THE MEANING OF RAMANUJAN NOW AND FOR THE FUTURE

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Abstract. December 22, 2010 marks the 123th anniversary of Ramanujan's birth. In this paper we pay homage to this towering figure whose mathematical discoveries so affected mathematics throughout the twentieth century and into the twenty-first.

1. Introduction

Whenever we remember Ramanujan, three things come most vividly to mind: (1) Ramanujan was a truly great mathematician; (2) Ramanujan's life story is inspiring; and (3) Ramanujan's life and work give credible support to our belief in the Universality of truth. We shall examine each of these topics in the next three sections. A careful examination of each topic should, at least, give us some inkling of the meaning of Ramanujan.

2. Ramanujan's Early Work

First and foremost in any assessment of Ramanujan are his research achievements. These are presented in his *Collected Papers* [38], his *Notebooks* [39], and the *Lost Notebook* [40]. The Collected Papers have recently been reissued by AMS-Chelsea [38] with 70 pages of commentary provided by Bruce Berndt. In addition, Berndt [19] has provided a five volume account of the *Notebooks* with all of Ramanujan's assertions proved in full. An edited version of the *Lost Notebook* [40] is being prepared by Berndt and me ([11] is the first of four volumes).

If I had to choose one word to characterize the impact of Ramanujan's work, it would be "surprise." This is perhaps most evident in Hardy's response to these two Ramanujan formulas:

$$\frac{1e^{-2\pi}e^{-4\pi}}{1+1+1+\cdots} = \left\{\sqrt{\left(\frac{5+\sqrt{5}}{2}\right)} - \frac{\sqrt{5}+1}{2}\right\} e^{i\pi} \tag{2.1}$$

$$\frac{1e^{-2\pi\sqrt{5}}e^{-4\pi\sqrt{5}}}{1+1+1+\cdots} = \left[\frac{\sqrt{5}}{1+\sqrt[5]{\left\{5^{\frac{3}{4}}\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{5}{2}}-1\right\}}} - \frac{\sqrt{5}+1}{2}\right] e^{\frac{2\pi}{\sqrt{5}}} \tag{2.2}$$

Hardy commented on (2.1) and (2.2) as follows [26; p. 9]:

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> "I had never seen anything in the least like them before. A single look at them is enough to show that they could only be written down by a mathematician of the highest class. They must be true because, if they were not true, no one would have had the imagination to invent them. Finally (you must remember that I knew nothing whatever about Ramanujan, and had to think of every possibility), the writer must be completely honest, because great mathematicians are commoner than thieves or humbugs of such incredible skill."

We have subsequently learned from G. N. Watson [53] that these two formulas are rather intricate corollaries of the Rogers-Ramanujan identities.

$$1+\frac{q}{1-q}+\frac{q^4}{(1-q)(1-q^2)}+\frac{q^9}{(1-q)(1-q^2)(1-q^3)}+\dots = \frac{1}{(1-q)(1-q^6)\dots(1-q^4)(1-q^9)\dots} \tag{2.3}$$

$$1+\frac{q^2}{1-q}+\frac{q^6}{(1-q)(1-q^2)}+\frac{q^{12}}{(1-q)(1-q^2)(1-q^3)}+\dots = \frac{1}{(1-q^2)(1-q^7)\dots(1-q^3)(1-q^8)\dots} \tag{2.4}$$

While it is true that Rogers anticipated these formulas [45], it is also true that no one before Ramanujan appreciated the immense implications of (2.3) and (2.4).

Why are they surprising? For one thing they appear to be first cousins of identities like Euler's result [2; eqs. (1.2.5) and (2.2.6) at $t = q$].

$$1+\frac{q}{1-q}+\frac{q^3}{(1-q)(1-q^2)}+\frac{q^6}{(1-q)(1-q^2)(1-q^3)}+\dots = \frac{1}{(1-q)(1-q^3)\dots(1-q^5)\dots} \tag{2.5}$$

Euler's result is very easy to prove; the Rogers-Ramanujan identities were surprisingly troublesome as Hardy notes [26; p. 91].

