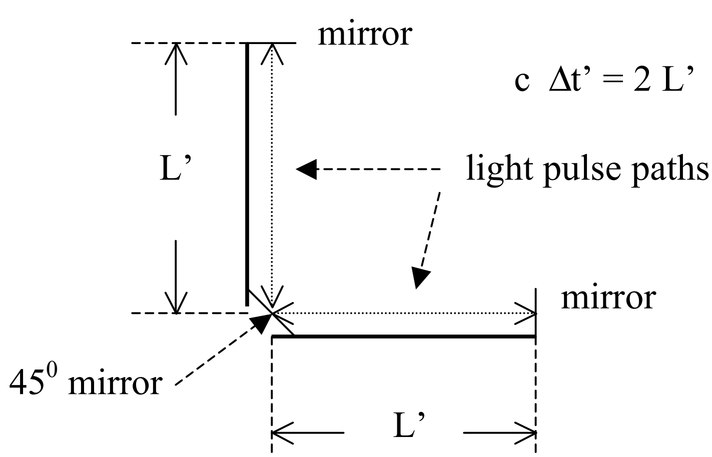
Class IV: Relativity and space (mostly)

1. Length contraction

Having learned that 'moving' clocks fall behind compared to the 'stationary' clocks they pass as they go, we now ask what, if anything, happens to the lengths of 'rigid' rods when they 'move'? The invariant light speed has to do with how much distance something (light in vacuum) covers in a given time. If that principle combined with relativity can alter the flow of time in moving reference frames, perhaps it also alters spatial distances in moving reference frames.

To find out we imagine an idealized 'light clock', ticking away as a consequence of light pulses bouncing back and forth between mirrors at the ends of 'rigid' rods (**Fig. 1.1**). We already know how the ticking changes when the clock moves (the ticking slows down) and from that we can find out how the rod lengths (the distances between the mirrors) change.



$$c\ \Delta t' = 2\ L' \qquad (1.1)$$

Fig. 1.1: Structure of light clock from the perspective of its rest frame, F'. The time interval between consecutive reflections from the 45^0 mirror at the join of the vertical and horizontal rods is $\Delta t'$.

A precaution, however, before we jump right in. In our previous analysis, $\Delta t'$ was a time interval measured by a single clock, C', at rest in F' and moving in F. Δt, on the other hand, was a time interval measured by a pair of synchronized clocks, C_1 and C_2, each read as C' passed by them. $\Delta t'$ is always smaller than Δt.

When we calculate the time of a cycle of reflections in our 'light clock' at rest, that will correspond to $\Delta t'$ since one clock, coincident with our 'light clock', can measure it. And when we calculate the time of such a cycle for our 'light clock' in motion, that will correspond to Δt, since the time will be read off the 'stationary' synchronized clocks the light clock passes just at the end points of the cycle. Without these comments being kept in mind, it's easy to mistakenly switch the roles of $\Delta t'$ and Δt (It will probably remain easy even with these comments).

From the perspective of the inertial frame, F, in which the light clock is moving with speed, v, horizontally to the right (**Fig. 1.2**), the light pulses travel longer paths between consecutive reflections from the 45^0 mirror. But, according to Einstein, they still move with the universal speed, c.

Consequently the time interval in F between consecutive reflections at the central mirror, Δt, is longer than $\Delta t'$, the exact relationship being given by (2.8) in **Class III**, i.e.,

$$\Delta t = \Delta t' / [1 - (v/c)^2]^{1/2} = \gamma\, \Delta t' \qquad (1.2)$$

From the top diagram in **Fig. 1.2** the light pulse moving between the central and top mirror now, in F, moves along two oblique paths, each the hypotenuse of a right triangle, each taking half the time interval and so, each of length, c (Δt /2). The base of these triangles is due to the motion of the light clock and so has length, v (Δt /2). The height of the triangles is the vertical rod length, $L_\perp$, and so, by Pythagoras' theorem,

$$c^2\, (\Delta t/2)^2 = v^2\, (\Delta t/2)^2 + L_\perp^{\;2}, \qquad (1.3a)$$

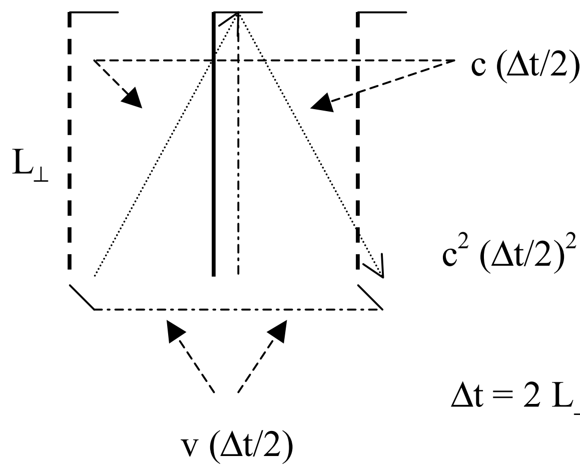
or

$$\Delta t = 2\, L_\perp / [c^2 - v^2]^{1/2} .= \gamma\, 2L_\perp/c \qquad (1.3b)$$

