The Physical Forces of Everyday Life: III.

1. Torque or the Moments of Forces

To prepare us for the statement of the last basic Rule for Forces, we need, first, to introduce the concept of **torque** or the **moment of force**. This concept provides a measure of the turning tendency or twisting strength of a force. That turning tendency depends not only on the strength and direction of the force itself but also on *where* the force is applied relative to the axis about which the turning may occur.

The concept is most simply encountered in the experience of opening a heavy door or riding a child's see-saw or the varied sizes of the chain sprockets of a multi-geared bicycle or the long arms of an old fashioned lug wrench for removing wheels from a vehicle, etc.

It's a lot easier to push the door out of your way if you press against the door far from the hinges rather than close to the hinges (where the rotation occurs). A lightweight child on the see-saw can balance an adult on the opposite side if the child sits suitably farther from the axle (where the rotation occurs) than the adult. Pedaling uphill is made easier by shifting to the larger radius sprockets on the back wheel, which applies the chain's force farther from the axle (where the rotation occurs). It is also made easier by shifting to the smaller sprockets at the pedals, which increases the ratio of the distances from the pedal axle (where the rotation occurs), of the applied force on the pedal to the resisting force of the chain. Only very strong people can remove tight wheel lugs with short arm wrenches as opposed to long arm wrenches with the force applied towards the ends of the arm(s) far from the lug bolt (where the rotation occurs).

All of these examples and many more give rise to the notion that some combination of the strength of a force and the distance of its' application from a turning axis would be a suitable measure of the turning effectiveness of an applied force. Careful study of the matter yields the concept of the **moment of a force** relative to an axis as the appropriate measure. It's a rather complicated quantity and we will define it in stages.

First there is the **moment arm**.

FOR ANY FORCE, APPLIED AT SOME POINT, P, AND ANY STRAIGHT LINE, A, CONSIDERED AS A ROTATION AXIS, THE **MOMENT ARM** OF THE FORCE RELATIVE TO THAT AXIS IS THE LINE, PERPENDICULAR TO THE AXIS, FROM THE AXIS TO THE POINT, P (**Fig. III. 1**).
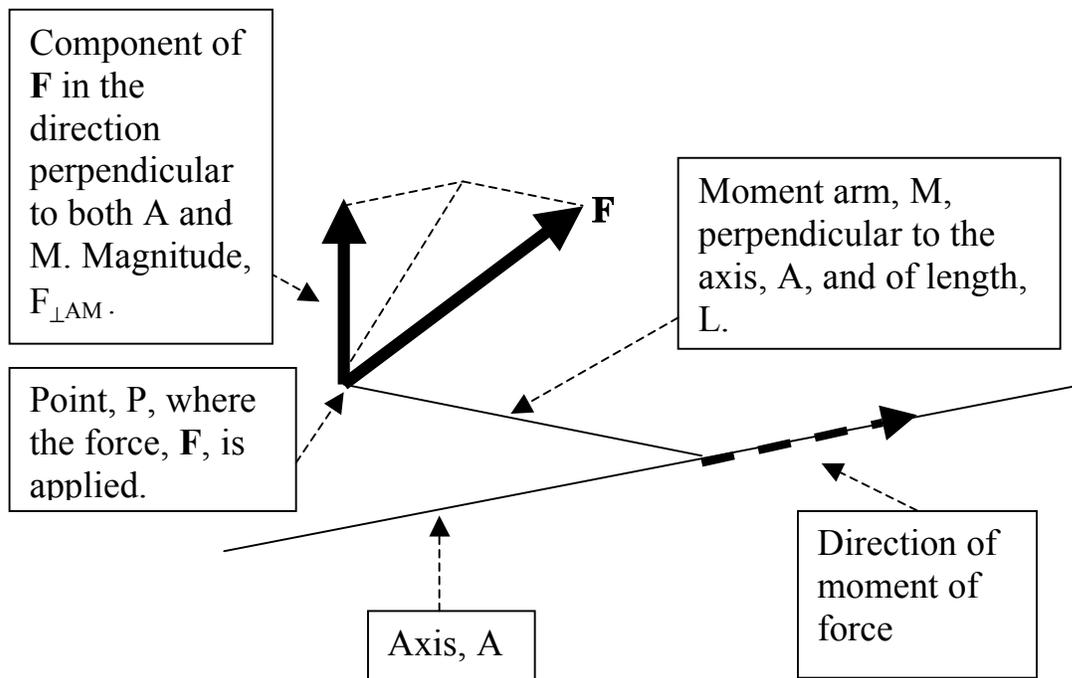
Component of **F** in the direction perpendicular to both A and M. Magnitude, $F_{\perp AM}$.

Moment arm, M, perpendicular to the axis, A, and of length, L.

Point, P, where the force, **F**, is applied.

F

Direction of moment of force

Axis, A

Fig. III. 1: Moment of a force relative to an axis. The *magnitude* of the moment, T_A, is given by $T_A = L\ F_{\perp AM}$. The *direction* of the moment is, by convention, parallel to the axis and given by the thumb of the right hand if the fingers are curled in the direction of the forced rotation.

The next part of the definition is even more involved.

FOR ANY FORCE, APPLIED AT SOME POINT, P, AND ANY STRAIGHT LINE, A, CONSIDERED AS AN AXIS, THE *MAGNITUDE* OF THE **MOMENT OF THE FORCE** RELATIVE TO THAT AXIS IS THE PRODUCT OF THE LENGTH OF THE MOMENT ARM AND THE MAGNITUDE OF THE COMPONENT OF THE FORCE PERPENDICULAR TO BOTH THE AXIS AND THE MOMENT ARM.