> "The formulas have a very curious history. They were found first in 1894 by Rogers, a mathematician of great talent but comparatively little reputation, now remembered mainly from Ramanujan's rediscovery of his work. Rogers was a fine analyst, whose gifts were, on a smaller scale, not unlike Ramanujan's; but no one paid much attention to anything he did, and the particular paper in which he proved the formulae was quite neglected.
>
> Ramanujan rediscovered the formulae sometime before 1913. He had then no proof (and knew that he had none), and none of the mathematicians to whom I communicated the formulae could find one. They are therefore stated without proof in the second volume of MacMahon's *Combinatory analysis.*
>
> The mystery was solved, trebly, in 1917. In that year Ramanujan, looking through old volumes of the Proceedings of the London Mathematical Society, came accidentally across Rogers's paper. I can remember very well his surprise, and the admiration which he expressed for Rogers's work. A correspondence followed in the

> course of which Rogers was led to a considerable simplification of his original proof. About the same time I. Schur, who was then cut off from England by the war, rediscovered the identities again. Schur published two proofs, one of which is "combinatorial" and quite unlike any other proof known."

Of course, we cannot omit Hardy's own favorite Ramanujan formula [38; p. xxxv]:

$$p(4) + p(9) + p(14)x^2 + \ldots = 5\frac{\left\{\left(1-x^5\right)\left(1-x^{10}\right)\left(1-x^{15}\right)\ldots\right\}^5}{\left\{(1-x)\left(1-x^2\right)\left(1-x^3\right)\ldots\right\}^6} \tag{2.6}$$

where p(n) is the number of partitions of n.

The factor 5 is enough by itself to surprise anyone. An immediate corollary of this result is that every fifth value of the partition function $p(n)$ is divisible by 5. Identity (2.6) may be seen as the precursor of countless magnificent results that stretch through the twentieth century and that culminated in A. O. L. Atkin's conquest [17] of the Ramanujan conjecture for congruences associated to the partition function. In the last two decades, Ken Ono and his collaborators have tremendously extended and expanded the theory of congruences for $p(n)$ (see for example, [37]).

3. Surprises in the Lost Notebook

This section begins an aspect of Ramanujan's work that has only recently been verified [9], [10]. It concerns two identities that are so surprising to an insider that I avoided studying them for twenty five years out of fear. Here are these formidable identities. First from the middle of page 26 in Ramanujan's Lost Notebook [40]

$$\sum_{n=0} a^n q^{n^2} = \prod_{n=1}^{\infty} \left(1 + aq^{2n-1}(1 + y_1(n) + y_2(n) + \cdots)\right), \tag{3.1}$$

where

$$y_1(n) = \frac{\sum_{j=n}^{\infty}(-1)^j q^{j(j+1)}}{\sum_{j=0}^{\infty}(-1)^j(2j+1)q^{j(j+1)}} \tag{3.2}$$

and

$$y_2(n) = \frac{\left(\sum_{j=n}^{\infty}(j+1)(-1)^j q^{j(j+1)}\right)\left(\sum_{j=n}^{\infty}(-1)^j q^{j(j+1)}\right)}{\left(\sum_{j=0}^{\infty}(-1)^j(2j+1)q^{j(j+1)}\right)^2}. \tag{3.3}$$

Our second result is the third identity on page 57 of [40]

$$\sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q;q)_n} = \prod_{n=1}^{\infty}\left(1 + \frac{aq^{2n-1}}{1 - q^n y_1 - q^{2n} y_2 - q^{3n} y_3 - \cdots}\right), \tag{3.4}$$

where

$$y_1 = \frac{1}{(1-q)\psi^2(q)}, \tag{3.5}$$

$$y_2 = 0, \tag{3.6}$$

$$y_3 = \frac{q+q^3}{(1-q)(1-q^2)(1-q^3)\psi^2(q)} - \frac{\sum_{n=0}^{\infty} \frac{(2n+1)q^{2n+1}}{1-q^{2n+1}}}{(1-q)^3\psi^6(q)}, \tag{3.7}$$

$$y_4 = y_1 y_3, \tag{3.8}$$

$$\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \tag{3.9}$$

and where

$$(A;q)_n = (1-A)(1-Aq)(1-Aq^2)\cdots(1-Aq^{n-1}). \tag{3.10}$$

In both identities $0 < q < 1$.

