.e. (by Pythagoras' theorem),

$$u'^2 = |\mathbf{u}'|^{\,2} = u_{||}'^2 + |\mathbf{u}_\perp'|^{\,2} = u_{||}'^2 + u_\perp'^2$$

$$= [(u_{||} - v)^{\,2} \,/\, (1 - (u_{||}v/c^2))^{\,2}] + u_\perp^2\,[(1 - (v/c)^2)\,/\,(1 - (u_{||}v/c^2))^{\,2}]. \qquad (2.4)$$

Manipulation of this expression (details in **Appendix**) leads to an upper and lower bound on u' given by (**Fig. 2.2**),

$$|\,u - v\,| \,/\, [1 - (u\,v\,/\,c^2)] \;\le\; u' \;\le\; (\,u + v\,) \,/\, [1 + (u\,v\,/\,c^2)]\ , \qquad (2.5)$$

and all of these expressions are less than or equal to c if u and v are. In fact if u approaches c for fixed v then both the lower and upper limit in (2.5) approach c and u' is forced to c. Thus c is an invariant speed for anything moving that fast, not only for light in vacuum.

The pre-relativistic upper and lower bounds on u' are just the numerators in (2.5), i.e.,

$$|\,u - v\,| \;\le\; u' \;\le\; (\,u + v\,)\ , \qquad (2.6)$$

and by comparison we see that the relativistic lower bound is never smaller and usually larger than | u – v |, while the relativistic upper bound is never larger and usually smaller than (u + v). Relativistic velocity composition is always more constrained than pre-relativistic velocity composition.

Finally, we note that the velocities, **u** and **u**', in this discussion, being the velocities of objects relative to inertial frames, need not be constant. They can be changing, accelerated velocities. The velocity, **v**, of the frame, F', relative to the frame, F, however, must be constant in time if both frames are to be inertial frames.