THE *DIRECTION* OF THE MOMENT OF THE FORCE IS PARALLEL TO THE AXIS IN THE DIRECTION OF THE THUMB OF THE RIGHT HAND IF THE FINGERS OF THE RIGHT HAND ARE CURLED IN THE DIRECTION THE FORCE WILL PUSH THE ROTATION (**Fig. III. 1**).

We represent the magnitude of the moment of the force by T because another name for the moment is **torque**. The arrow representing both the magnitude and direction of the moment, or torque, is denoted by the boldface **T**.

1a. Torque relative to a point*

There is a concept of torque of a force relative to a point rather than to an axis which we will not need here, but which is regarded as more fundamental and will be described in this special topic subsection. As we will see, the new concept is closely related to our concept of torque relative to an axis.

If a force, **F**, is applied at a point, P, then the torque of that force relative to a point, Q, is just the torque of that force relative to the particular axis passing through Q which is perpendicular to the direction of **F**. So on one hand torque relative to a point is a special case of torque relative to an axis! On the other hand, it turns out that given the torque of **F** at P relative to Q, the torque of **F** at P relative to an axis, A, passing through Q, is just the *component* parallel to A of the torque relative to Q.

Thus from the torque relative to the *point* Q, we can obtain the torque relative to any *axis* through Q.

2. Cycling up hill

Well, this has been pretty complicated! Let's look at some examples.

An interesting example is provided by the bicycle sprockets case mentioned above. This case is also simple because all the axes and moment arms are perpendicular to the forces.
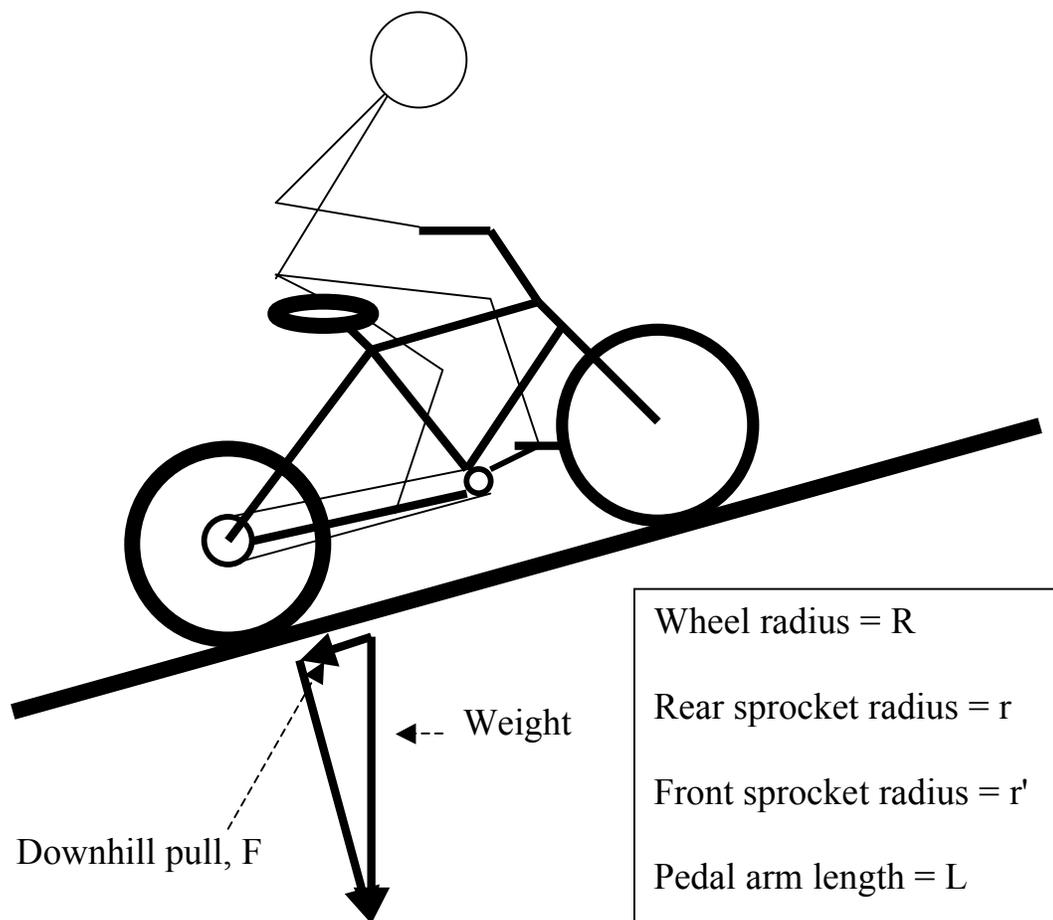
Fig. III. 2: Cyclist pedaling uphill