It seems to have taken me forever to recognize that both of these results lie in the realm of entire funcitons of the variable a.

To understand something of the depth of these discoveries it is necessary to provide at least an intuitive introduction to entire functions. First of all, an entire funciton is an analytic function of z that has no singularities in the finite portion of the z plane. Consequently, it has a power series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

that converges for all z.

We now turn to E. T. Copson [24; p. 158] who succinctly describes the analogy of entire functions and polynomials:

> "The most important property of a polynomial is that it can be expressed uniquely as a product of linear factors of the form
>
> $$Az^p\left(1-\frac{z}{z_1}\right)\left(1-\frac{z}{z_2}\right)\cdots\left(1-\frac{z}{z_n}\right),$$
>
> where A is a constant, p is a positive integer or zero, and z_1, z_2, $\ldots$, z_n the points, other than the origin, at which the polynomial vanishes, multiple zeros being repeated in the set according to their order. Conversely, if the zeros are given, the polynomial is determined apart from an arbitrary constant multiplier.
>
> Now a polynomial is an integral function [i.e. entire function] of a very simple type, its singularity at infinity being a pole. We naturally ask whether it is possible to exhibit in a similar manner the way in which any integral function depends on its zeros."

This prologue leads to the central fundamental theorem on entire functions, Hadamard's Factorization Theorem, [24; p. 174]. The full theorem is not necessary for us. In fact, we need only the following special case.

Hadamard's Factorization Theorem (weak case)**.** *Suppose $f(z)$ is an entire function with simple zeros at $z_1, z_2, z_3, \ldots, f(0) = 1$ and $\sum_{n=1}^{\infty} |z_n|^{-1} < \infty$, then*

$$f(z) = \prod_{n=1}^{\infty}\left(1-\frac{z}{z_n}\right).$$

In each of (3.1) and (3.4), we note that each series defines an entire function of the complex variable a. In each identity, Ramanujan is presenting the Hadamard Factorization of the function in question and additionally is specifying explicit formulas, or at least approximations, for each of the zeros.

Once one understands that this is what is going on, generally we are still far from understanding why Ramanujan is able to make these assertions about the zeros of these functions.

The full details are presented in [9] and [10], but the main idea involved relies on polynomial approximations of these functions. For (3.4), the relevant polynomial sequence is

$$K_n(a) = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q q^{j^2} a^j, \tag{3.11}$$

where

$$\begin{bmatrix} n \\ j \end{bmatrix}_q = \begin{cases} 0 & \text{if } j \leq 0 \text{ or } j > n \\ 1 & \text{if } j = 0 \text{ or } n \\ \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-j+1})}{(1-q^j)(1-q^{j-1})\cdots(1-q)} & \text{otherwise} \end{cases} \tag{3.12}$$

For (2.1), the relevant polynomial sequence is

$$p_n(a) = (q^2;q^2)_\infty(-aq;q^2)_n \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_{q^2} \frac{q^{2j}}{(-aq;q^2)_j}. \tag{3.13}$$

It turns out that

$$\lim_{n\to\infty} K_n(a) = \sum_{m=0}^{\infty} \frac{a^m q^{m^2}}{(q;q)_m}, \tag{3.14}$$

and

$$\lim_{n\to\infty} p_n(a) = \sum_{m=0}^{\infty} q^{m^2} a^m. \tag{3.15}$$

The sequence $K_n(a)$ are, in fact, the Stieltjes–Wigert polynomials. G. Szegö [51] had studied these at length as an interesting family of orthogonal polynomials. Indeed, his paper concludes by noting that the limiting funciton

$$\sum_{n=0}^{\infty} \frac{q^{n^2} a^n}{(q;q)_n}$$

has (for $0 < q < 1$) real, negative simple zeros. This then is the obvious starting point for proving (3.4), and this is the tack taken in [9].

Proving (3.1) turns out to be much more problematic. Knowledge of q-hypergeometric series leads one inexorably to the sequence $p_n(x)$. However, they do *not* form a family of orthogonal polynomials. Consequently all of their important features must be deduced ex nihilo. It must be noted that the radii of convergence given in [9] and [10] are too large. What can actually be proved by standard methods requires $0 < q < 0.00406$, a much smaller interval than $0 < q < 1/4$. This was previously pointed out in [12].