Comparing this to (1.1) and (1.2) we find that

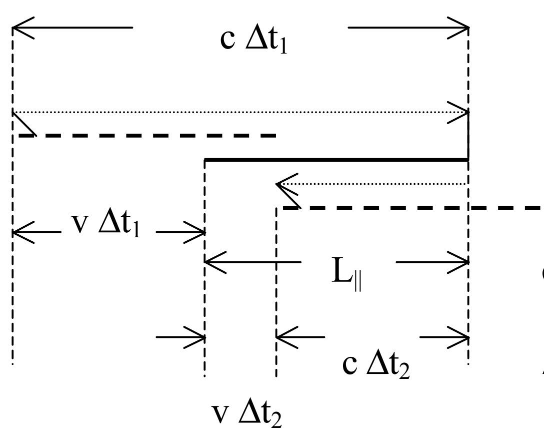
$$L_\perp = L', \qquad (1.4)$$

and *the dimensions of rods in directions perpendicular to their uniform motion do not change!*



$$c^2\,(\Delta t/2)^2 = L_\perp{}^2 + v^2\,(\Delta t/2)^2$$

$$\Delta t = 2\,L_\perp / [c^2 - v^2]^{1/2}$$



$$c\,\Delta t_1 = L_{||} + v\,\Delta t_1$$

$$c\,\Delta t_2 = L_{||} - v\,\Delta t_2$$

$$\Delta t = \Delta t_1 + \Delta t_2$$

$$= 2\,c\,L_{||} / [c^2 - v^2]$$

Fig. 1.2: The light clock from the perspective of the inertial frame, F, in which the clock moves with speed, v, to the left. The top diagram describes the motion of the vertical rod and the up and down light pulse path. The bottom diagram describes the motion of the horizontal rod and the right and left light pulse path.

From the lower diagram in **Fig. 1.2** the light pulse first moves to the right chasing a receding mirror for a time, Δt_1, so that

$$c\,\Delta t_1 = L_{\|} + v\,\Delta t_1\,, \qquad (1.5a)$$

or

$$\Delta t_1 = L_{\|} / (c - v). \qquad (1.5b)$$

The reflected light pulse then moves to the left towards an oncoming mirror for a time, Δt_2, so that

$$c\,\Delta t_2 = L_{\|} - v\,\Delta t_2\,, \qquad (1.5c)$$

or

$$\Delta t_2 = L_{\|} / (c + v)\,. \qquad (1.5d)$$

These equations yield,

$$\Delta t = \Delta t_1 + \Delta t_2 = 2\,c\,L_{\|} / [c^2 - v^2], \qquad (1.6)$$

which, when combined with (1.1) and (1.2), yields,

$$L_{\|} = [1 - (v/c)^2]^{1/2}\,L', \qquad (1.7)$$

i.e., *the dimensions of rods in the direction of uniform motion are diminished from their value at rest by the factor $[1 - (v/c)^2]^{1/2}$*. This is called **length contraction**. The length of a rod in its rest frame is called its **proper length**.

Just as time dilation implies that of two relatively moving inertial observers, each would assess the clocks of the other to be slower than their own, so length contraction implies that each of the observers would assess the standard measuring rods of the other to be shorter than their own by amounts depending on the orientation of the rods with respect to their relative motion. Remember the **Class I** discussion of Jean judging me and my passing car to be shrunk while I judge Jean and the roadside telephone poles to be thinner and closer together.

In both the preceding italicized assertions we made explicit reference to the uniformity of the motion of the moving rods. This is essential for the derivations since without the moving clocks of (1.1) or the moving rods of (1.4) and (1.7) being at rest in *some* inertial frame, the derivations would not go through. It is not uncommon for students and others to apply these results to cases where v is a *changing* instantaneous speed. That is an error!

Finally, we notice in passing that in our light clock, while the total cycle time for each light pulse, the vertical and the horizontal, are equal in F' and in F, the time intervals from the 45^0 mirror to the next mirror for each light pulse are equal in F' but not in F. That's because the starting location and ending location, at the 45^0 mirror, is the same for each light pulse and coincident simultaneity is frame independent. The reflections from the end mirrors, however, are separated events and, while simultaneous in F', are not simultaneous in F, i.e., $\Delta t_1 > \Delta t/2 > \Delta t_2$.

2. The docking starship paradox

Just as the counterintuitive nature of the relativity of simultaneity and time dilation suggests the paradoxical nature of different aging of identical but differently traveling twins, so the counterintuitive nature of length contraction suggests a paradoxical feature of a starship passing through a maintenance dock (**Fig. 2.1a, b**). If the relative speed between the starship

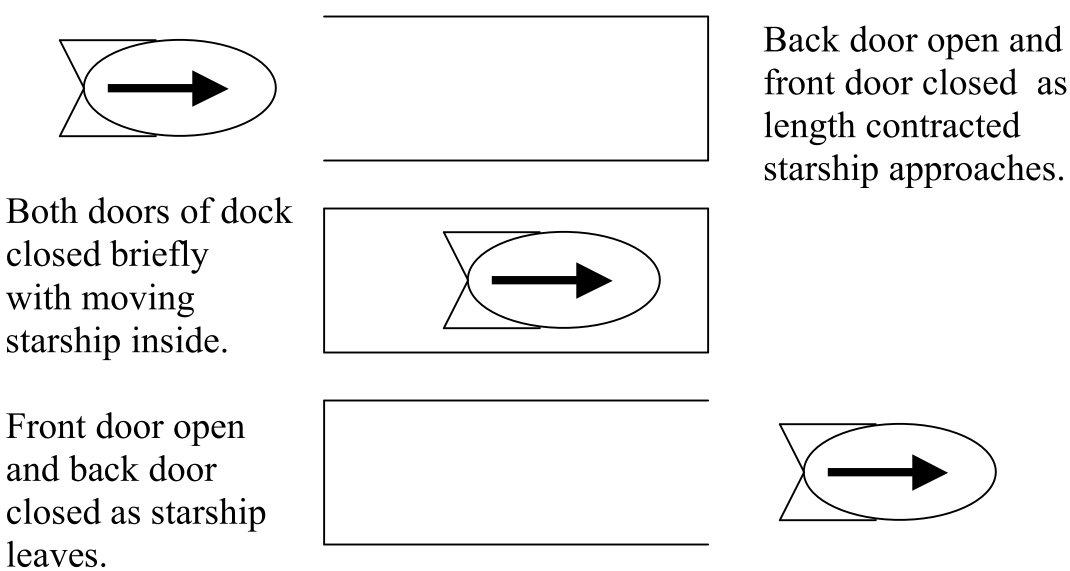
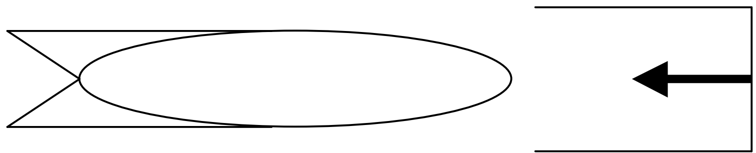
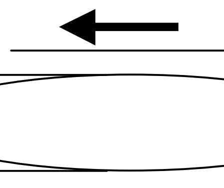