Because the combined weight of rider and bicycle pulls vertically down, that weight has a component pulling down the hill that must be overcome to move up the hill. The ground must push up the hill on the rear wheel at least as strongly as that weight component pulls down the hill. Remember that that happens in reaction to the wheel pushing down the hill against the ground by that amount. Because the rear wheel radius, R, is the moment arm of the forces, **F**, at the ground contact relative to the rear axle, the torque of those forces has magnitude, RF. For the case of the ground force on the wheel that torque must be matched or overcome by the torque of the cycle chain pulling forward on the top of the rear sprocket. Calling the tension force in the chain, t, the chain torque on the rear wheel is r t. So we must have $RF \leq r\,t$ if the chain torque is to overcome the ground force torque on the rear wheel. That same chain tension pulls backwards on the top of the

front sprocket with a torque, $r' t$ and that torque must be matched or overcome by the torque of the pedal force, F', acting on the pedal arm, a torque of LF'. That means we must have $r' t \leq LF'$. From the first inequality,

$$RF \leq r t, \qquad \text{we have (divide both sides by r),} \qquad (RF / r) \leq t ,$$

and from the second inequality,

$$r' t \leq LF', \quad \text{we have (divide both sides by r'),} \qquad t \leq (LF' / r') .$$

Combining these inequalities yields, $(RF / r) \leq (LF' / r')$, or

$$(r' / r)(R / L)F \leq F'.$$

The force, F, is determined by the weight of bike and rider and the slope of the hill. It increases with both. The ratio, R/L, is fixed by the design of the bike. Only the ratio, r'/r, is variable by changing gears. The smaller we make it the less force we have to exert on the pedals to climb the hill*.

We've neglected internal bike friction but if we maintain our bike well that should be very small.

3. The leaning ladder

For the next example we return to a case where we don't want movement and friction is essential - the leaning ladder (**Fig. III. 3**)!

There is friction at the wall-ladder contact as well as at the floor-ladder contact. But the wall friction is not nearly as important as the floor friction (because gravity is pulling the ladder towards the floor, not the wall) and so, to simplify the analysis, we will *pretend* there's no wall friction at all (*Can we do that!?*). That being the case the wall reaction, **f**, is perpendicular to the wall and is just compensated by the oppositely directed static friction force, −**f**, at the floor contact. Similarly the downward pull of gravity, **W**, is compensated by the upward reaction of the floor, −**W**.

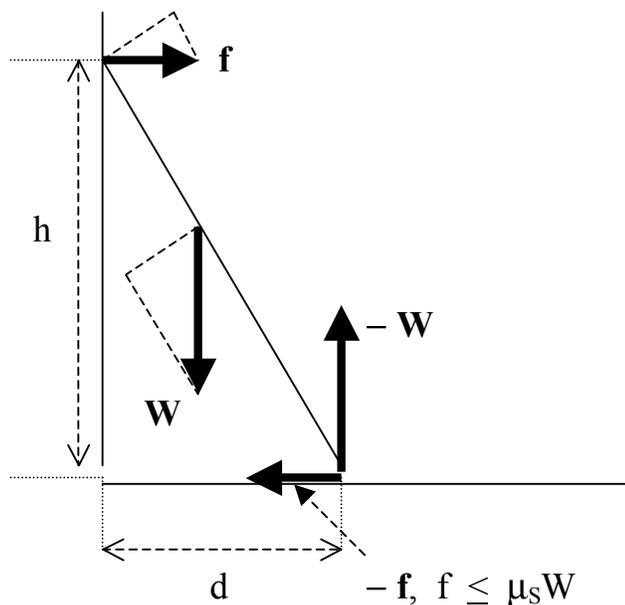* See the Appendix for more details

Fig. III. 3: Forces *on* a stable leaning ladder

But what about rotation? Consider the floor contact as an axis (coming out of the page). The torques of the various forces relative to that axis had better cancel if no rotation is to occur there. Notice that if the ladder slides it will be rotating about the (now moving) floor contact. So a stationary ladder requires the torques to cancel each other out as well as the forces. The friction force at the floor has no torque because it has no moment arm relative to the floor contact because it's applied *at* the floor contact. Similarly for the upward reaction force of the floor. But the other two forces have moment arms and torques relative to the floor axis. Because of the identical *shape* of the triangles in the picture it turns out that the magnitude of the torque, $\mathbf{T}_f$, of the wall reaction is just given by $T_f = h \, f$ (Its' direction is into the page.). Similarly the opposite torque (directed out of the page) of the ladder weight, which acts as if it were concentrated at the CM of the ladder has magnitude, $T_W = (d / 2)W$. These two torques must balance each other. Notice that as the ladder leans more and more, h gets smaller and d gets larger. So with shallower lean the wall reaction, f, must get larger. But since the friction force at the floor balances the wall reaction, balance can be maintained only if the wall reaction is less than the static friction limit, $\mu_S W$. In other words, for a stable ladder we must have,

$$f \leq \mu_S W \qquad \text{and} \qquad h\, f = (\, d\, /\, 2\,)W \qquad \text{or,}$$

$$(\, d\, /\, 2\,)W = h\, f \leq h\, \mu_S W \qquad \text{or (cancel the W's),}$$

$$(\, d\, /\, 2\,) \qquad \leq \qquad h\, \mu_S \, .$$

If the lean is too shallow, d is too large and h is too small to satisfy the inequality and the ladder falls! Note that the critical lean is independent of the weight of the ladder! However, if someone climbs the ladder, the critical lean depends on how high they go on the ladder*.

4. Momentum, Angular Momentum and the last Rule for Forces

When we apply torques to objects that are free to rotate about an axis and there are no other torques resisting the motion, the objects start rotating. And once rotating, if we let them go, they keep rotating until they encounter resisting torques that slow down the rotation. The simplest example is that of spinning a large beach ball in your hands and then throwing it while still spinning. It will lose very little spin until it hits the ground or some obstacle.