The implications and extensions of these results are explored by Huber [27], Huber and Yee [28] and Ismail [29].

We now move on to another set of surprises.

In the early part of the 18th century, L. Euler observed that for each integer n, the number of partitions of n into odd parts equals the number of partitions of n into distinct parts. He provided a proof through generating functions. For subsequent purposes, we name the function in question $S(q)$. Euler observed

$$S(q) := \prod_{n=1}^{\infty}(1+q^n) = \prod_{n=1}^{\infty}\frac{(1-q^{2n})}{(1-q^n)} = \prod_{n=0}^{\infty}\frac{1}{1-q^{2n+1}}. \tag{3.16}$$

Here is what Ramanujan does with $S(q)$ (taken from mid-page 14 of [40]).

$$\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(-q;q)_n} = 1+\sum_{n=0}^{\infty}q^{n+1}(-1)^n(q;q)_n \tag{3.17}$$

$$= S(q) + 2\sum_{n=0}^{\infty}(S(q)-(-q;q)_n) - 2S(q)\sum_{n=1}^{\infty}\frac{q^n}{1-q^n} \tag{3.18}$$

$$= S(q) + 2\sum_{n=0}^{\infty}\left(S(q)-\frac{1}{(q;q^2)_n}\right) - 2S(q)\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}. \tag{3.19}$$

The last two lines here truly stunned me when I first gazed at them. The infinite series involving $S(q)$ are quite unlike anything I had seen before. In each of these series, Ramanujan is taking the difference between $S(q)$ and a partial product that converges to $S(q)$. Why in the world should this produce anything as orderly as these identities?

With a sense of great exhilaration I managed to prove these results in January of 1985. The proof appeared in 1986 [4].

Many years later, in 2000, Ken Ono rekindled my interest by noting that Don Zagier had proved a very similar result [55] and had applied it to an interesting L-function evaluation. Our subsequent explorations led us to the following very general result [14; p. 403]:

Proposition 1. *Suppose that*

$$f(z) = \sum_{n=0}^{\infty}\alpha_n z^n$$

is analytic for $|z|<1$. If α is a complex number for which

(i) $\sum_{n=0}^{\infty}(\alpha-\alpha_n) < +\infty$,

(ii) $\lim_{n\to+\infty} n(\alpha-\alpha_n) = 0$,

then

$$\lim_{z\to 1^-}\frac{d}{dz}(1-z)f(z) = \sum_{n=0}^{\infty}(\alpha-\alpha_n).$$

From this general result, identities (3.17)–(3.19) are not that hard to prove.

In the ensuing time, Coogan, Lovejoy and Ono [22], [23], [34] have found many further applications of this result. Recently, P. Freitas and I [13] have extended this result to the following theorem involving higher derivatives:

Proposition 2. *Let*

$$f(z) = \sum_{n=0}^{\infty}\alpha_n z^n$$

be analytic for $|z| < 1$, and assume that for some positive integer p we have that

$$\sum_{n=0}^{\infty}\left[\prod_{j=1}^{p}(n+j)\right](\alpha_{n+p}-\alpha_{n+p-1})$$

converges;

$$\sum_{n=0}^{\infty}\left[\prod_{j=1}^{p-1}(n+j)\right](\alpha-\alpha_{n+p-1})$$

converges, where α is a fixed complex number such that

$$\lim_{n\to\infty}\left[\prod_{j=1}^{p}(n+j)\right](\alpha_{n+p}-\alpha)=0.$$

Then

$$\frac{1}{p}\lim_{z\to 1^-}\left\{\frac{d^p}{dz^p}[(1-z)f(z)]\right\}=\sum_{n=0}^{\infty}\left[\prod_{j=1}^{p-1}(n+j)\right](\alpha-\alpha_{n+p-1})$$

