A Note On a Method of Erdős and the Stanley-Elder Theorems

by
George E. Andrews and Emeric Deutsch

December 30, 2015

Abstract

An enumeration method of Erdős is applied to provide a massive generalization of the theorems of Stanley and Elder on integer partitions.

Key words: Partition, Stanley’s Theorem, Elder’s Theorem

Subject Classification Code: 05A17, 05A19, 11P81

1 Introduction

In [4], Erdős provided the asymptotics of the partition function $p(n)$ by elementary means. His starting point was the identity of Ford [7] (probably going back to Euler):

\[ np(n) = \sum_{j=1}^{n} p(n - j)\sigma(j), \tag{1.1} \]

where $\sigma(j)$ is the sum of divisors of $j$. The standard proof of (1.1) is by logarithmic differentiation of ([7], also [1, p.98])

\[ \sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1 - q^n}. \tag{1.2} \]
However, Erdős wanted to avoid even this amount of analysis. So he rewrote \((1.1)\) as follows
\[
np(n) = \sum_{v \geq 1} \sum_{k \geq 1} vp(n - kv),
\]
and then he remarked: "We easily obtain \((1.3)\) by adding up all the partitions of \(n\), and noting that \(v\) occurs in \(p(n - v)\) partitions." We assume he is telegraphing that \(v\) appears twice in \(p(n - 2v)\) partitions, etc.

This same counting method makes transparent a very general theorem in partitions.

**Definition 1.** A partition configuration, \(A\), is a non-decreasing sequence of non-negative integers, \((a_1, \ldots, a_k)\) with length \(k\) and weight \(w(A) = a_1 + a_2 + \cdots + a_k\).

**Definition 2.** A partition, \(\lambda: \lambda_1 + \lambda_2 + \cdots + \lambda_m\) (\(1 \leq \lambda_1 \leq \lambda_2 \cdots \leq \lambda_m\)) is said to have a partition configuration \(A\) if there is a subset of parts of \(\lambda\) of the form \(a_1 + j, a_2 + j, \ldots, a_k + j\) for some \(j \geq 1\).

For example, the partition \((2 + 4 + 4 + 5 + 8 + 9)\) contains an instance of \(A = (0, 3, 6, 7)\) because the parts 2, 5, 8, 9 exceed by 2 the successive entries of \(A\).

**Theorem 1.** Given a partition configuration \(A\), in each partition of \(n\) we count the number of distinct configurations \(A\) therein and then sum over all partitions of \(n\). Call this sum \(p_A(n)\). Then
\[
p_A(n) = p(k; n - w(A)),
\]
where \(p(k; n)\) is the total number of appearances of \(k\) in the partitions of \(n\).

As an example of Theorem 1, we take \(A: (0, 1, 2)\) (having length \(k = 3\) and weight \(w(A) = 3\)) and \(n = 10\). The partitions of 10 containing the partition configuration \(A\) are \(1 + 1 + 1 + 1 + 1 + 2 + 3, 1 + 1 + 1 + 2 + 2 + 3, 1 + 2 + 2 + 2 + 3, 1 + 1 + 2 + 3 + 3\) and \(1 + 2 + 3 + 4\) which contain \(A\) \(1 + 1 + 1 + 1 + 2 = 6\) times. So \(p_A(10) = 6\). As for \(p(3; 10 - 3) = p(3; 7)\) we see that the partitions of 7 containing 3’s are \(1 + 1 + 1 + 1 + 3, 1 + 1 + 2 + 3, 2 + 2 + 3, 1 + 3 + 3, 3 + 4\). So \(p(3; 7) = 1 + 1 + 1 + 2 + 1 = 6\), the total number of 3’s in the partitions of 7.

In section 2, we use the Erdős method to provide a short proof of Theorem 1 together with the theorems of Elder and Stanley. In section 3, we extend these ideas to a question concerning divisibility restrictions on parts. We conclude with some general observations.
2 Proof of Theorem 1.