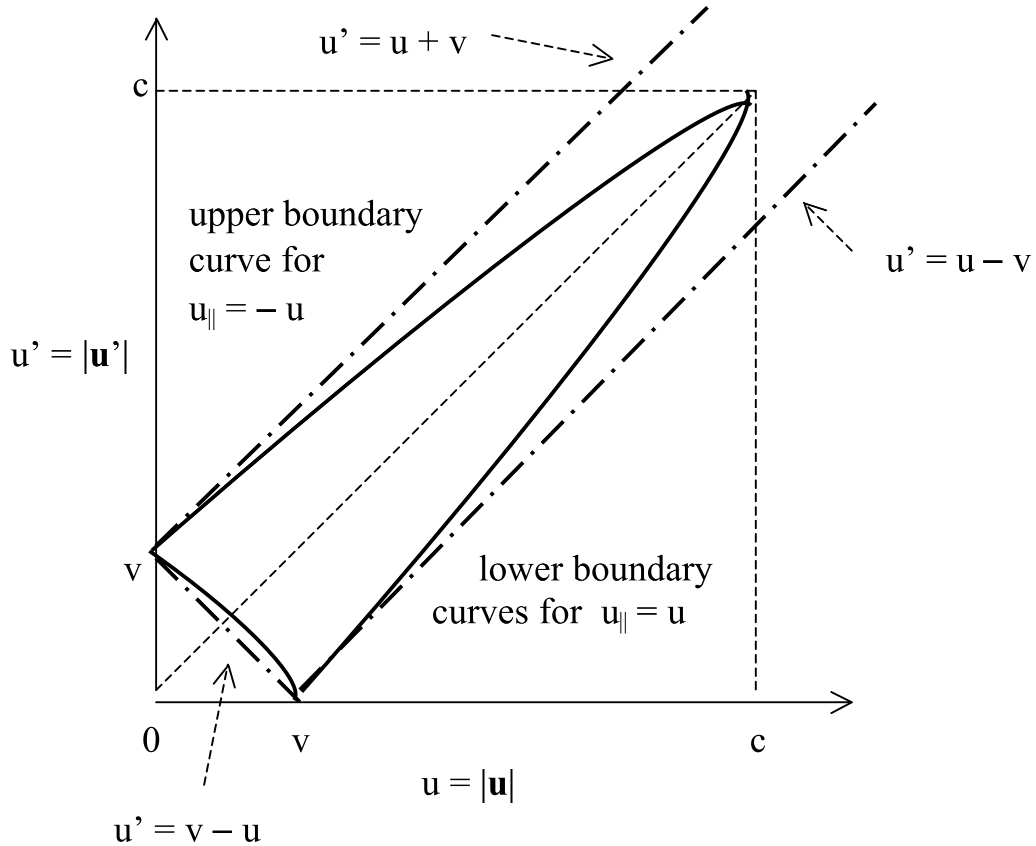


Fig. 2.2: Graphical representation of the dependence of the magnitude of **u**’, u’ = |**u**’|, on the magnitude of **u**, u = |**u**|. The bold-dashed lines are the pre-relativistic boundary of the possible values for u’ for a given u. The bold-solid curves are the relativistic boundary for u’ given u. In both cases the possible u’ values must lie on or within the boundaries. The upper boundary corresponds to **u** pointing in the opposite direction to **v**, i.e., $u_{||} = - u$ and $\mathbf{u}_{\perp} = 0$. The lower boundaries correspond to **u** pointing in the same direction as **v**, i.e., $u_{||} = u$ and $\mathbf{u}_{\perp} = 0$. The points lying *within* the boundaries correspond to $|\mathbf{u}_{\perp}| > 0$. The most important feature of the relativistic boundary is the increasing constraint on u’ as u approaches c, ultimately forcing u’ to c as well.

3: Momentum and mass

After analyzing, from the two principles of relativity and invariant light speed, what the consequences were for the properties of time, space and velocity measurements, Einstein was ready to tackle the modifications required in our understanding of the forces of nature and how they accelerate the motions of mass carrying objects, whether tiny particles or large, extended bodies. Unfortunately, we can not continue to follow his original approach to these topics as closely as we have up to now. The reason's are that he relied heavily on his understanding of electromagnetism for his analysis and that he suffers some lapses in judgement concerning the interpretation of his results. No genuine errors of calculation or analysis are involved. It's just that Einstein's heretofore unerring judgement of the meaning of the details is lacking and his assessments confuse the issues for awhile.

Clarification and simplification were brought to bear just under two years later by the dean of German physics, Max Planck, and in the next year, 1908, the Americans, Lewis and Tolman, showed that the required analysis could be completely divorced from any dependence on the details of electromagnetism. Our discussion will be based on the ideas of Lewis and Tolman.

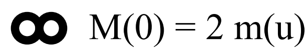
Lewis and Tolman argued that pre-relativistic physics was characterized by the great conservation principles of mass, momentum and energy, and the construction of relativistic physical principles should strive to preserve those conservation principles, if at all possible. The principles asserted that for any system that was *not interacting with its external environment*, the total mass, momentum and energy of that system would not change with time. The mass conservation principle was so basic it was almost never explicitly asserted, except by chemists who made great use of it, and the momentum and energy principles had been found to be related to the **homogeneity** of space and time, respectively, which appeared to be retained in Einstein's new conceptions of space and time. What homogeneity means in this context is that no location in space or moment in time is physically distinguishable from others by features of space and time, themselves.

So, starting with mass, m, and momentum, m**u**, for a particle moving with velocity, **u**, we ask whether the conservation of the total value of these quantities in a collision can hold in all inertial reference frames?