We have applied this more general result to a wide variety of Ramanujan style identities. For example,

$$\sum_{n=0}^{\infty}\left[\frac{(t)_n}{(t)_\infty}-1\right]=\sum_{n=1}^{\infty}\frac{t^n}{(q)_n(1-q^n)},$$

$$\sum_{n=0}^{\infty}\left[\frac{(t)_n}{(q)_n}-\frac{(t)_\infty}{(q)_\infty}\right]^2=\left[\frac{(t)_\infty}{(q)_\infty}\right]^2\sum_{n=1}^{\infty}\frac{(q/t)_n}{(q)_n}\left[\frac{(q)_n}{(t)_n}-1\right]\frac{t^n}{1-q^n},$$

$$\sum_{n=0}^{\infty}\left[1-\frac{(q)_\infty}{(q)_n}\right]^2=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{q^{n(n+1)/2}}{(q)_n}\frac{1-(q)_n}{1-q^n},$$

$$\sum_{n=0}^{\infty}\frac{(q)_\infty}{(q)_n}\left[1-\frac{(q)_\infty}{(q)_n}\right]=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{q^{n(n+1)/2}}{1-q^n}.$$

For our final set of surprises let us conclude with five formulas from the Lost Notebook [39; p. 13].

$$\frac{1}{1+q}-\frac{q^2(1-q)}{(1+q)(1+q^2)(1+q^3)}+\frac{q^6(1-q)\left(1-q^3\right)}{(1+q)(1+q^2)(1+q^3)(1+q^4)(1+q^5)}-\cdots$$
$$=1-q+q^3-q^6+\cdots \quad (3.20)$$

$$\frac{1}{1+q}+\frac{q(1-q)^2}{(1+q)(1+q^2)(1+q^3)}+\frac{q^2(1-q)^2\left(1-q^3\right)^2}{(1+q)(1+q^2)(1+q^3)(1+q^4)(1+q^5)}-\cdots$$
$$=1-q^2+q^6-q^{12}+\cdots \quad (3.21)$$

$$\frac{1}{1+q}+\frac{q(1-q)}{(1+q)(1+q^2)(1+q^3)}+\frac{q^2(1-q)\left(1-q^3\right)}{(1+q)(1+q^2)(1+q^3)(1+q^4)(1+q^5)}-\cdots$$
$$=1-q^3+q^9-q^{18}+\cdots \quad (3.22)$$

$$\frac{1}{1+q}+\frac{q\,(1-q)}{(1+q)\,(1+q^3)}+\frac{q^2\,(1-q)\left(1-q^3\right)}{(1+q)\,(1+q^3)\,(1+q^5)}-\cdots$$
$$=1-q^4+q^{12}-q^{24}+\cdots \quad (3.23)$$

$$\frac{1}{1+q}+\frac{q\,(1-q)}{(1+q)\,(1+q^3)}+\frac{q^2\,(1-q)\left(1-q^2\right)}{(1+q)\,(1+q^3)\,(1+q^5)}-\cdots$$
$$=1-q^6+q^{18}-q^{36}+\cdots \quad (3.24)$$

Here we have five formulas listed one after the other by Ramanujan. What's the surprise? Some are much deeper than others. The first is easy for any q-series researcher. The second and fourth are challenging exercises, but they fall relatively rapidly. The third and the fifth required on and off efforts for months (cf. [3; §6]). Subsequently, S.O. Warnaar and I have been able to unify the first four of these identities, and J. Lovejoy has pointed out that these four follow from some of his more general q-hypergeometric series identities.

Our choice of topics in this section has been dictated to some extent by the oddness or surprise of Ramanujan's discoveries. However, it should be stressed that there are numerous recent spectacular discoveries by Ono, Bringmann, Mahlburg, Zwegers and others related to Ramanujan's mock theta functions [20], [21], [36], [56].

4. Ramanujan and Computation

Anyone who studies Ramanujan's Lost Notebook comes away with some appreciation of the importance of computation in Ramanujan's work. Again, G. H. Hardy provides a relevant comment [38; p. xxxv]

> "His memory, and his powers of calculation, were very unusual, but they could not reasonably be called "abnormal". If he had to multiply two large numbers, he multiplied them in the ordinary way; he would do it with unusual rapidity and accuracy, but not more rapidly or more accurately than any mathematician who is naturally quick and has the habit of computation. There is a table of partitions at the end of our paper .... This was, for the most part, calculated independently by Ramanujan and Major MacMahon; and Major MacMahon was, in general, slightly the quicker and more accurage of the two."