Fig. 2.1a: Dock-workers perspective on rapidly moving length contracted starship passing through dock and briefly permitting simultaneous closing of both dock doors with starship inside. There is NO time when both doors are open!

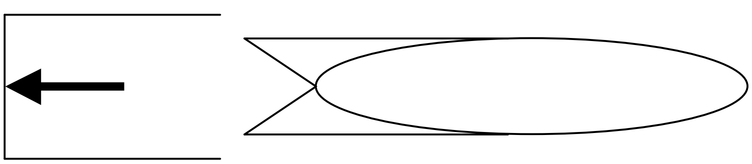
and the dock is high enough (for dock and starship of equal proper length, any speed will do) then for the dock-workers there will be a time interval, however short, during which the length contracted starship is completely inside the dock and the dock doors can be briefly closed (**Fig. 2.1a**). For the starship crew, however, the length contracted dock can never contain the starship, even for an instant, and there is no time when both dock doors can be closed with the starship inside (**Fig. 2.1b**)!



Starship crew wait anxiously for front door of approaching length contracted dock to open.



Starship crew watch length contracted dock slip over starship with both doors open.



Starship crew view closure of back door of length contracted dock as it recedes from starship.

Fig. 2.1b: Starship crews perspective as length contracted dock slips over starship which is too long for containment. There is NO time at which both dock doors are closed!

The dichotomy is maximized if one assumes the front door of the dock is initially closed while the back door is initially open. At some time the front door is opened and remains so and at some time the back door is closed and remains so. One can then adjust the timing of the opening and closing so that in the dock inertial frame there is no time at which both doors are open and in the starship inertial frame there is no time at which both doors are closed!

The resolution of the apparent paradox is that since the front and back doors are spatially separated, the temporal order of the front door opening (bow-front door coincidence) and the back door closing (stern-back door coincidence) can be frame dependent (**Fig. 2.2**). In the dock frame the back door is closed before the front door is opened, briefly enclosing the starship.

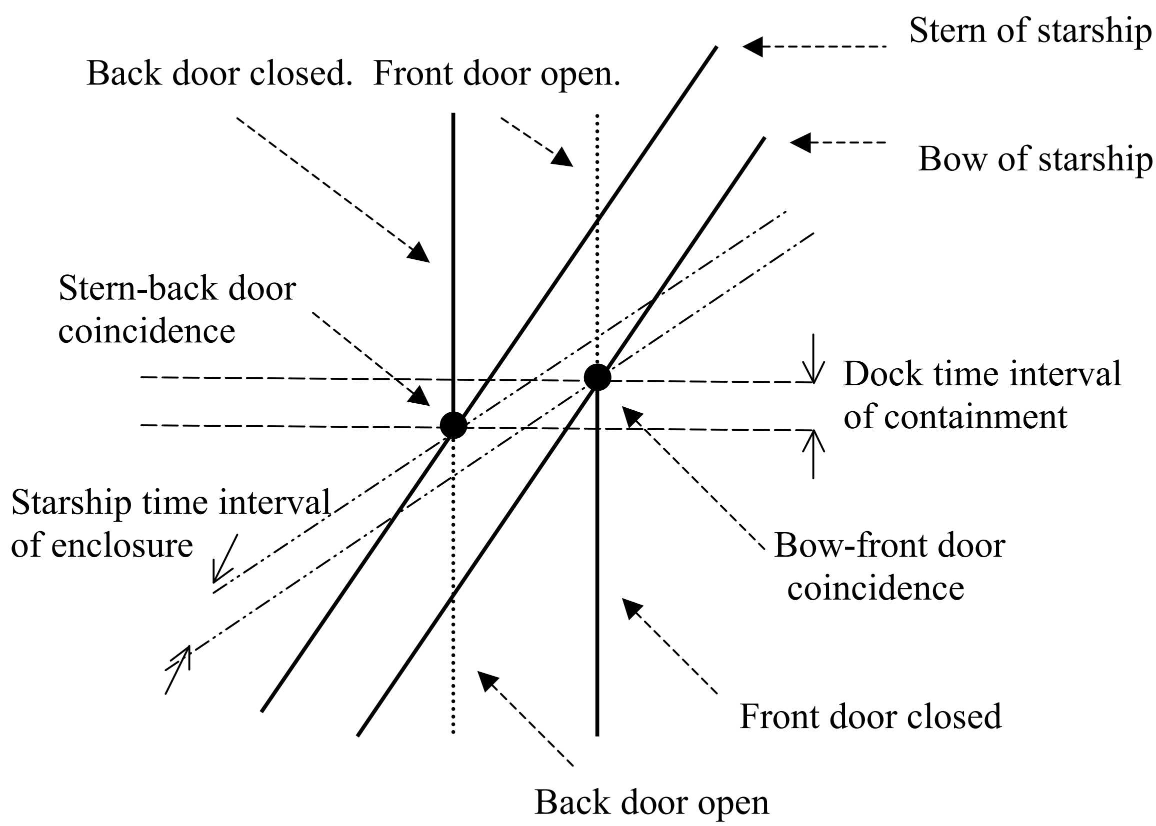


Fig. 2.2: Space and time diagram for relatively moving dock and starship relationships. Time evolves vertically for the dock frame and on a slope for the starship frame. The stern-back door coincidence precedes the bow-front door coincidence in the dock frame and the order is reversed in the starship frame.