Before the collision in F

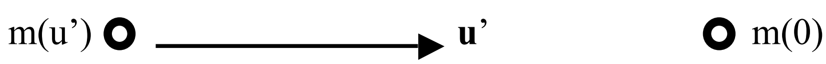
$\mathbf{P} = m(u)\,\mathbf{u} - m(u)\,\mathbf{u} = 0$

After the collision in F



$\mathbf{P} = 0$

Before the collision in F’



$\mathbf{P'} = m(u')\,\mathbf{u'}$

$u' = 2\,u\,/\,[1 + (u/c)^2]$

After the collision in F’

$M(u) = m(u') + m(0)$

$\mathbf{P'} = M(u)\,\mathbf{u} = m(u')\,\mathbf{u'}$

Hence, $m(u')\,(\mathbf{u'} - \mathbf{u}) = m(0)\,\mathbf{u}$, or $m(u') = m(0)\,/\,[1 - (u'/c)^2]^{1/2}$.

Fig. 3.1: Derivation of the velocity dependence of mass from the conservation of total momentum and total mass, with momentum of a particle defined by $\mathbf{p} = m(u)\,\mathbf{u}$.

Consider two particles, of the same mass, m, moving in opposite directions with the same speed, u, towards a head on collision, **Fig. 3.1**. Suppose that after the collision the particles do not rebound, but stick together, forming a composite particle with mass, M. If the particle velocities before the collision are **u** and – **u**, then, allowing for some speed dependence of the masses, the total momentum is,

$$m(u)\,\mathbf{u} - m(u)\,\mathbf{u} = 0. \tag{3.1a}$$

From conservation of momentum the compound particle left after the collision must have zero momentum and, therefore, must be at rest, i.e.,

$$M = M(0). \tag{3.1b}$$

On the other hand, the total mass is conserved through the collision and, therefore,

$$m(u) + m(u) = M(0). \tag{3.1c}$$

This is all from the perspective of an inertial frame, F.

Now consider the same collision from the perspective of a frame, F', moving with velocity, – **u**, relative to F, **Fig. 3.1**. In F' the initial particle moving with velocity **u** in F now, from (2.3a), has velocity,

$$\mathbf{u}' = 2\mathbf{u} \,/\, [1 + (u/c)^2] \tag{3.2}$$

while the initial particle moving with velocity – **u** in F is, in F', at rest. Furthermore, the final, compound particle, which was at rest in F is, in F', moving with velocity, **u**. Consequently, the momentum and mass conservation equations in F' are,

$$m(u')\,\mathbf{u}' = M(u)\,\mathbf{u}\,, \tag{3.3a}$$

and

$$m(u') + m(0) = M(u)\,. \tag{3.3b}$$

If we substitute the left side of (3.3b) into the right side of (3.3a), we get,

$$m(u')\,\mathbf{u}' = [m(u') + m(0)]\,\mathbf{u}. \tag{3.4a}$$

Next, collect together the m(u') terms to get,

$$m(u')\,(\mathbf{u}' - \mathbf{u}) = m(0)\,\mathbf{u}\,. \qquad (3.4b)$$

Now from (3.2) it follows that (details in **Appendix**),

$$\mathbf{u}' - \mathbf{u} = [1 - (u'/c)^2]^{1/2}\,\mathbf{u}\,, \qquad (3.5)$$

and if we substitute this result into the left side of (3.4b) and cancel the common factor of **u** on both sides, we find, $m(u')\,[1 - (u'/c)^2]^{1/2} = m(0)$, or

$$m(u') = m(0) / [1 - (u'/c)^2]^{1/2} = \gamma\, m_0\,. \qquad (3.6)$$

Having obtained this speed dependence of the mass of a particle from the consideration of a rather special kind of collision it would remain to be seen whether the result suffices to preserve mass and momentum conservation in all kinds of collisions between particles. In fact it does!