In a monograph on q-series [5; p. 87], I was led to speculate about Ramanujan and the age of computer algebra:

"Sometimes when studying his work I have wondered how much Ramanujan could have done if he had had MACSYMA or SCRATCH-PAD or some other symbolic algebra package. More often I get the feeling that he was such a brilliant, clever, and intuitive computer himself that he really did not need them."

However much fun such speculation may be; it is more important that we examine the ways in which Ramanujan's computations and calculations have guided more recent discoveries. Among the letters from Ramanujan that perhaps influenced Hardy's remark, we might mention the one on Ramanujan's empirical evidence for the Rogers-Ramanujan identities [40; pp. 360-361]. The study of the calculations led to a truncated version of the Bailey Chain method [7] which in turn led to positivity theorems about the differences of successive Gaussian polynomials [8].

However, there is perhaps a more revealing result in his early notebooks [19; Part V, p. 130]. In this identity, p_n denotes the nth prime. So $p_1 = 2$, $p_2 = 3$, $p_3 = 5, \ldots$.

$$\prod_{n=1}^{\infty} \frac{1}{1-q^{p_n}} = 1 + \sum_{j=1}^{\infty} \frac{q^{p_1+p_2+\cdots+p_j}}{(1-q)(1-q^2)\cdots(1-q^j)}. \tag{4.1}$$

It should be noted that this assertion is *false*, and also Ramanujan must have thought so because he drew a line through this formula. Here we face one of the few false results asserted by Ramanujan. The temptation is to forget (4.1) and move on to the countless valued discoveries by Ramanujan.

However, the question nags: Why would Ramanujan have written down (4.1) in the first place? In the late 1980's A. and J. Knopfmacher [33; Th. 1.4] proved the following theorem. Let $\mathcal{L} = \mathbb{C}((q))$ be the field of formal Laurent series over the complex numbers, $\mathbb{C}$. If

$$A = \sum_{n=\nu}^{\infty} \mathcal{C}_n q^n,$$

we call $\nu = \nu(A)$ the *order* of A and we define the *norm* of A to be

$$\|A\| = 2^{-\nu(A)}.$$

In addition, we define the *integral part* of A to be

$$[A] = \sum_{\nu \leqq n \leqq} \mathcal{C}_n \mathcal{Q}^n.$$

Theorem 3 (Extended Engel Expansion Theorem)**.** *Every $A \in \mathcal{L}$ has a finite or convergent (relative to the above norm) series expansion of the form*

$$A = a_0 + \sum_{n=1}^{\infty} \frac{1}{a_1 a_2 \cdots a_n},$$

where $a_n \in \mathbb{C}[q^{-1}]$, $a_0 = [A]$, $\nu(a_n) \leqq -n$, and $\nu(a_{n+1}) \leqq \nu(a_n) - 1$.

The series is unique for A (up to constants in $\mathbb{C}$), and it is finite if and only if $A \in \mathbb{C}(q)$. In addition, if

$$a_0 + \sum_{j=1}^{n} \frac{1}{a_1 \cdots a_j} = \frac{p_n}{q_n}, \quad \text{where} \quad q_n = a_1 a_2 \cdots a_n,$$

then

$$\left\| A - \frac{p_n}{q_n} \right\| \leqq \frac{1}{2^{n+1}\|q_n\|}$$

and

$$\nu\left(A - \frac{p_n}{q_n}\right) = -\nu(q_{n+1}) \geqq \frac{(n+1)(n+2)}{2}.$$

In fact, the a_n are given by

$$a_n = \left[\frac{1}{A_n}\right]$$

where $A_0 = A$, $a_0 = [A]$, and

$$A_{n+1} = a_n A_n - 1.$$

From this expansion theorem they observed empirically that the generalized Engel expansion for

$$\prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}$$

is, in fact,

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n},$$

i.e. they had found an empirical method that led to the First Rogers-Ramanujan identity (2.3). In subsequent work [15], it was shown that this observation can be rigorously established. In addition, while (4.1) is false, nonetheless, the generalized Engel expansion of the left side of (4.1) yields precisely the first four terms of the right side before chaos sets in in subsequent terms.