In the starship frame the front door is opened before the back door is closed and the starship is never enclosed! This is every bit as counterintuitive as the differently ageing identical twins and every bit as permitted, indeed demanded by STR! As with the twins, a space and time diagram (**Fig. 2.2**) enables us to see a bit more deeply into HOW it can be so.

3. Classifying Space and Time intervals

When, in the previous class, we discussed time intervals for relatively moving inertial clocks, we were, in effect, examining pairs of localized events (marking the beginning and end of a time interval) that could be *connected* by a single inertial clock. The sense of the *connection* is that there could exist an inertial clock for which both events occurred at the spatial location of that one clock. In the inertial frame of that clock the spatial interval between the events is zero, $\Delta\mathbf{x}' = 0$. The time interval between the events, however, while frame dependent, is non-zero in all frames if non-zero in any frame. Furthermore, if one of the events is later than the other in one inertial frame it is the later of the two in all inertial frames. These conclusions are all contained in the equation (1.2), above.

Such pairs of events are called **time-like separated** and the space and time intervals between them are also called time-like. *For a time-like separated pair of events there is always some inertial frame in which the events are spatially coincident and separated only by time.*

The spatial interval between the events, in the inertial frame in which the clock moves with velocity, **v**, is, from (1.2), given by,

$$\Delta\mathbf{x} = \mathbf{v}\,\Delta t = \mathbf{v}\,\gamma\Delta t'. \qquad (3.1)$$

In *this* class we have been discussing the lengths of relatively moving inertial rods and thus, in effect, have been examining pairs of localized events (marking the endpoints of a distance interval) that could be *connected* by a single inertial rod. The sense of the *connection* is that there could exist an inertially moving rod for which the events occur simultaneously at the endpoints of the rod, $\Delta t = 0$. In such a case the distance between the events is the length of the rod. Both the time interval and the distance interval between the events will be frame dependent and only in the rest frame of the rod, where simultaneity may well be lost, is the distance interval guaranteed to, once again, be the length of the rod. But if the distance interval is non-zero in the 'simultaneity' frame it will be non-zero in all frames and the earlier of the events in the rest frame will be the event located at the leading end of the rod in the 'simultaneity' frame.

Such pairs of events are called **space-like separated** and the space and time intervals between them are also called space-like. *For any pair of space-like separated events there is always some inertial frame in which the events are simultaneous and separated only by distance.*

Suppose F is the inertial frame in which a pair of space-like separated events are simultaneous, $\Delta t = 0$, and the spatial separation between them is

$$\Delta \mathbf{x} = (\Delta x_{\|}, \Delta \mathbf{x}_{\perp}) \qquad (3.2)$$

where $\Delta x_{\|}$ is the component of $\Delta \mathbf{x}$ that is parallel to the velocity, $\mathbf{v}$, of a frame, F', relative to F and $\Delta \mathbf{x}_{\perp}$ is the component perpendicular to $\mathbf{v}$.. In F' the events may not be simultaneous. Can we determine the time interval, $\Delta t'$, between them in F'? From our results (1.4) and (1.7) we can!

Imagine two rods, one moving in F with velocity, $\mathbf{v}$, (rod A) and the other at rest in F (rod B), but both having the same length in F and arranged so that their respective endpoints are coincident simultaneously at the locations of the space-like separated events (**Fig. 3.1a**). In F' rod A will be at rest and rod B will be moving with velocity, $-\mathbf{v}$. Consequently, in F' the rods will not have the same length or the same orientation (**Fig. 3.1b**). The reason is that rod A will have it's parallel dimension, $\Delta x'_{\|}(A)$, in F' larger than it is in F since it is moving in F and at rest in F', while rod B will have it's parallel dimension, $\Delta x'_{\|}(B)$, in F' shorter than it is in F since it is at rest in F and moving in F'. From (1.7),

$$\Delta x'_{\|}(A) = \gamma \, \Delta x_{\|} \, , \qquad (3.3a)$$

$$\Delta x'_{\|}(B) = \gamma^{-1} \, \Delta x_{\|} \; . \qquad (3.3b)$$

The perpendicular components, from (1.4), will not change,

$$\Delta \mathbf{x}'_{\perp}(A) = \Delta \mathbf{x}'_{\perp}(B) = \Delta \mathbf{x}_{\perp} \; . \qquad (3.3c)$$

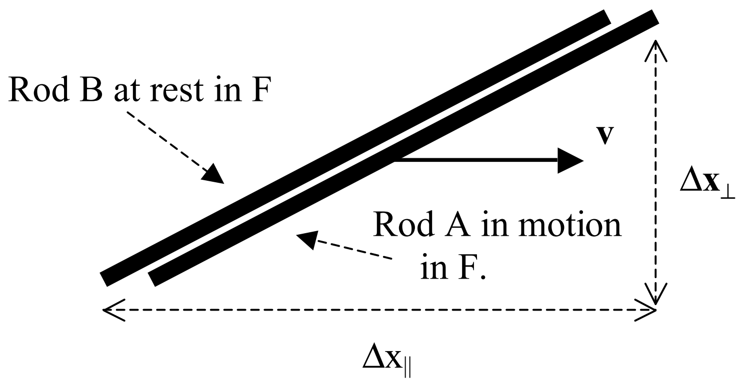


Fig. 3.1a: Rods A and B in frame, F, *just* after the simultaneous coincidence of their respective endpoints as A, moving with velocity, **v**, passes B.