We note that attributing a speed dependence to the mass of an object completely undermines the interpretation of mass as measuring an intrinsic property of the object. For from the perspective of two inertial reference frames, F and F', moving relative to one another, any given object will have different velocities and, therefore, could have a different speed and mass. There was, however, a second way of interpreting mass before STR and this second way survived the acquisition of speed dependence by mass. The second interpretation was that mass measured the resistance a body offered to acceleration by forces. The common term for this resistance is **inertia**. Thus, the larger m is, the smaller the acceleration, **a**, will be for a given force, **f**. What (3.6) tells us is that an object's resistance to acceleration increases with the objects speed. This is still a bit counterintuitive because the increased speed can come from just viewing the object from a different reference frame and why should *that* change the resistance to acceleration? In part, the answer comes from realizing that when the mass change comes from changing the reference frame perspective, the acceleration is also being assessed from that new perspective. In any case these conclusions about mass, momentum, acceleration and force have since been corroborated many times to very high precision.

In particular, we notice that (3.6) requires that the mass becomes larger without limit as the speed, u, approaches vacuum light speed, c. Consequently, the forces required to increase the speed will also grow without limit as u approaches c. This increasing difficulty in accelerating massive particles closer and closer to light speed is encountered every day at all the world's high energy accelerator laboratories like Fermilab in Illinois or CERN in Europe. Nevertheless, speeds so close to light speed are commonly reached that the speedy electrons or protons, as the case may be, are then thousands of times more massive than the electrons or protons in your body. Still, *infinite* forces being unavailable, no particle or any other massive object can ever be accelerated beyond or even to the speed, c! The vacuum speed of light appears to be a universal speed limit!

4: Energy and mass

a: Kinetic energy

If a total force, **F**, acts on a particle with momentum, **p**, the momentum changes with time according to the equation,

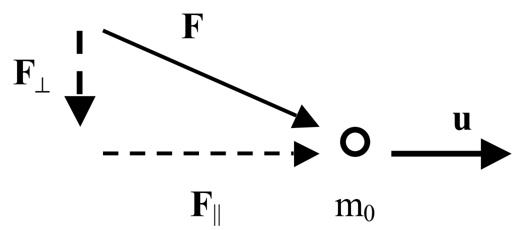
$$d\mathbf{p} / dt = \mathbf{F}, \qquad (4.1)$$

where the use of d rather than Δ to indicate a change in a quantity is to emphasize that *very small changes* are being considered so that they occur over very short time intervals, dt. This equation is fundamental in both pre-relativistic physics and STR. It becomes more than just a definition of force when we can identify features of an environment of a particle that, when present, do, indeed, result in the momentum of the particle changing with time.

In general **F** will have components both parallel and perpendicular to **p**. At any moment the parallel component will tend to change the magnitude of **p** while leaving the direction unchanged and the perpendicular component will tend to change the direction of **p** while leaving the magnitude unchanged. Consequently we can write,

$$dp / dt = F_{\|} . \qquad (4.2)$$

The product of the parallel component of the total force and the magnitude of the particle velocity, $F_{||}$ u, is called the **power** of the force or the **rate of doing work**. On the left side of the equation, the corresponding product, u dp / dt, is called the rate of change of **kinetic energy**, K, or energy of motion (**Fig. 4.1**). This is the first of many kinds of energy which we will not have the opportunity to discuss in any depth. What connects all kinds of energy is that they can all be used to generate forces that will do work. Energy is sometimes defined as *the capacity to do work* which, ultimately, means the capacity to push things around.



Non-relativisitically, i.e., with constant mass,

$$dK_{||} / dt = u F_{||} = u\, d(m_0 u) / dt = m_0 u\, du / dt = m_0\, d(u^2/2)/dt$$

$$= d(m_0 u^2 / 2) / dt .$$

Therefore, $$K = (1/2)\, m_0 u^2.$$

Fig. 4.1: Kinetic energy of motion of a particle with constant mass (non-relativistic).