Thus while we cannot be certain how Ramanujan came upon (4.1), we can note that our knowledge that “something’s going on here” provides inspiration for the exploitation of results like the generalized Engel expansion.

As a final comment in this section, I hark back to Hardy’s comparison of the computational skills of Ramanujan and P. A. MacMahon. Certainly at the beginning of the twentieth century MacMahon was the one other major researcher applying computational studies to problems in the theory of partitions.

MacMahon was perhaps most proud of his method of Parition Analysis [35; Sec. VIII]. The method fell into disuse because of the massive calculations necessary in its employment. However, with the advent of computer algebra systems, this method has flourished. The most direct applications of MacMahon’s ideas are developed in [16]. Related systems have been developed by Stembridge [50] implementing the ideas of Stanley [49] and DeLoera [25] implementing Barvinok’s work [18].

5. The Life of Ramanujan

Almost anyone interested enough in Ramanujan to be reading these words knows the broad outline of Ramanujan’s life. There have been many biographies written. In India, there are books by S. R. Ranganathan [43], Suresh Ram [41], and K. Srinivasa Rao [44] to name only three. There are booklets for students by T. Soundararajan [47] and P. K. Srinivasan [48]. S. Ramaseshan published a magnificent article in *Current Science* [42]. This is a small sampling of the interest in and affection shown to the memory of Ramanujan by his countrymen.

In North America, Robert Kanigel’s *The Man Who Knew Infinity* [30] was widely read and translated into German [31] and Korean [32].

Ramanujan was born to a poor Brahmin family in Erode on December 22, 1887 and grew up in Kumbakonam. He took an early interest in mathematics and always excelled in his study thereof. In 1904, he entered the Government College at Kumbakonam. He continued to do well at mathematics but neglected other subjects. As a result, he lost his scholarship and left the college. By 1913 he was married and employed in a dead-end job as a clerk at the Madras Port Trust. Nonetheless, his mathematical studies continued unabated. At the suggestion of friends, he communicated some of his work to English mathematicians, among them G. H. Hardy.

Hardy very quickly perceived the depth and brilliance of Ramanujan’s achievements. Hardy arranged for Ramanujan to come to England in 1914. For the next several years, these two collaborated on truly pathbreaking research. They found the formula for the partition function, literally founded probabilistic number theory, and made clear the major role that modular forms could play in several aspects of number theory.

In 1917, Ramanujan became ill. It was thought at the time that he had tuberculosis. After convalescing in England through much of 1918, he returned to India in 1919 and died in 1920.

It was during this final year that the *Lost Notebook* (actually a collection of approximately 100 sheets of paper) was written.

In speaking of that last year, S. R. Ranganathan quotes Janaki Ammal, Ramanujan's widow [43; MT6]:

> "He returned from England only to die, as the saying goes. He lived for less than a year. Throughout this period, I lived with him without break. He was only skin and bones. He often complained of severe pain. In spite of it he was always busy doing his Mathematics. That, evidently helped him to forget the pain. I used to gather the sheets of paper which he filled up. I would also give the slate whenever he asked for it. He was uniformly kind to me. In his conversation he was full of wit and humour. Even while mortally ill, he used to crack jokes. One day he confided in me that he might not live beyond thirty-five and asked me to meet the event with courage and fortitude. He was well looked after by his friends. He often used to repeat his gratitude to all those who had helped him in his life."

Ramanujan's life is truly inspiring. The rise from poverty to international acclaim is breathtaking. His courage facing death was magnificent. However, his contributions were not just for his time; they affected mathematics and mathematicians for the next century.

It is, of course, possible to be so overwhelmed by the Ramanujan story that one loses the feeling that it might possess lessons for one's own life. S. R. Ranganathan [43] tells the following appealing story which is more down to earth:

> "It was February 1912. K S Srinivasan, popularly called "Sandow" by his friends and a classmate of mine in the Madras Christian College, had known Ramanujan intimately while at Kumbakonam. He called on Ramanujan at Summer House one evening.