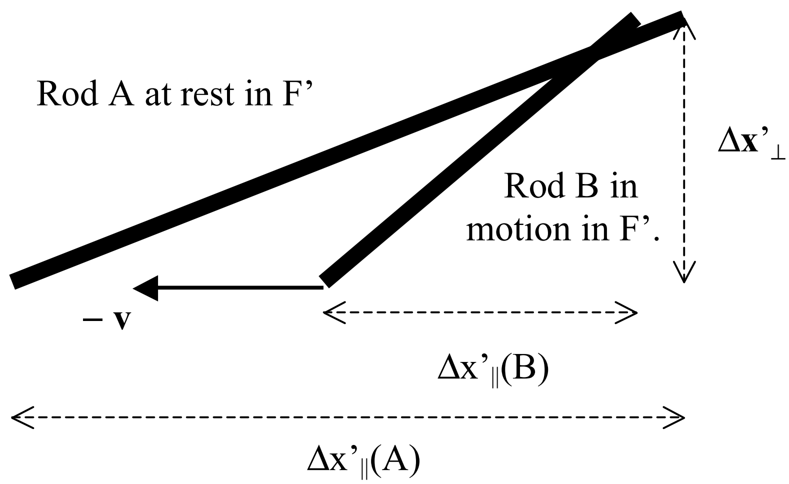


Fig. 3.1b: Rods A and B in frame, F', *just* after the initial coincidence of their upper endpoints as B, moving with velocity, – **v**, passes A. The coincidence of their lower endpoints will occur $[\Delta x'_{||}(A) - \Delta x'_{||}(B)]/v$ later.

In F', then, the upper endpoints (**Fig. 3.2b**) will meet first and the lower endpoints will meet later by the amount

$$\Delta t' = [\Delta x'_{||}(A) - \Delta x'_{||}(B)] / (-v)$$

$$= [\gamma\, \Delta x_{||} - \gamma^{-1}\, \Delta x_{||}] / (-v) = [\gamma^{-1} - \gamma](\Delta x_{||} / v)$$

$$= -\gamma(v/c^2)\, \Delta x_{||}\,. \qquad (3.4)$$

The minus sign in the final result is due to the fact that if we measure $\Delta x_{||}$ as positive going from the lower-left endpoint to the upper-right endpoint then $\Delta t'$ will be positive if the lower-left coincidence precedes the upper-right coincidence. But the reverse is the case. Hence the minus sign.

A third kind of pair of events is a pair that can be connected by a light pulse moving in vacuum. In these cases the time interval, Δt, can never be zero since it will always take some time for light to pass from one of the events to the other. And the distance between the events, $|\Delta \mathbf{x}|$, can never be zero since light in vacuum always moves with the same speed. Instead, the relationship,

$$c^2\Delta t^2 - \Delta \mathbf{x}^2 = 0, \qquad (3.5a)$$

is always satisfied, i.e., it is satisfied in every inertial frame. Pairs of events such as these and the space and time intervals between them are called **light-like**.

Now suppose we examine the quantity on the left side of (3.5a) for time-like and space-like intervals. For the time-like case we have, from (1.2) and (3.1),

$$c^2\Delta t^2 - \Delta \mathbf{x}^2 = c^2\, \gamma^2\, \Delta t'^2 - \mathbf{v}^2\, \gamma^2\, \Delta t'^2 = [c^2 - v^2 / 1 - (v/c)^2]\, \Delta t'^2$$

$$= c^2\, \Delta t'^2 = c^2\, \Delta t'^2 - 0 = c^2\Delta t'^2 - \Delta \mathbf{x}'^2. \qquad (3.5b)$$

The quantity, $c^2\Delta t^2 - \Delta \mathbf{x}^2$, appears to have the same value in any inertial frame (since v is arbitrary) as it does in the frame where the events are spatially coincident. In the time-like case this quantity is positive, not zero as in (3.3a), but, as in the light-like case, it is an *invariant*.

Finally, for space-like pairs of events and their intervals, using (1.2), (3.2), (3.3a) and (3.3c), for rod A, and (3.4), we find,

$$c^2\Delta t'^2 - \Delta\mathbf{x}'^2 = c^2\,\gamma^2(v/c^2)^2\,\Delta x_{||}^2 - \gamma^2\,\Delta x_{||}^2 - \Delta\mathbf{x}_{\perp}^2$$

$$= \gamma^2\,[(v/c)^2 - 1]^{1/2}\,\Delta x_{||}^2 - \Delta\mathbf{x}_{\perp}^2 = -\,\Delta\mathbf{x}^2$$

$$= 0 - \Delta\mathbf{x}^2 = c^2\Delta t^2 - \Delta\mathbf{x}^2, \qquad (3.5c)$$

and once again the quantity, $c^2\Delta t^2 - \Delta\mathbf{x}^2$, appears to have the same value in any inertial frame as it does in the frame where the events are simultaneous. In this space-like case the quantity is negative, but as for light-like and time-like intervals it is *invariant*.