In any case, for a particle in pre-relativistic physics, with a constant mass, m_0, and momentum, m_0 **u**, the rate of change of kinetic energy is,

$$dK / dt = u\, dp / dt = u\, d(m_0 u) / dt = m_0 u\, du / dt$$

$$= m_0\, d(u^2 / 2) / dt = d(m_0 u^2 / 2) / dt . \qquad (4.3)$$

It follows, if we take K to be zero when u = 0, that, pre-relativistically,

$$K = m_0 u^2 / 2 . \qquad (4.4)$$

So K, while dependent on mass, is more dependent on speed and appears to be a very different kind of quantity than mass. In particular, inert mass, by itself, can not do work. Such, at least, was the thinking before Einstein.

But now consider the same calculation for a particle in STR (**Fig. 4.2**).

$$dK / dt \ = \ u \, dp / dt \ = \ u \, d(m_0 \, u / [1 - (u/c)^2]^{1/2}) / dt$$

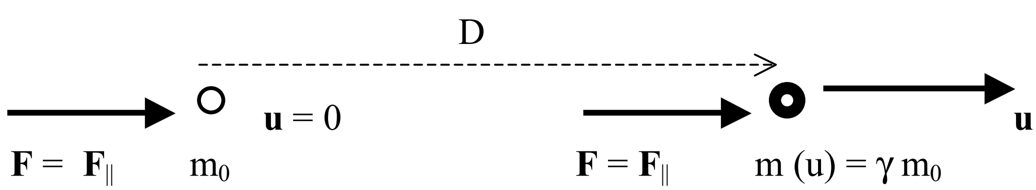
$$= \ d(m_0 \, c^2 / [1 - (u/c)^2]^{1/2}) / dt \ = \ d(m(u) \, c^2) / dt \ ; \quad \text{Trust me!} \qquad (4.5)$$

If we again take K to be zero when u = 0, we get,

$$K \ = \ m(u) \, c^2 - m_0 \, c^2 \ = \ m_0 \, c^2 \, [\gamma - 1], \qquad (4.6)$$

And now the variation in K is entirely accountable from the variation in mass, a much closer connection between kinetic energy and mass! The two results, (4.4) and (4.6) are compatible since for $u \ll c$ we have,

$$m(u) \ \simeq \ m_0 + (m_0 / 2) \, (u/c)^2. \qquad (4.7)$$



$$F_{||} \, D \ = \ K \ = \ m \, (u) \, c^2 - m_0 \, c^2 \ = \ m_0 \, c^2 \, [\gamma - 1]$$

Fig. 4.2: Constant total force acting on a relativistic particle starting from rest. The product of the force and the distance through which it acts is the kinetic energy which increases proportional to the change in mass.

b: Radiant energy

Consider a system, S, at rest in an inertial frame, F, which emits equal amounts of electromagnetic radiation energy, E, of definite frequency, f, in opposite directions simultaneously (**Fig. 4.3a**). Employing another one of Einstein's 1905 innovations (for which he received the Nobel prize in 1921) we can think of the emissions as each consisting of some number, N, of **photons** (Einstein's particle-like quantum constituents of light) each of which have energy, hf, and momentum, hf/c, where h is a constant (**Planck's constant**).