This is the rotational analogue to the fact that once we use forces to put a stationary object in motion, it will keep moving (the same way) unless and until forces come into play again that change the motion, possibly stopping it. When we throw a ball it arcs in flight because after the ball leaves our hand Earth's gravity keeps acting on the ball, gradually converting its forward and upward velocity to a forward and downward velocity. Wind resistance acts as well but is not as important as gravity (except for baseball pitchers, as we will see later). But if a hockey player hits a puck that slides on the ice, Earth's gravity is cancelled by the upward support of the ice and only surface friction and air resistance act on the puck until it hits something. Those forces (for smooth wet ice) are sufficiently small that to a good approximation the sliding puck has constant velocity from when it leaves the hockey stick until it hits something.

We captured these features of motion in the equation,

$$\mathbf{F} = M\mathbf{A},$$

* See the Appendix for more details.

and there is an even 'better' way to express the same thing. Imagine the moving system, whatever it is, as made up of very tiny pieces of matter. Each of these pieces has a mass, m say, and a velocity, **v**. The product of m and **v**, m**v**, is called the **momentum** of the piece and, having magnitude and direction, can be represented by an arrow. The momentum of the whole system, called the **Total Momentum** of the system and denoted by **P**, is represented by just the *resultant* of all the arrows representing the momenta of the pieces. This Total Momentum turns out to be M**V**, where M is the **Total Mass** of the system and **V** is the velocity of the CM of the system. In other words,

$$\mathbf{P} = M\mathbf{V} \ .$$

If the masses of the individual pieces don't change with time (the usual case) then the time rate of change of **P** is just M**A**, where **A**, the acceleration of the CM, is the time rate of change of the velocity, **V**. Consequently, our third Rule of Forces can be expressed as,

$$\boxed{\mathbf{F} = \text{Time rate of change of } \mathbf{P}}$$

This equation is actually *more fundamental* than $\mathbf{F} = M\mathbf{A}$ since it can also handle the cases when M changes with time, in which case $\mathbf{F} = M\mathbf{A}$ turns out to be *wrong*.

But now back to rotational motion. We are going to construct a quantity, $\mathbf{J}_A$, that has magnitude and direction, like so many other quantities we've discussed, and in terms of which we can state the Fourth Rule for Forces. As an equation the Fourth Rule is similar to our previous, fundamental, version of the Third Rule for Forces. It is,

$$\boxed{\mathbf{T}_A = \text{Time rate of change of } \mathbf{J}_A}$$

where $\mathbf{T}_A$ is the resultant of all the torques on the system relative to the same axis, A.

The equation is simple. The definition, however, of $\mathbf{J}_A$, called the **Total Angular Momentum**, is as involved as the definition of torque was. In fact it's almost a verbatim repeat of that definition. We start with the definition of the **angular momentum**, $\mathbf{j}_A$, of a small moving particle relative to an axis, A, (**Fig. III. 4**).

The **Total Angular Momentum**, $\mathbf{J}_A$, of a system relative to an axis, A, is just the resultant of the angular momenta of all the small pieces of matter in the system relative to the same axis. The Fourth Rule of Forces then is that

THE TIME RATE OF CHANGE OF THE TOTAL ANGULAR MOMENTUM WITH RESPECT TO AN AXIS, A, IS EQUAL TO THE TOTAL TORQUE WITH RESPECT TO THE SAME AXIS.
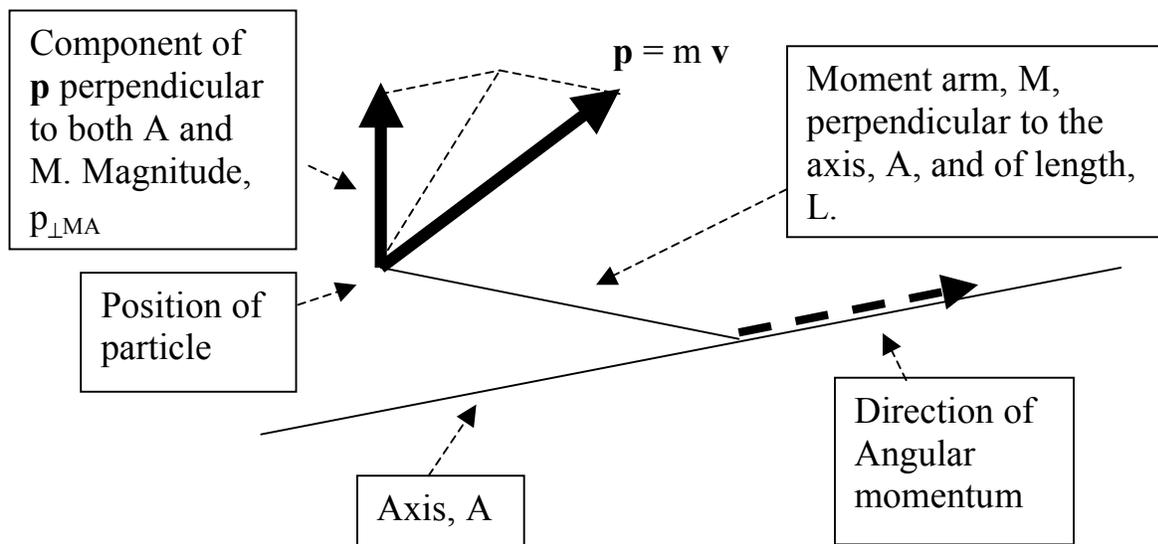

Component of **p** perpendicular to both A and M. Magnitude, $p_{\perp MA}$

p = m **v**

Moment arm, M, perpendicular to the axis, A, and of length, L.

