1. Introduction. In [1] and [2], a large number of $q$-identities have been deduced from a single identity of basic hypergeometric type. In particular, most of the third order mock theta function identities [14; 63], all of the fifth order mock theta function identities [15; 277-279], several of the identities given by Fine in [7], and Heine's original transformation of basic hypergeometric series [8; 106] are all deducible from the Fundamental Lemma of [1]. The object of this paper is to show that many other $q$-identities which have not seemed to fit into any of the general theorems on basic hypergeometric series (c.f. Sears [11], [12], or Slater [13]) are actually deducible from the Fundamental Lemma of [1]. We shall utilize the following notation

$$\Pi_n(x, q) = \prod_{j=0}^{m-1} (1 + xq^j)$$
$$\Pi_n(x, q) = \prod_{j=0}^{m} (1 + xq^j).$$

We shall prove

(A1) \[ \sum_{n=0}^{\infty} \frac{q^{(m+1)n}}{\Pi_n(-q, q) \Pi_{m+1}(-q, q)} = q \sum_{p=0}^{\infty} (-1)^p q^{p(p+3)}, \]

(A2) \[ \sum_{n=0}^{\infty} q^{n+1} \Pi_n(-q, q) \Pi_{m+1}(-q, q) = q \sum_{p=0}^{\infty} (-1)^p q^{p(p+3)}, \]

(A3) \[ \sum_{n=0}^{\infty} q^{\frac{1}{2}(m+1)(m+2)} \Pi_n(-q, q) \Pi_{m+1}(-q, q) = q \sum_{p=0}^{\infty} q^{p^2+3p}, \]

(C1) \[ \frac{\sum_{n=0}^{\infty} (-1)^n q^{\frac{1}{2}(n-1)a^n} \Pi_n(-q, q)}{\Pi_n(-q, q) \Pi_n(-q, q)} = \frac{\Pi_n(-q, q) \Pi_n(-q, q)}{\Pi_n(-q, q) \Pi_n(-q, q)} \]

(C2) \[ \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}(n+1)} \Pi_n(-a, q)}{\Pi_n(-q, q)} = \Pi_n(q, q) \sum_{n=0}^{\infty} \frac{(-1)^n q^{a^n}}{\Pi_n(-q, q^2)}, \]

(C3) \[ \sum_{n=0}^{\infty} (-1)^n q^{a^n} \Pi_n(-a, q^2) = \Pi_n(-q, q^2) \sum_{n=0}^{\infty} q^{2n-a^n} \Pi_n(-q, q^2), \]

(R1) \[ \sum_{n=0}^{\infty} q^{2n-2n} \Pi_n(-q, q^2) = \Pi_n(-q, q^2) \sum_{n=0}^{\infty} q^{2n} \Pi_n(-q, q) \Pi_n(-q, q^2) \]

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\[
(R2) \quad \sum_{n=0}^{m} \frac{q^n z^n}{\prod_{n=0}^{m} (-q^n, q^n)} = \prod_{n=0}^{m} (e^{\beta q^n}, q^n) \sum_{n=0}^{m} \frac{q^n z^n}{\prod_{n=0}^{m} (-q^n, q^n)} \prod_{n=0}^{m} (e^{\beta q^n}, q^n).
\]

Identities (A1), (A2), and (A3) were proved by Auluck [3; 681 Equation (11); 683 Equation (20); 685 Equation (28)]. (C1) was proved by Carlitz in [5]; (C2) and (C3) were posed as problems by Carlitz [6] in the American Mathematical Monthly (actually there was a slight error in the original statement for (C3) which has been corrected here). (R1) and (R2) were originally proved in a slightly different form in L. J. Rogers's famous paper of 1894 [10; 330, Equations (2), (3), and (4)]. Also (R1) and (R2) were proved by Watson in [16; 46–47] while special cases of (R1) and (R2) had already been proved by Watson in [14; 59–60] and [15; 275].

We shall prove generalizations of all of these identities in §3. In §2, we shall show that all of the above identities are special cases of our more general identities.