In fact, this quantity is so important in STR that it is called (confusingly) the **invariant interval**, notwithstanding its construction out of the difference of *squares* of intervals. Since the value of the 'invariant interval' can only be positive, negative or zero, the classification of pairs of events and their intervals into the three types of time-like, space-like and light-like is exhaustive. No other kinds of pairs of events and intervals exist. The invariance of the 'invariant interval' is what replaces the invariant time interval of pre-relativistic physics. See **Fig. 3.2** for a space and time diagrammatic representation of time-like, space-like and light-like intervals.

Once again one might wonder why we could have believed for so long that it was Δt that was invariant rather than $c^2\Delta t^2 - \Delta\mathbf{x}^2$, which looks so different?! The answer, like the answer to similar questions we've asked before, has to do with the enormity of the vacuum speed of light, c. That c^2 multiplying the Δt^2 in the first term enhances that first term so much over the second term that, for intervals, Δt and $\Delta\mathbf{x}$, encountered up to and through the 19th century, the $\Delta\mathbf{x}$ made no noticeable difference. The invariance of the 'invariant interval' seemed to be the invariance of the first term alone, and that would mean the invariance of Δt. For example, if $\Delta t = 1$ sec, then $c^2\Delta t^2 \sim 34.6$ billion mi.^{2}. Even if $|\Delta\mathbf{x}|$ varied from zero to 1,000 mi. it would only change the value of the 'invariant interval' by less than 0.003% !

4. The Lorentz transformations

So now, suppose we want to calculate the intervals, $\Delta t'$ and $\Delta\mathbf{x}'$, between *any* two events in some inertial frame, F', when we know that the intervals between those events are Δt and Δ**x** in an inertial frame, F, and we know that F' is moving relative to F with velocity, **v**. The equations which tell us how

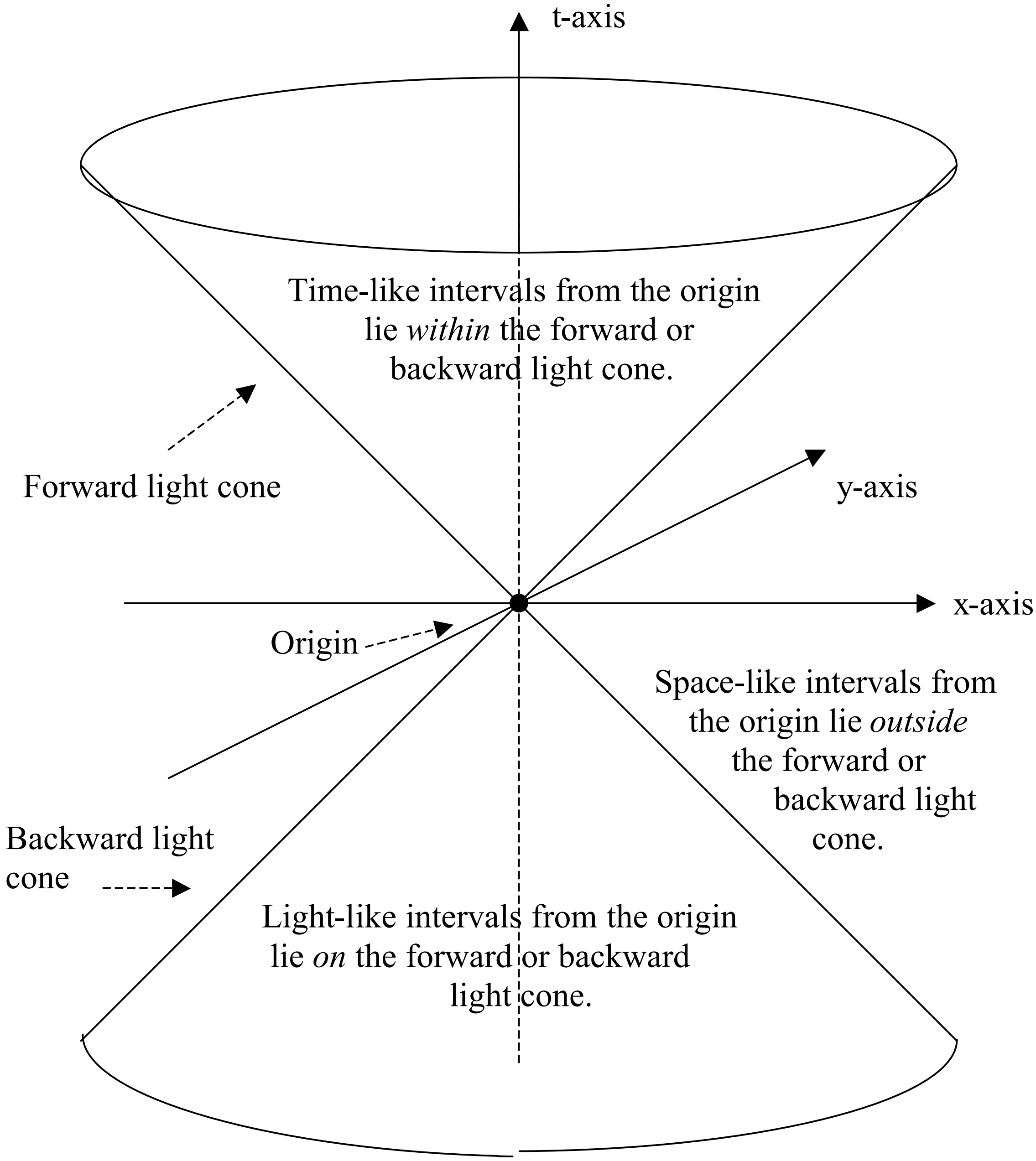


Fig. 3.2: Classification of space and time intervals from any event (here called origin). All light-like intervals define and lie on a *hypersurface* called the **light cone** , composed of the **forward** and **backward** light cones. Time-like intervals lie *within* the light cone and space-like intervals lie *outside* the light cone.

to calculate $\Delta t'$ and $\Delta \mathbf{x}'$ in terms of Δt and $\Delta \mathbf{x}$ are called the **Lorentz transformation equations** because they were first obtained by the eminent Dutch physicist, Hendrick Anton Lorentz, before the development of Einstein's STR and with a very different interpretation than Einstein's theory placed upon them. Lorentz embraced the Aether view and regarded the Δt and $\Delta \mathbf{x}$ measured in the Aether frame as 'correct' and the other $\Delta t'$ and $\Delta \mathbf{x}'$ as convenient illusions. Once STR was introduced, it did not take Lorentz long to see the advantages of embracing it.