Position of particle

Direction of Angular momentum

Axis, A

Fig. III. 4: Angular momentum of a moving particle relative to an axis. The *magnitude* , j_A, is given by $j_A = L\,p_{\perp MA}$. The *direction* of the angular momentum is given by the thumb of the right hand if the fingers are curled around A in the direction indicated by $\mathbf{p}_{\perp MA}$. (Compare to **Fig. III. 1**)

4a. Angular momentum relative to a point*

Just as with torques, there is a concept of angular momentum relative to a point rather than an axis. For a particle at point, P, moving with momentum, **p**, the angular momentum relative to the point, Q, $\mathbf{j}_Q$, is the angular

momentum relative to that axis through Q that is perpendicular to **p**. Just as with torques the angular momentum relative to any axis, A, through Q is the component of $\mathbf{j}_Q$ parallel to A. The total angular momentum of a system relative to Q, $\mathbf{J}_Q$, is the vector sum of the angular momenta of all its particles relative to Q. And just as with the relationship between total torques and angular momenta relative to axes, we have,

$$\mathbf{T}_Q = \text{Time rate of change of } \mathbf{J}_Q .$$

5. The spinning skater

To illustrate this complicated stuff consider, first, a spinning ice skater (**Fig. III. 5**). She begins her spin (by generating a vertical torque on her body with her skates) with her arms extended. Then, spinning freely, i.e. no more external torque and very little friction with the ice, she pulls her extended arms in close to her torso. If she kept spinning at the same angular speed (so many rotations per second), the matter composing her arms would reduce its' angular momentum as their moment arm relative to the rotation axis
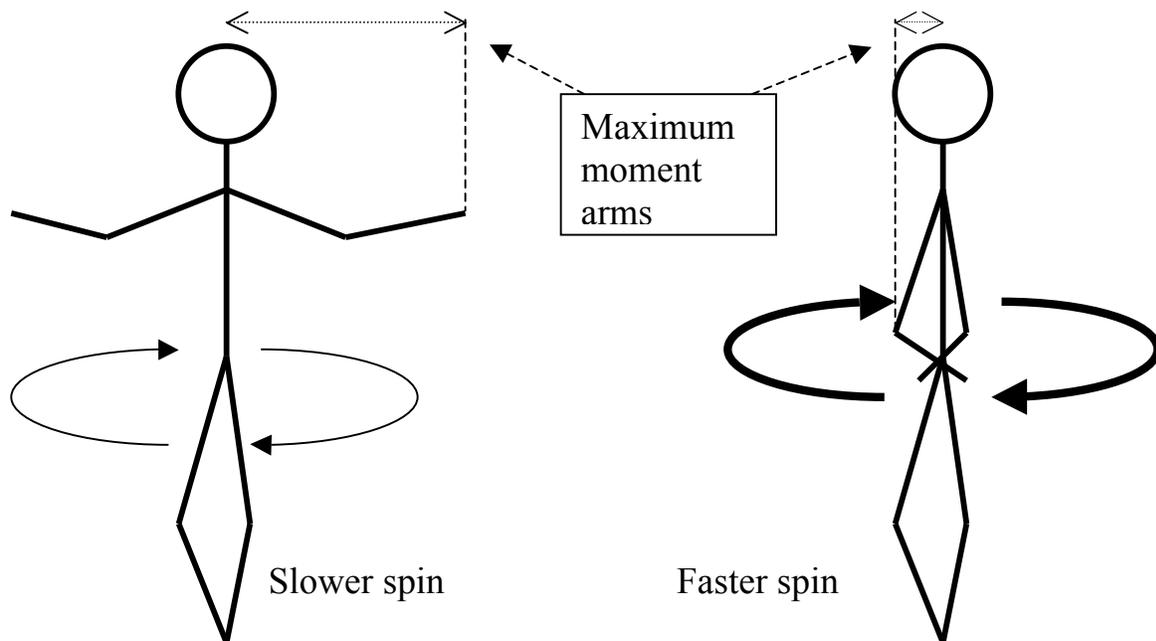
Maximum
moment
arms

Slower spin

Faster spin

Fig. III. 5: Spinning skater with variable moment arms

decreases. But the total angular momentum can't decrease in the absence of external torque. So, to preserve the angular momentum, her torso and arms together spin faster. She doesn't have to do anything else besides pull her arms in (which does, however, require more effort than it would if she were standing still). It happens automatically due to the conservation of angular momentum. Once she's finished impressing the audience, she re-extends her arms which dramatically slows down the spinning and makes it easy to stop altogether with a little external torque.

6. The prevailing mid-latitude westerlies

A more serious example of the conservation of angular momentum in the absence of external torque is the global phenomenon of the mid-latitude prevailing westerly winds! The detailed behavior of wind on a *local* scale is one of the most complicated phenomena one can encounter. But the prevailing westerlies, if one ignores some details, is rather simple (**Fig. III. 6**).

Imagine a rotating Earth initially carrying its' atmosphere with it so that there are no winds on the Earth's surface. The Sun then preferentially heats the air between the tropics. That air expands and rises, due to the heating, into both more northerly and southerly latitudes. As it does, surface air is pulled towards the equator to replace the rising, heated air. Focusing now on the Northern hemisphere, this circulation in the low latitudes; rising air flowing northward at altitude and surface air flowing southward, constitutes a so-called Meridional cell. As we enter the mid-latitudes the high altitude air has cooled and descends towards the surface, some remaining with the first cell and flowing southward while the rest flows northward along the surface through the mid-latitudes joining the oppositely circulating mid-latitude Meridional cell. Finally there is a polar cell circulating in the same sense as the low latitude cell. Why there are just three oppositely circulating Meridional cells is a *very* complicated business, but now we focus on the surface level flow in a given cell.