2. Derivation of original identities. This section is devoted to deducing the identities given in §1 from the following six identities.

\[
(I1) \quad \sum_{n=0}^{m} \frac{\prod_{n=0}^{m} (-\alpha, q) \prod_{n=0}^{m} (-\beta, q) \prod_{n=0}^{m} (-\gamma, q)}{\prod_{n=0}^{m} (-\beta, q) \prod_{n=0}^{m} (-\alpha, q) \prod_{n=0}^{m} (-\gamma, q)} = \prod_{n=0}^{m} (-\alpha, q) \prod_{n=0}^{m} (-\beta, q) \prod_{n=0}^{m} (-\gamma, q) \sum_{n=0}^{m} \prod_{n=0}^{m} (-\gamma/\beta, q) \prod_{n=0}^{m} (-\gamma, q) \prod_{n=0}^{m} (-\alpha, q).
\]

\[
(I2) \quad \sum_{n=0}^{m} \frac{\prod_{n=0}^{m} (-\alpha, q) \prod_{n=0}^{m} (-\beta, q) \prod_{n=0}^{m} (-\gamma, q)}{\prod_{n=0}^{m} (-\beta, q) \prod_{n=0}^{m} (-\gamma, q) \prod_{n=0}^{m} (-\alpha, q)} = \prod_{n=0}^{m} (-\beta, q) \prod_{n=0}^{m} (-\gamma, q) \prod_{n=0}^{m} (-\alpha, q) \cdot \sum_{n=0}^{m} \prod_{n=0}^{m} (-\beta, q) \prod_{n=0}^{m} (-\alpha, q) \prod_{n=0}^{m} (-\gamma, q) \prod_{n=0}^{m} (-\beta, q).
\]

\[
(I3) \quad \sum_{n=0}^{m} \frac{\prod_{n=0}^{m} (q/\tau, q) \prod_{n=0}^{m} (-\beta, q) \prod_{n=0}^{m} (-\gamma, q)}{\prod_{n=0}^{m} (-\beta, q) \prod_{n=0}^{m} (-\gamma, q)} = \prod_{n=0}^{m} (q, q) \sum_{n=0}^{m} \frac{(-1)^n q^n \gamma^n}{\prod_{n=0}^{m} (-q^n, q^n)} + \prod_{n=0}^{m} (q, q) \prod_{n=0}^{m} (-\beta q, q^n) \sum_{n=0}^{m} \prod_{n=0}^{m} (-\beta q^2, q^n) \prod_{n=0}^{m} (-q^n, q^n).
\]

\[
(I4) \quad \sum_{n=0}^{m} \frac{\prod_{n=0}^{m} (-q/\tau, q^2) \prod_{n=0}^{m} (-\beta q^3, q^2) \prod_{n=0}^{m} (-\gamma, q^2) \prod_{n=0}^{m} (-\beta, q^2) \prod_{n=0}^{m} (-\gamma, q^2)}{\prod_{n=0}^{m} (-q, q^2) \prod_{n=0}^{m} (-q^2, q^2)} = \prod_{n=0}^{m} (-q, q^2) \sum_{n=0}^{m} \frac{q^{2n} \gamma^n \beta^n}{\prod_{n=0}^{m} (-q^n, q^n)} + \frac{1}{2} \prod_{n=0}^{m} (-q, q^2) \prod_{n=0}^{m} (-\beta, q^2) \prod_{n=0}^{m} (-\gamma, q^2) \sum_{n=0}^{m} \prod_{n=0}^{m} (-\beta q, q^2) \prod_{n=0}^{m} (-\gamma q, q^2).
\]
(I5) \[ \sum_{n=0}^{\infty} \frac{\prod_{n}(\tau \beta, q)^{2n}}{\prod_{n}(\tau, q)^{2n}} = \frac{\prod_{n}(\tau \beta, q)}{\prod_{n}(\tau, q)} \sum_{n=0}^{\infty} \frac{\prod_{n}(\tau \beta, q)^{2n}}{\prod_{n}(\tau, q)^{2n}}. \]

(I6) \[ \sum_{n=0}^{\infty} \frac{\prod_{n}(-\tau, q^{2})^{2n}}{\prod_{n}(-\tau, q)^{2n}} = \frac{\prod_{n}(-\tau, q^{2})^{2n}}{\prod_{n}(-\tau, q)^{2n}} \sum_{n=0}^{\infty} \frac{\prod_{n}(-\tau, q^{2})^{2n}}{\prod_{n}(-\tau, q)^{2n}}. \]

(I1) was originally given by Heine [8; 106]. (I2) is easily deduced from (I1); however, it appears to have been first explicitly stated by Rogers [9; 171, Equation (2)]. (I3), (I4), (I5), and (I6) are new to my knowledge.

To obtain (A1) set \( \alpha = -q^{2}/r^{4}, \beta = -q/r^{4}, \gamma = q^{3} \) in (I2) and let \( \tau \to 0 \). (A2) is a special case of (C1) obtained by setting \( b = 0 \) and \( a = q^{2} \) in (C1). (A3) is a special case of (R1) obtained by replacing \( q \) by \( q^{3} \) in (R1) and then setting \( z = q^{4} \). (C1) is obtained from (I1) by setting \( r = x, \alpha = b, \) and \( \gamma = a \) and letting \( \beta \to 0 \). (C2) is obtained from (I3) by setting \( \beta = a \) and letting \( r \to 0 \). (C3) is obtained from (I4) by setting \( \beta = a \) and letting \( r \to 0 \). (R1) is obtained from (I5) by replacing \( q \) throughout by \( q^{2} \), then setting \( \tau \beta = -2q \), and letting \( r \to 0 \). (R2) is obtained from (I6) by setting \( \tau = -q^{2}/\beta \) and letting \( \beta \to 0 \).