The hypotheses invoked by Einstein (in addition to the principle of relativity and the light speed principle) to carry out a derivation of the Lorentz transformations were not novel and were universally accepted in the physics community. They were,

(1) If $\Delta t = \Delta \mathbf{x} = 0$, then $\Delta t' = \Delta \mathbf{x}' = 0$, i.e., coincident simultaneity is invariant.

(2) If $\Delta t = \Delta t_1 + \Delta t_2$ and $\Delta \mathbf{x} = \Delta \mathbf{x}_1 + \Delta \mathbf{x}_2$, then $\Delta t' = \Delta t'_1 + \Delta t'_2$ and $\Delta \mathbf{x}' = \Delta \mathbf{x}'_1 + \Delta \mathbf{x}'_2$, i.e., sums of intervals are transformed into sums of intervals. This is sometimes described as the assumption that space and time are homogeneous, i.e., the transformation can not depend on where or when the intervals occur or on how large or small they are.

These hypotheses severely restrict the complexity of the transformation equations and when we then require the equations to be compatible with the invariance of vacuum light speed and what we've already learned about time dilation and length contraction we obtain the results (**Appendix 1**),

$$\Delta t' = \gamma[\Delta t - (v/c^2)\, \Delta x_{\|}] , \qquad (4.1a)$$

$$\Delta x'_{\|} = \gamma[\Delta x_{\|} - v\, \Delta t] , \qquad (4.1b)$$

$$\Delta \mathbf{x}'_{\perp} = \Delta \mathbf{x}_{\perp} , \qquad (4.1c)$$

where $\Delta x_{\|}$ and $\Delta \mathbf{x}_{\perp}$ are the components of $\Delta \mathbf{x}$ parallel and perpendicular, respectively, to $\mathbf{v}$. The equations, (4.1), are the Lorentz transformations.

Remembering that $\gamma = 1 / [1 - (v/c)^2]^{1/2}$, we see that as (v/c) gets very small γ gets very close to 1 and the equations approximate,

$$\Delta t' = \Delta t \,, \tag{4.2a}$$

$$\Delta x'_{||} = \Delta x_{||} - v\, \Delta t \,, \tag{4.2b}$$

$$\Delta \mathbf{x}'_{\perp} = \Delta \mathbf{x}_{\perp} \,, \tag{4.2c}$$

which are the pre-relativistic, so-called Galilean transformation equations.

Suppose a clock is at rest in F', and, therefore is moving through F with velocity, **v**. For two events occurring *at* that clock we have $\Delta\mathbf{x}' = 0$, and so, from (4.1b), $\Delta x_{||} = v\, \Delta t$. When this is substituted into the right side of (4.1a) we get,

$$\Delta t' = \gamma[\Delta t - (v/c)^2 \Delta t] = [1 - (v/c)^2]^{1/2} \Delta t \,, \tag{4.3}$$

i.e., time dilation. The time interval measured by the moving clock in F, $\Delta t'$, is less than that measured by the synchronized clocks at rest in F, Δt.

Suppose a rod is at rest in F', and, therefore, is moving through F with velocity, **v**. For two events occurring at the endpoints of the rod, the distance between them in F will be the length of the rod in F if and only if the events are simultaneous in F, i.e., $\Delta t = 0$. From (4.1b, c) this leads to

$$\Delta x'_{||} = \gamma\, \Delta x_{||} = \Delta x_{||} / [1 - (v/c)^2]^{1/2}, \tag{4.4a}$$

$$\Delta \mathbf{x}'_{\perp} = \Delta \mathbf{x}_{\perp} \,, \tag{4.4b}$$

i.e., length contraction. The dimension of the rod parallel to the motion is larger in the rest frame, $\Delta x'_{||}$, than in the frame where the rod is moving, $\Delta x_{||}$, and the dimensions of the rod perpendicular to the motion are the same in both frames.

Finally, it follows from (4.1) that for all intervals (**Appendix 2**),

$$c^2 \Delta t'^2 - \Delta \mathbf{x}'^2 = c^2 \Delta t^2 - \Delta \mathbf{x}^2, \tag{4.5}$$

the 'invariant interval' is, indeed, invariant.

Appendix 1

Suppose we split the 3 dimensional vector, $\Delta\mathbf{x}$, up into components that are parallel or perpendicular to the velocity, **v**, of F' relative to F. The parallel component will be denoted by $\Delta x_{||}$. The two perpendicular components will be denoted by Δx_1 and Δx_2 and they are perpendicular to each other as well as to **v**. Then the two preceding hypotheses require the transformations to have the form,

$$\Delta t' = A\,\Delta t + B\,\Delta x_{||} + C\,\Delta x_1 + D\,\Delta x_2\,, \tag{A1.1a}$$

$$\Delta x'_{||} = A_{||}\,\Delta t + B_{||}\,\Delta x_{||} + C_{||}\,\Delta x_1 + D_{||}\,\Delta x_2\,, \tag{A1.1b}$$

$$\Delta x'_1 = A_1\,\Delta t + B_1\,\Delta x_{||} + C_1\,\Delta x_1 + D_1\,\Delta x_2\,, \tag{A1.1c}$$

$$\Delta x'_2 = A_2\,\Delta t + B_2\,\Delta x_{||} + C_2\,\Delta x_1 + D_2\,\Delta x_2\,, \tag{A1.1d}$$

where the A's, B's, C's and D's are constants depending only on the relationship between the frames, F and F', but not on the intervals.