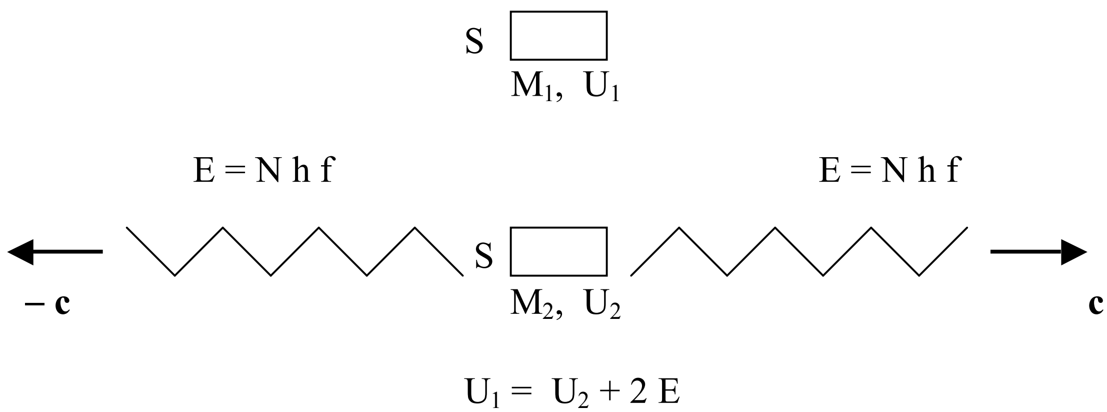


Fig. 4.3a: System, S, at rest in F, initially with mass, M_1, and energy, U_1, simultaneously emits N photons of energy hf in each of two opposite directions, thereby lowering its energy to U_2. The mass after the emissions is M_2.

The total energy emitted in one direction is just the sum of the energies of the photons moving in that direction and so,

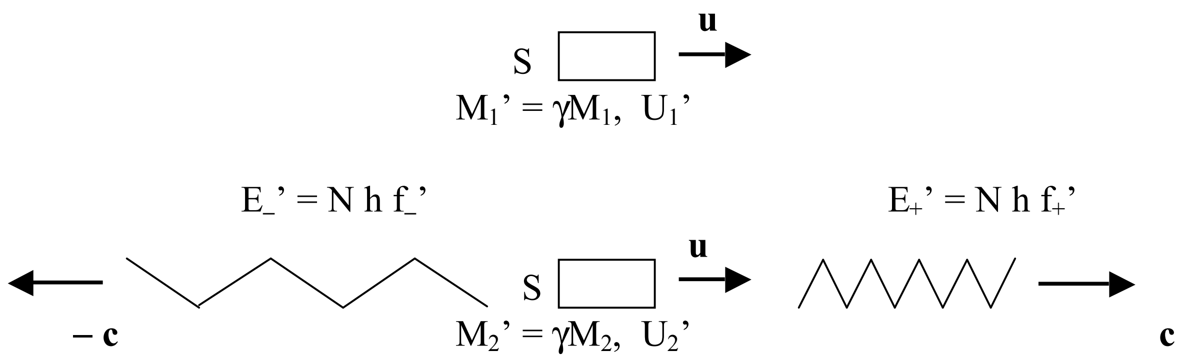
$$E = N\,hf \tag{4.8}$$

Since the same number of photons are moving in each of the two opposite directions, the momentum change of the system, S, is zero. If U_1 is the energy content of S before the emissions and U_2 the energy content afterwards, then energy conservation requires,

$$U_1 = U_2 + 2\,E\,. \tag{4.9}$$

Next we asses this process from the perspective of an inertial frame, F', in which S is moving in the direction of one of the emissions with speed, u

(**Fig. 4.3b**). We will use the abbreviation, $\beta := u/c$. Here the radiation in the direction of the motion of S is Doppler shifted up in frequency to the value,



$$U_1' = U_2' + E_+' + E_-'$$

$$(f_+' / f) = (E_+' / E) = [1 + \beta / 1 - \beta]^{1/2}, \quad \text{where} \quad \beta := u/c ,$$

$$(f_-' / f) = (E_-' / E) = [1 - \beta / 1 + \beta]^{1/2},$$

$$[1 + \beta / 1 - \beta]^{1/2} + [1 - \beta / 1 + \beta]^{1/2} = 2 / [1 - \beta^2]^{1/2} = 2 \gamma ,$$

$$E_+' + E_-' = 2 \gamma E ,$$

$$U_1' - U_1 = K_1 = M_1 c^2 [\gamma - 1] , \quad U_2' - U_2 = K_2 = M_2 c^2 [\gamma - 1] ,$$

$$K_1 = K_2 + 2 E [\gamma - 1] ,$$

$$M_1 c^2 = M_2 c^2 + 2 E .$$

Fig. 4.3b: System, S, in motion in F', initially with mass, M_1' and energy, U_1', emits N (Doppler shifted) photons of energy hf_+' in its direction of motion and energy hf_-' in the opposite direction, thereby lowering its energy to U_2'. The mass of S after the emissions is M_2'. The energy differences, U' – U are kinetic energies of S and comparison with the radiant energy emitted shows that the radiant energy carried off a mass equivalent.