In the mid-latitudes, the surface flow is northward and the moment arm to the rotation axis is decreasing. So if the flowing air continues to rotate *with* the Earth, it will lose angular momentum. But being air, there's almost no external torque to diminish its' angular momentum. The Earth's gravity on the air, directed towards the 'center' of the Earth, has no such torque (no component perpendicular to both the rotation axis and moment arm).

Consequently, the angular momentum can not decrease and the air moving towards the poles must rotate *faster* than the Earth to maintain its' angular momentum. The Earth rotates towards the East and so the flowing air must rotate towards the East even faster. To conserve angular momentum the flow must acquire an eastward component and that is called (oddly?) the prevailing westerlies!

If the surface flow is southward in the Northern hemisphere, as it is in the low latitude and polar latitude Meridional cells, then the moment arm from the rotation axis of the Earth is increasing with the flow. This would result in an increase of the angular momentum of the surface flowing air which, again, can not be generated by available forces. Consequently the angular momentum of the surface flow must remain essentially constant and that requires the southward flow to acquire a westward component. That results in the prevailing easterly Trade Winds in the low latitudes and the polar easterlies in the high latitudes.
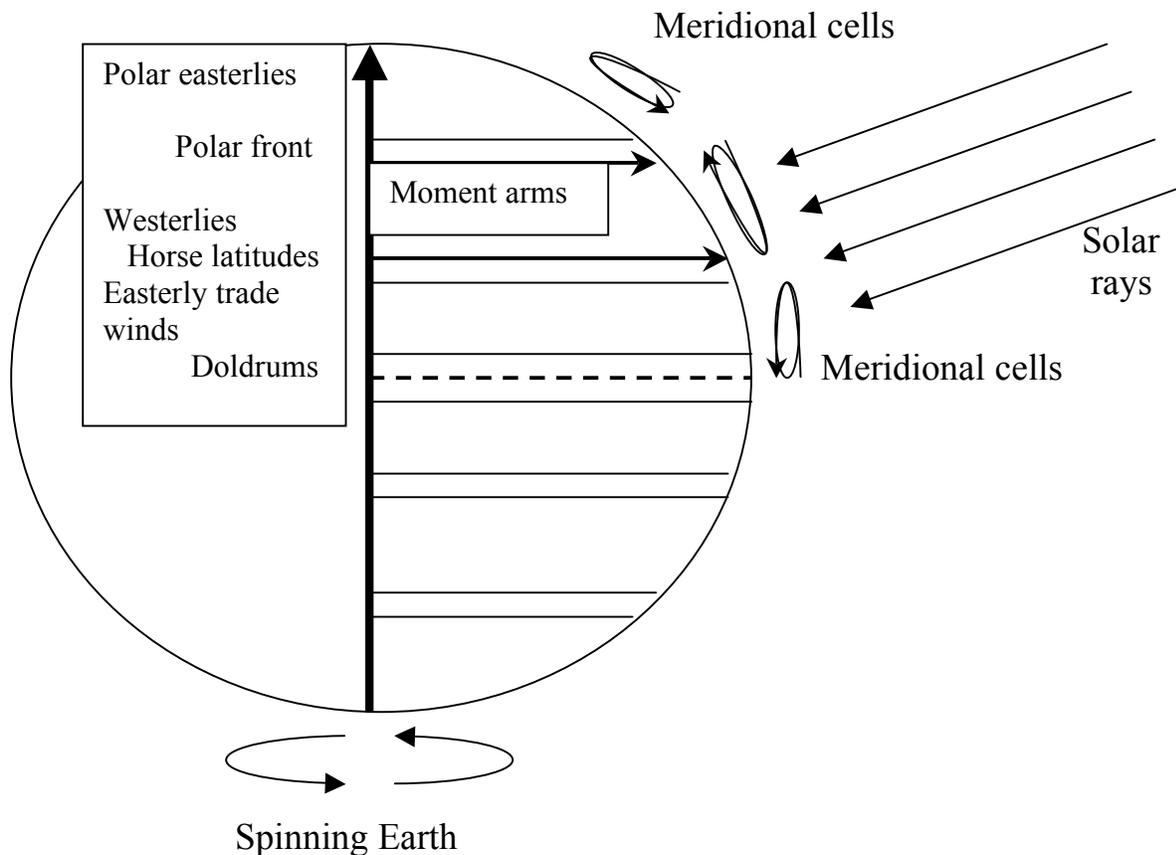
Fig. III. 6: Source of the prevailing mid-latitude Westerlies from Solar heating on a spinning, spherical Earth.

The narrow bands of latitudes between adjacent Meridional cells have no prevailing east-west flow and are called the Doldrums near the equator and the Horse latitudes between the low and mid-latitude cells.

Next we consider an example with the spinning Earth in which torque is exerted to keep a system in step with the Earth's rotation.

7. Wear on North-South rail lines and truck tires

Consider a train traveling North from the Southern USA. As it moves North the moment arm from the train to the rotation axis of the Earth is decreasing (**Fig. III. 6**). But the track is fixed in the ground so the rotation rate of the train stays fixed with the Earth at 360^0 in 24 hours. Consequently the angular momentum of the train relative to the Earth's rotation axis is decreasing. The torque required to decrease this angular momentum is provided by the force with which the Eastside rail presses Westward on the train. Had the train been going South from the Northern USA, its' angular momentum would be increasing and that would be accomplished by the Westside rail pressing Eastward on the train. Accordingly North-South train tracks that carry only or mostly North bound trains have their Eastside rails wear out faster. North-South tracks carrying mostly South bound trains have their Westside rails wear out faster.