Now consider a pair of events which are simultaneous in F, i.e., $\Delta t = 0$. This means that the spatial interval between the events in F can be the spatial dimensions of a rod moving in F if the events occur at the endpoints of the rod. Let the rod be at rest in F' and, therefore, moving with velocity, **v**, in F. From our discussions in sections **1** and **3** leading to the length contraction results, (1.4), (1.7), (3.3a), (3.3c) and (3.4) we must have,

$$\Delta t' = -\,\gamma\,(v/c^2)\,\Delta x_{||}\,, \tag{A1.2a}$$

$$\Delta x'_{||} = \gamma\,\Delta x_{||}\,, \tag{A1.2b}$$

$$\Delta x'_1 = \Delta x_1\,, \tag{A1.2c}$$

$$\Delta x'_2 = \Delta x_2\,. \tag{A1.2d}$$

But to obtain the results (A1.2) from (A1.1) just by substituting $\Delta t = 0$ into the right hand side of (A1.1) requires that we have,

$$C = D = C_{\|} = D_{\|} = B_1 = D_1 = B_2 = C_2 = 0\ , \tag{A1.3a}$$

$$C_1 = D_2 = 1\ , \tag{A1.3b}$$

$$B_{\|} = \gamma\ . \tag{A1.3c}$$

$$B = -\,\gamma\,(v/c^2)\ . \tag{A1.3d}$$

The equations (A1.1) have now been reduced to,

$$\Delta t' = A\ \Delta t - \gamma\,(v/c^2)\ \Delta x_{\|}\ , \tag{A1.4a}$$

$$\Delta x'_{\|} = A_{\|}\ \Delta t + \gamma\ \Delta x_{\|}\ , \tag{A1.4b}$$

$$\Delta x'_1 = A_1\ \Delta t + \Delta x_1\ , \tag{A1.4c}$$

$$\Delta x'_2 = A_2\ \Delta t + \Delta x_2\ . \tag{A1.4d}$$

Now consider a pair of events which are spatially coincident in F, i.e., $\Delta\mathbf{x} = 0$. Such events could occur at the location of a clock at rest in F and that clock would be moving in F' with the velocity, $-\,\mathbf{v}$. Consequently we must have, $\Delta\mathbf{x}' = -\,\mathbf{v}\ \Delta t'$, which means,

$$\Delta x'_{\|} = -\ v\ \Delta t'\ , \tag{A1.5a}$$

$$\Delta x'_1 = 0\ , \tag{A1.5b}$$

$$\Delta x'_2 = 0\ . \tag{A1.5c}$$

Furthermore, from our discussion of time dilation in class **III**, resulting in (1.2) (but with the role of Δt and $\Delta t'$ reversed from the present discussion), we have,

$$\Delta t' = \gamma\,\Delta t\ . \tag{A1.5d}$$

To obtain the results (A1.5) just from substituting $\Delta\mathbf{x} = 0$ into the right hand side of (A1.4) requires that we have,

$$A_1 = A_2 = 0\,, \qquad \text{(A1.6a)}$$

$$A = \gamma\,, \qquad \text{(A1.6b)}$$

$$A_{\|} = -\,v\,\gamma\,. \qquad \text{(A1.6c)}$$

The transformation equations have now been reduced to,

$$\Delta t' = \gamma\,\Delta t - \gamma\,(v/c^2)\,\Delta x_{\|} = \gamma\,[\Delta t - (v/c^2)\,\Delta x_{\|}]\,, \qquad \text{(A1.7a)}$$

$$\Delta x'_{\|} = -\,v\,\gamma\,\Delta t + \gamma\,\Delta x_{\|} = \gamma[\Delta x_{\|} - v\,\Delta t]\,, \qquad \text{(A1.7b)}$$

$$\Delta x'_1 = \Delta x_1\,, \qquad \text{(A1.7c)}$$

$$\Delta x'_2 = \Delta x_2\,, \qquad \text{(A1.7d)}$$

i.e., the Lorentz transformation equations.

Appendix 2

From the general Lorentz transformation equations, (4.1) or (A1.7), we have,

$$c^2\,\Delta t'^2 - \Delta\mathbf{x}'^2 = c^2\,\Delta t'^2 - \Delta x'^{\,2}_{\|} - \Delta\mathbf{x}'_{\perp}$$

$$= c^2\,\gamma^2\,[\Delta t - (v/c^2)\,\Delta x_{\|}]^2 - \gamma^2\,[\Delta x_{\|} - v\,\Delta t]^2 - \Delta\mathbf{x}_{\perp}^{\,2}$$

$$= \gamma^2\,(c^2 - v^2)\,\Delta t^2 - \gamma^2\,2(c^2(v/c^2) - v)\,\Delta t\,\Delta x_{\|} - \gamma^2\,((v/c)^2 - 1)\,\Delta x_{\|}^{\,2} - \Delta\mathbf{x}_{\perp}^{\,2}$$

$$= c^2\,\Delta t^2 - \Delta x_{\|}^{\,2} - \Delta\mathbf{x}_{\perp}^{\,2} = c^2\,\Delta t^2 - \Delta\mathbf{x}^2. \qquad \text{(A2.1)}$$

The ‘invariant interval’ is, indeed, invariant.