We remark following Erdős that to obtain \( p_A(n) \) there must be \( p(n - ((a_1 + j) + \cdots + (a_k + j))) \) partitions which contain the partition configuration \( A \) in the form

\[
(a_1 + j) + (a_2 + j) + \cdots + (a_k + j).
\]

Hence

\[
\sum_{n \geq 0} p_A(n)q^n = \sum_{j=1}^{\infty} \frac{q^{(a_1+j)+(a_2+j)+\cdots+(a_k+j)}}{\prod_{n=1}^{\infty} (1 - q^n)}
= \frac{q^{w(A)} \sum_{j=1}^{\infty} q^{kj}}{\prod_{n=1}^{\infty} (1 - q^n)}
= \frac{1 - q^k}{(1 - q^k)^2 \prod_{n=1}^{\infty} (1 - q^n)}
= q^{w(A)} \left(q^k + 2q^{2k} + 3q^{3k} + \cdots\right) \prod_{n=1}^{\infty} (1 + q^n + q^{2n} + q^{3n} + \cdots)
= q^{w(A)} \sum_{n \geq 0} p(k, n)q^n,
\]

and Theorem 1 follows by comparing coefficients of \( q^n \) in the extremes of (2.1).

\[\square\]

**Corollary 2** (Stanley’s Theorem [2],[8]). The number of 1’s in the partitions of \( n \) is equal to the number of parts that appear at least once in a given partition of \( n \), summed over all partitions of \( n \).

**Proof.** Take \( A : (0) \) in Theorem 1.

A more general theorem is attributed to Paul Elder.

**Corollary 3** (Elder’s Theorem [2][8]). The number of \( j \)'s appearing in the partitions of \( n \) is equal to the number of parts that appear at least \( j \) times in a given partition of \( n \), summed over all partitions of \( n \).

**Proof.** Take \( A : (0, 0, \ldots, 0) \) of length \( j \) in Theorem 1.

**Corollary 4.** In each partition of \( n \) count the number of sequences of consecutive integers of length \( k \). Then sum these numbers over all partitions of \( n \). This equals the number of appearances of \( k \) in the partitions of \( n - k(k-1)/2 \).

**Proof.** In Theorem 1 take \( A : (0, 1, \ldots, k - 1) \).

\[\square\]
3 Divisibility of Parts

The method of Erdős can be further extended in many ways.

**Theorem 5.** Given $k \geq 1$. In each partition of $n$ we count the number of times a part divisible by $k$ appears uniquely (i.e. is not a repeated part); then sum these numbers over all the partitions of $n$. The result is equal to the number of appearances of $2k$ in the partitions of $n + k$.

**Example.** $k = 1$, $n = 5$. There are eight singletons in the partitions of 5: 5, 4 + 1, 3 + 2, 3 + 1 + 1, 2 + 2 + 1, 2 + 1 + 1 + 1, 1 + 1 + 1 + 1 + 1. There are eight 2’s in the partitions of 6: 4 + 2, 3 + 2 + 1, 2 + 2 + 2, 2 + 2 + 1 + 1, 2 + 1 + 1 + 1 + 1.

**Remark.** The case $k = 1$ was published as a problem in [3].

**Proof.** The generating function for multiples of $k$ being unique parts is

$$
\sum_{j=1}^{\infty} \prod_{n \neq kj} q^{kj} = \frac{1}{\prod_{n=1}^{\infty} \left(1 - q^n\right)} \sum_{j=1}^{\infty} q^{kj} \left(1 - q^{kj}\right)
$$

$$
= \frac{1}{\prod_{n=1}^{\infty} \left(1 - q^n\right)} \left( \frac{q^k}{1 - q^k} - \frac{q^{2k}}{1 - q^{2k}} \right)
$$

$$
= \frac{q^k}{(1 - q^{2k})} \cdot \prod_{n=1}^{\infty} \left(1 - q^n\right)
$$

$$
q^{-k} \left( q^{2k} + 2q^{3k} + 3q^{4k} + \cdots \right) \prod_{n=1 \atop n \neq 2k}^{\infty} \frac{1}{1 - q^n},
$$

and this last expression is the generating function for the number of appearances of $2k$ in the partitions of $n + k$.  

4 Conclusion

It is clear that the scope of Theorem 1 could be generalized to account for results like Theorem 4. We should also note that Dastidar and Gupta [2] have generalized the Stanley and Elder theorems where they add what they term ”packets” of size $k$ to partitions, and this count equals the number of appearances of $k$ in the partitions of $n + k$. 

4
Finally we note the charming survey "A Fine Rediscovery" by R. Gilbert [8], which provides a detailed history of the Stanley and Elder theorems and points out that N. J. Fine was the original discoverer of both theorems [5],[6].

References


The Pennsylvania State University University Park, PA 16802 gea1@psu.edu

Polytechnic Institute of New York University, Brooklyn, NY, 11201 emericdeutsch@msn.com