Similarly, the Eastern sides of tires of long haul Northbound 18 wheelers and the Western sides of tires of long haul Southbound 18 wheelers wear a bit faster.

8. Mouse on circular treadmill

Finally let us return to the hapless mouse running inside the circular treadmill that we used as an illustration of the first rule for forces (**Fig. III. 7**).

To begin with he's at the bottom of the treadmill. When he begins to run he exerts a backwards force on the treadmill which, given his moment arm with respect to the rotation axis, creates a torque (into the page of the diagram) relative to the axis. The reaction force of the treadmill on the mouse propels the mouse forward and a little up the side of the circular mill. If there is no friction at the axle then as the mouse acquires the speed of running he is comfortable with he will relax the backward thrust of his legs, reducing the

acceleration of the treadmill as his body moves back down to his starting position and he just keeps his legs moving fast enough to keep up with the turning treadmill that passes underneath. Conservation of angular momentum in the absence of external torque will keep the treadmill turning steadily.
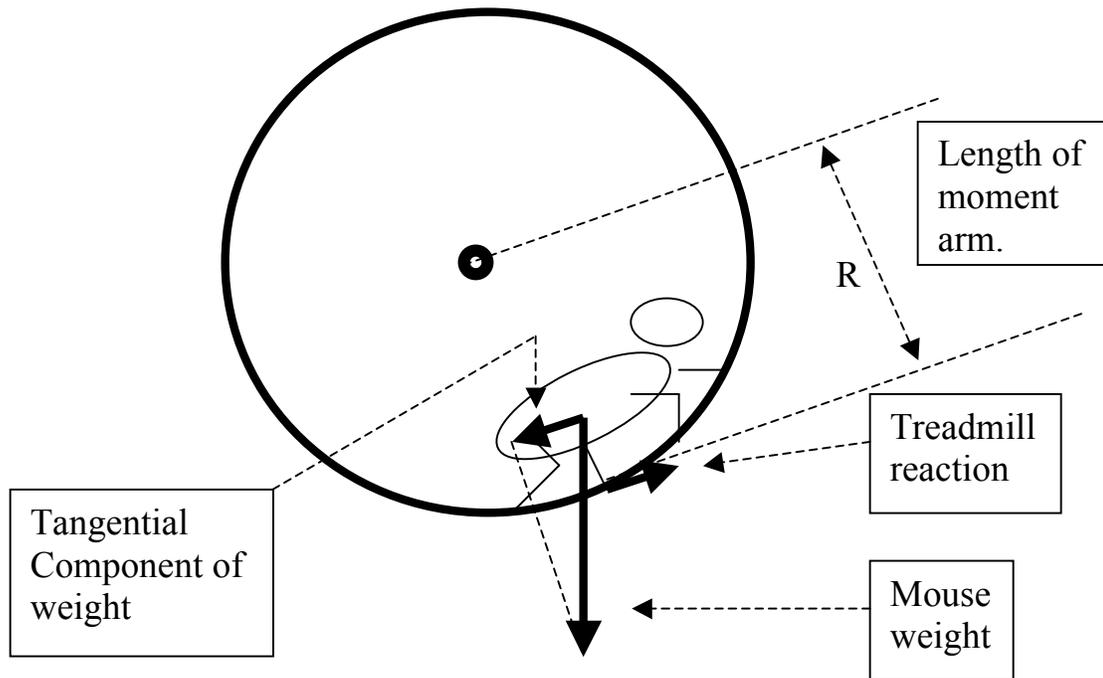

Fig. III. 7: Mouse running steadily on treadmill with friction

Realistically, however, there *will* be some friction at the axle. This will require the mouse to stay slightly up the side of the treadmill even once the angular acceleration stops and the rotation rate is constant. For now the backward thrust of his legs is producing the torque needed to just cancel the torque due to friction. But the reaction force on him (which is tangential to the rim), instead of accelerating him, just cancels the component of his weight that is tangential to the rim. If he were at the exact bottom of the treadmill, there would be no tangential component of his weight for the reaction force to balance and the unbalanced reaction force would accelerate the mouse.

So the angular displacement from the vertical that the mouse has to climb in order to keep the treadmill turning at a constant rate is a measure of the friction torque generated at the axle.

Appendices:

A1: Cycling uphill

The slope of the hill is measured by the ratio of the change in elevation to the horizontal distance covered, i.e., h / d in **Fig. III. A1**. The triangle made by a segment of the hill surface, s, the horizontal distance covered, d, and the elevation change, h, has the same shape as the triangle made by the weight vector, $\mathbf{W}$, its component perpendicular to the hill surface, $\mathbf{W}_\perp$, and the component parallel to the hill surface, $\mathbf{W}_\parallel$.
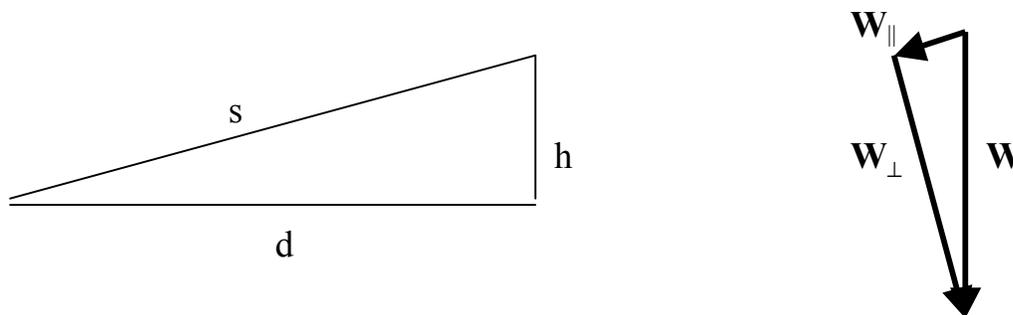