$$f_+' = [1 + \beta / 1 - \beta]^{1/2} f, \qquad \text{where} \quad \beta := u/c, \tag{4.10a}$$

and the radiation in the opposite direction is Doppler shifted down in frequency to the value,

$$f_-' = [1 - \beta / 1 + \beta]^{1/2} f. \tag{4.10b}$$

But this increases the energy and momentum of the photons moving in the direction of S and decreases the energy and momentum of the photons moving in the opposite direction by just these Doppler shift factors.

Consequently, if U_1' and U_2' are the energy contents of S in F' before and after the emissions, then,

$$U_1' = U_2' + [1 + \beta / 1 - \beta]^{1/2}E + [1 - \beta / 1 + \beta]^{1/2}E$$

$$= U_2' + 2\,E / [1 - \beta^2]^{1/2} = U_2' + 2\,E\,\gamma. \tag{4.11}$$

Now $U_1' - U_1$ is just the energy change in S, before the emissions, due to acquiring motion with speed u. In other words it's the kinetic energy, K_1, due to the speed u before the emissions, i.e., from (4.6),

$$U_1' - U_1 = K_1 = M_1c^2\,[\,\gamma - 1]\,, \tag{4.12a}$$

where M_1 is the rest mass of S prior to the emissions.

Similarly, $U_2' - U_2$ is just the kinetic energy of S due to the speed u after the emissions,

$$U_2' - U_2 = K_2 = M_2c^2\,[\gamma - 1]\,, \tag{4.12b}$$

Where M_2 is the rest mass of S after the emissions. But if we subtract (4.9) from (4.11) we find,

$$U_1' - U_1 = U_2' - U_2 + 2\,E\,[\,\gamma - 1]\,, \tag{4.13}$$

And, upon substituting (4.12a, b) into (4.13), we have,

$$M_1 c^2 [\gamma - 1] = M_2 c^2 [\gamma - 1] + 2 E [\gamma - 1] , \tag{4.14a}$$

or, canceling out the common factor of $[\gamma - 1]$,

$$M_1 c^2 = M_2 c^2 + 2 E , \tag{4.14b}$$

and the emitted radiation energy, 2 E, (in F) comes from the change in mass of S, $(M_1 - M_2)$, multiplied by the square of c.

If we recall that in F' the two opposing radiation pulses had different momentum magnitudes and, therefore, must have transmitted momentum to S as a reaction. But the speed of S in F' never changes, so where did the momentum go? Remember that the momentum of S is $\gamma M u$. Since u and γ don't change all the momentum transfer to S is due to the change in M. The net momentum to the right, carried off by the radiation, resulted in a *decrease* in the momentum to the right of S due to the decrease in mass.

c: Heat energy and energy in general

As bodies are heated, the heat energy goes primarily into kinetic energy of the molecules that compose the bodies. The temperature of a body is a measure of the average kinetic energy of the molecules. Since, as we've seen, the kinetic energy of moving objects is proportional to the change in the mass of the object it follows that heat energy is also accompanied by a proportional change in mass.

Since all the different forms of energy are largely convertible into one another with only the conservation of the total amount as a constraint, it is clear that consistency demands that all forms of energy are associated with corresponding quantities of mass in accordance with the famous equation,

$$\mathbf{E = M c^2}.$$

Appendix

a: derivation of (2.3a)

$u_{\|}' = \Delta x_{\|}' / \Delta t' = \gamma\,[\Delta x_{\|} - v\,\Delta t] / \gamma\,[\Delta t - (v/c^2)\,\Delta x_{\|}]$