3. Derivation of (I1)-(I6). We shall need the following identity

\[ \sum_{n=0}^{\infty} \frac{\prod_{n}(-a, q^{2})^{2n}}{\prod_{n}(-a, q)^{2n}} = \frac{\prod_{n}(-a, q^{2})^{2n}}{\prod_{n}(-a, q)^{2n}} \quad [4; 66]. \]

The identities (I1)-(I6) are obtained by repeated application of the following three lemmas. Since all these lemmas are special cases of the Fundamental Lemma of [1], we shall give a detailed proof only of Lemma 2; the other lemmas are proved similarly.

**Lemma 1.** If \( k \geq 1 \) is an integer,

\[ \sum_{n=0}^{\infty} \frac{\prod_{n}(-a, q^{2})^{2n}}{\prod_{n}(-a, q)^{2n}} = \frac{\prod_{n}(-a, q^{2})^{2n}}{\prod_{n}(-a, q)^{2n}} \]

Proof. In Theorem A of [1], take \( r = 1, s = l = 0, e = k, d = 1 \).

**Lemma 2.**

\[ \sum_{n=0}^{\infty} \frac{\prod_{n}(-a, q^{2})^{2n}}{\prod_{n}(-a, q)^{2n}} \]

\[ = \frac{1}{2} \frac{\prod_{n}(-b, q^{2})}{\prod_{n}(-b, q)} \frac{\prod_{n}(-at, q)}{\prod_{n}(-at, q)} \sum_{n=0}^{\infty} \frac{\prod_{n}(-c/b, q^{2})}{\prod_{n}(-c/b, q^{2})} \frac{\prod_{n}(-t, q)^{2n}}{\prod_{n}(-t, q)^{2n}} \]

\[ + \frac{1}{2} \frac{\prod_{n}(-b, q^{2})}{\prod_{n}(-b, q)} \frac{\prod_{n}(-at, q)}{\prod_{n}(-at, q)} \sum_{n=0}^{\infty} \frac{\prod_{n}(-c/b, q^{2})}{\prod_{n}(-c/b, q^{2})} \frac{\prod_{n}(t, q)^{2n}}{\prod_{n}(t, q)^{2n}}. \]
Proof.

\[
\sum_{m=0}^{\infty} \prod_{n=0}^{2m} (-a, q) \prod_{n=0}^{2m} (-b, q^2)^{\tau}\n= \prod_{m=0}^{\infty} (-b, q^2) \sum_{m=0}^{\infty} \prod_{n=0}^{2m} (-a, q) \prod_{n=0}^{2m} (-c, q^2)^{\tau}
\]

\[
= \prod_{m=0}^{\infty} (-b, q^2) \sum_{m=0}^{\infty} (1 + (-1)^m) \prod_{n=0}^{\infty} (-a, q) \prod_{n=0}^{\infty} (-c q^2, q^{2m})
\]

This concludes the proof of Lemma 2.

Lemma 3.

\[
\sum_{n=0}^{\infty} \prod_{n=0}^{\infty} (-a, q^n) \prod_{n=0}^{\infty} (-b, q^n) \prod_{n=0}^{\infty} (-c, q^n)
\]

\[
= \prod_{m=0}^{\infty} (-b, q^2) \sum_{m=0}^{\infty} \prod_{n=0}^{2m} (-c/b, q^2) \prod_{n=0}^{2m} (-t, q)^{\tau}
\]

(by (H1))

\[
= \prod_{m=0}^{\infty} (-b, q^2) \prod_{m=0}^{\infty} (-atq, q) \prod_{m=0}^{\infty} (-atq, q)
\]

(by (H1))

\[
+ \prod_{m=0}^{\infty} (-b, q^2) \prod_{m=0}^{\infty} (at, q) \prod_{m=0}^{\infty} (at, q)
\]