Fig. III. A1: Similar displacement and force triangles

Consequently,

$$W_\parallel / W \;=\; h / s \;=\; h / [\, h^2 + d^2\,]^{1/2} \qquad\qquad \text{or}$$

$$W_\parallel \;=\; (\, h / s\,)W.$$

But $W_\parallel$ is the force, F, from our cycling discussion, i.e., the component of the weight of the bicycle and rider pulling down the hill and needing to be overcome at the contact point of the rear tire on the road. So the force needed on the pedal is,

$$F' = (\, r' / r\,)(\, R / L\,)(\, h / s\,)W.$$

For my bike, W ~ 240 lbs, R / L ~ 14"/ 7" = 2 and r' / r $\geq$ 0.75. So for a mild slope of h / s ~ 0.1 or about a 6^0 slope, I need to exert a force of

$$F' \sim 0.75 \text{ x } 2 \text{ x } 0.1 \text{ x } 240\text{lbs} \;=\; 0.15 \text{ x } 240 \text{ lbs} \;=\; 36 \text{ lbs}$$

on the pedal. For a slope of 30^0, however, h / s = 1/2 and

F' ~ 0.75 x 240 lbs = 180 lbs !!

A2: Climbing a ladder

Here we modify **Fig. III. 3** by inserting the weight vector, $\mathbf{W}_C$, of the climber at her location on the ladder (**Fig. III. A2**). To anticipate the result notice that if the climber is halfway up the ladder, the new weight vector contributes at the same place as the ladder weight, i.e., the CM of the ladder. Consequently the critical slope of the ladder will be the same as before. If the climber is below the halfway up mark, then the torque generated by her weight will be less than if she was halfway up and the critical slope will be greater than before. If she is more than halfway up, the torque she generates will be greater than when halfway up and the critical slope will be less than before. The rate of change of the critical slope as the climber climbs will increase with the ratio of the climbers weight to the ladders weight.

Let the climber's weight be W_C and the climber's vertical ascent on the ladder be h_C. Then the climber's horizontal distance from the ladder-floor contact point is $d_C = (h_C / h)d$ and the new torque balance equations are,
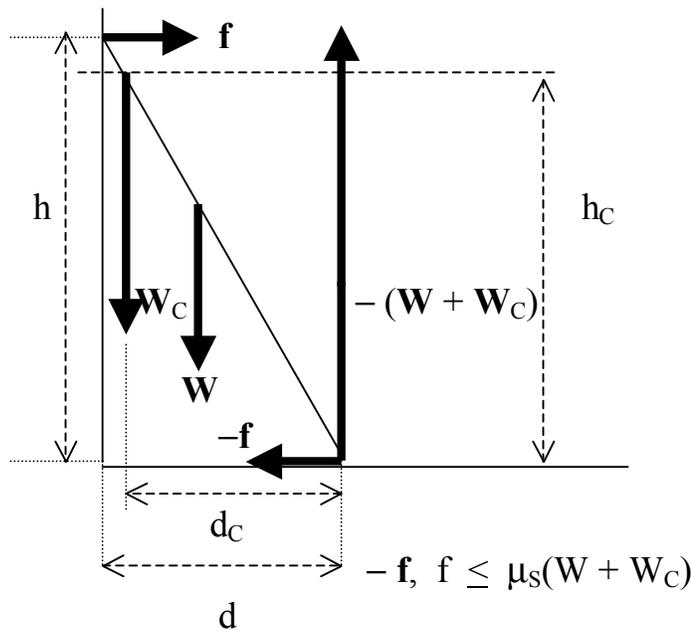

Fig. III. A2: Forces *on* a stable leaning ladder with climber

$$h f = (d / 2) W + d_C W_C = (d / 2)W + (h_C / h)d W_C$$

or $$f = (d / h)[(1/2) W + (h_C / h) W_C].$$

But we must have $f \leq \mu_S (W + W_C)$. Therefore, we must have,

$$(d / h)[(1/2) W + (h_C / h) W_C] \leq \mu_S (W + W_C),$$

or $$(d / h) \leq \mu_S (W + W_C) / [(1/2) W + (h_C / h) W_C] .$$

Writing, $W_C = r W$, and canceling out the common factor, W, we get,

$$(d / h) \leq \mu_S (1 + r) / [(1/2) + (h_C / h) r].$$

If the climber is at the bottom of the ladder,

$$(d / h) \leq \mu_S (1 + r) / (1/2) = 2 \mu_S (1 + r).$$

If at the top,

$$(d / h) \leq \mu_S (1 + r) / [(1/2) + r] .$$

Suppose r = 5 (easy for an aluminum ladder) ?? Then,

$$12 \mu_S \geq (d / h)_{crit} \geq (1 + (1/11)) \mu_S$$

at the   bottom   and   top,   respectively.