$= [\Delta x_{\|} - v\,\Delta t] / [\Delta t - (v/c^2)\,\Delta x_{\|}] = [(\Delta x_{\|} / \Delta t) - v] / [1 - (v/c^2)(\Delta x_{\|} / \Delta t)]$

$$= (u_{\|} - v) / [1 - (v\,u_{\|} / c^2)]\ .$$

b: derivation of (2.3b)

$\mathbf{u}_{\perp}' = \Delta \mathbf{x}_{\perp}' / \Delta t' = \Delta \mathbf{x}_{\perp} / \gamma\,[\Delta t - (v/c^2)\,\Delta x_{\|}]$

$= (\Delta \mathbf{x}_{\perp} / \Delta t) / \gamma\,[1 - (v/c^2)(\Delta x_{\|} / \Delta t)] = \mathbf{u}_{\perp} / \gamma\,[1 - (v\,u_{\|} / c^2)]$

$$= \mathbf{u}_{\perp}\,(1 - (v/c)^2)^{1/2} / [1 - (v\,u_{\|} / c^2)]\ .$$

c: derivation of upper and lower bounds on |u’|

$u'^2 = |\mathbf{u}'|^{\,2} = u_{\|}'^2 + |\mathbf{u}_{\perp}'|^{\,2} = u_{\|}'^2 + u_{\perp}'^2$

$= [(u_{\|} - v)^{\,2} / (1 - (u_{\|}v/c^2))^{\,2}] + u_{\perp}^{\,2}\,[(1 - (v/c)^2) / (1 - (u_{\|}v/c^2))^{\,2}].$

$= [u^2 + v^2(1 - (u_{\perp}/c)^2) - 2u_{\|}v] / (1 - (u_{\|}v/c^2))^{\,2}$

$= [u^2 + v^2(1 - u^2/c^2) + v^2u_{\|}^2/c^2 - 2u_{\|}v] / [1 - (u_{\|}v/c^2)]^2$

$= \{\,[u^2 + v^2 - v^2u^2/c^2 - c^2] + c^2[1 - (u_{\|}v/c^2)]^2\,\} / [1 - (u_{\|}v/c^2)]^2$

$= c^2\,\{1 - (1 - u^2/c^2)(1 - v^2/c^2) / [1 - (u_{\|}v/c^2)]^2\,\}.$

Now the smallest and largest possible values of $[1 - (u_{\|}v/c^2)]^2$ are $[1 - (uv/c^2)]^2$ and $[1 + (uv/c^2)]^2$, corresponding to $u_{\|} = u$ and $u_{\|} = -u$, respectively. Therefore,

$$u'_{min}{}^2 = c^2\ \{1 - (1 - u^2/c^2)(1 - v^2/c^2) / [1 - (uv/c^2)]^2\ \}$$

$$= (u - v)^2 / [1 - (uv/c^2)]^2,$$

$$u'_{max}{}^2 = c^2\ \{1 - (1 - u^2/c^2)(1 - v^2/c^2) / [1 + (uv/c^2)]^2\ \}$$

$$= (u + v)^2 / [1 + (uv/c^2)]^2 .$$

d: derivation of (3.5)

We start from (3.2),

$$\mathbf{u}' = 2\,\mathbf{u} / [1 + (u/c)^2]$$

From this it follows that (we use $\beta := u/c$ and $\beta' := u'/c$),

$$1 - \beta'^2 = 1 - [4\beta^2 / (1 + \beta^2)^2]$$

$$= [(1 + \beta^2)^2 - 4\beta^2] / (1 + \beta^2)^2 = (1 - \beta^2)^2 / (1 + \beta^2)^2$$

$$= [(1 - \beta^2) / (1 + \beta^2)]^2,$$

or

$$[1 - \beta'^2]^{1/2} = (1 - \beta^2) / (1 + \beta^2)$$

$$= [2 / (1 + \beta^2)] - 1.$$

Hence,

$$\mathbf{u}' - \mathbf{u} = [1 - (u'/c)^2]^{1/2}\,\mathbf{u} .$$