(by Lemma 1 with \(a = -g/\tau, k = 1, b = \beta, c = 0\))
\[ q \text{-IDENTITIES} \]

\[
\prod_{n=0}^{m} \frac{(-\beta, q^{2m} \tau^{2m})}{\prod_{\beta} = (-\tau, q)} = \left\{ \prod_{n=0}^{m} \frac{(-\tau, q)}{\prod_{\beta} = (-\beta, q^{2m} \tau^{2m})} \right\} + \frac{\prod_{n=0}^{m} (-\tau, q)}{\prod_{\beta} = (-\beta, q^{2m+1} \tau^{2m+1})} \\
\text{(by Lemma 3 with } a = c = 0, b = r, t = \beta) \\
= \prod_{n=0}^{m} (-\beta, q^{2m} \tau^{2m}) \prod_{\beta} = (-\tau, q) + \prod_{n=0}^{m} (-\beta, q^{2m} \tau^{2m}) \prod_{\beta} = (-\tau, q) + \prod_{n=0}^{m} (-\beta, q^{2m+1} \tau^{2m+1}) \\
\text{(by (H1) applied to } \prod_{n=0}^{m} (-\beta, q^{2m} \tau^{2m})). \text{ We now derive (I4).}
\]

\[
\sum_{n=0}^{m} \prod_{n=0}^{m} (-\frac{q^{n}-\beta^{n}}{q^{n}}} - \prod_{n=0}^{m} (-\frac{q^{n}-\beta^{n}}{q^{n}}) = \prod_{n=0}^{m} (-\beta, q^{2m} \tau^{2m}) \prod_{\beta} = (-\tau, q) + \prod_{n=0}^{m} (-\beta, q^{2m+1} \tau^{2m+1}) \\
\text{(replacing } q \text{ by } q^{2} \text{ in Lemma 1, then setting } k = 1, a = q/r, b = \beta, c = 0, t = \tau) \\
= \prod_{n=0}^{m} (-\beta, q^{2m} \tau^{2m}) \prod_{\beta} = (-\tau, q) + \prod_{n=0}^{m} (-\beta, q^{2m+1} \tau^{2m+1}) \\
\text{(by Lemma 2 with } a = c = 0, b = r, t = \beta) \\
= \frac{1}{2} \prod_{n=0}^{m} (-\beta, q^{2m+1} \tau^{2m+1}) + \frac{1}{2} \prod_{n=0}^{m} (-\beta, q^{2m+1} \tau^{2m+1}) \\
\text{(I4) is now obtained by noting that (H1) implies}
\]

\[
\prod_{n=0}^{m} (-\beta, q^{2m+1} \tau^{2m+1}) = 2 \prod_{n=0}^{m} (-\beta, q^{2m+1} \tau^{2m+1}) 
\]
We now prove (I5).

\[ \sum_{n=0}^{\infty} \frac{\prod_{s=0}^{n} (-\beta, q)^{2n}}{\prod_{s=0}^{n} (-q^2, q^2)} = \frac{\prod_{s=0}^{m} (-\beta, q)^{2m}}{\prod_{s=0}^{m} (-q, q)} \sum_{m=0}^{\infty} \frac{\prod_{m=0}^{n} (-q^2, q^2)}{\prod_{m=0}^{n} (-q, q)} \beta^m \]

(by Lemma 1 with \( k = 2, a = c = 0 \), \( t = \tau^2, b = \beta \))

\[ = \frac{\prod_{s=0}^{m} (-\beta, q)}{\prod_{s=0}^{m} (-q, q)} \sum_{m=0}^{\infty} \frac{\prod_{m=0}^{n} (-\tau, q)}{\prod_{m=0}^{n} (-q, q)} \beta^m \]

(by Lemma 1 with \( k = 1, a = -\tau, b = \tau, c = 0, t = \beta \)). Simplifying this last expression, we obtain (I5).

Finally for (I6), we have

\[ \frac{\prod_{s=0}^{m} (-\beta, q)}{\prod_{s=0}^{m} (-\tau, q^2)} \sum_{s=0}^{\infty} \frac{\prod_{s=0}^{m} (-\tau, q^2)}{\prod_{s=0}^{m} (-q, q)} \beta^m \]

(by Lemma 1 with \( k = 2, a = c = 0, t = \tau, b = \beta \))

\[ = \sum_{s=0}^{\infty} \frac{\prod_{s=0}^{m} (-\beta q, q^2)}{\prod_{s=0}^{m} (-q^2, q^2)} \beta^m \]

\[ = \frac{\prod_{s=0}^{m} (-\beta, q^2)}{\prod_{s=0}^{m} (-\tau, q^2)} \sum_{m=0}^{\infty} \frac{\prod_{s=0}^{m} (-\tau, q^2)}{\prod_{s=0}^{m} (-q^2, q^2)} \beta^m \]

(replacing \( q \) by \( q^2 \) in Lemma 1, then setting \( k = 1, a = \beta q, b = \beta, c = 0, t = \tau \)). Simplifying this last expression, we obtain (I6).

References


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