SANDOW: Ramanju, they all call you a genius.

RAMANUJAN: What! Me a genius! Look at my elbow, it will tell you the story.

SANDOW: What is all this Ramanju? Why is it so rough and black?

RAMANUJAN: My elbow has become rough and black in making a genius of me! Night and day I do my calculation on slate. (It) is too slow to look for a rag to wipe it out with. I wipe out the slate almost every few minutes with my elbow."

6. The Universality of Truth

In 1991 in *Minerva*, the sociologist Edward Shils published an article entitled Reflections on Tradition, Centre and Periphery and the Universal Validity of Science: The Significance of the Life of S. Ramanujan [46]. Roughly the first half of

Shils' article is devoted to an account of various movements (mostly arising in the West) which question the universality of science. Shils' italicized sub-headings are enough to give the flavor:

Ambivalent Marxism: an early forerunner of the denial of the universality of science.

Sociologists' disregard for any questions regarding the validity of scientific knowledge Meta-history as a denial of the possibility of truthful knowledge

Does every civilization have its own criteria of scientific validity?

Shils (and I) are firmly convinced of the universal validity of science, and Shils turns to Ramanujan to make the case [46; pp. 407, 408, 413]:

> "From thinking about Ramanujan, I have concluded that there are no territorial or social or religious or ethnic limitations on the validity of what a scientist discovers. The discoveries of a scientist of one civilization or nationality can be received, assessed and assimilated by scientists of any other civilization or nationality, assuming, of course, that the recipient scientist is sufficiently informed regarding the state of the subject and has the intelligence and scientific training to comprehend what is offered to him.
>
> The mathematics which Ramanujan did in India could be assessed by British mathematicians of the highest order to the extent that they could re-trace the steps which his intuitive powers had enabled him to leap over. It was not the "Indianness" of Ramanujan's mathematics which baffled the first British mathematicians, Hobson and Baker, whom he approached; it was their exceptionally advanced originality. It required two mathematicians of the very high quality of Hardy and Littlewood to appreciate, to learn from and to contribute to Ramanujan's work. As the years passed, and his notebooks have been studied, his mathematics have been interpreted, proved and assimilated by Western mathematicians.
>
> I have not read any references to the specifically Indian character of Ramanujan's mathematics. His mathematics are mathematics, indifferently of the place and circumstances of their creation
>
> Contrary to the Romantic idea that tradition and genius, or tradition and originality, are invariably antithetical to each other, the opposite is the case. A tradition receives into itself the product of the exertions of individuals of powerful intelligence, imagination, courage and sensibility. Genius takes its point of departure in tradition; it extends and elaborates what is given by tradition. It begins in tradition and departs from it and reaches destinations hitherto unreached. It begins in tradition and its subsequent advances from tradition bear within themselves traits of its point of departure in tradition. It never cuts loose from them completely, however far it advances from its starting point. This accounts for some of the difficulties Littlewood encountered when he tried to bring Ramanujan's use of more recently developed mathematical methods up to the level attained in Europe after 1880. Nevertheless, the achievements of the genius take their place in the tradition which becomes significantly modified by those advances.

The intimate reciprocal relationships between genius and tradition are evident in the case of Ramanujan. He made a connection with some older and incomplete condensations of the tradition of the mathematics developed in Europe; he retraced, in his own way and in ignorance of them, paths of the tradition which had already been traversed in Europe. In other respects he shot well ahead of the points reached by the movement of the tradition in Europe. It is unlikely that the tradition of Hindu belief in which Ramanujan participated steadfastly obstructed his advances from the tradition of mathematics which had developed in Europe. Ramanujan thought that the Hindu goddess in whom he believed had in fact inspired his mathematical advances.

It seems to this outsider that the scientific and mathematical traditions are alive and well in India. Long may they prosper! In reflecting upon and honoring the life and work of Ramanujan, one finds numerous themes to support the nobler ambitions of humanity."

7. Conclusion

In his Presidential address to the London Mathematical Society in 1936, G. N. Watson spoke movingly of his emotional response to Ramanujan's achievements. I close by quoting his last few paragraphs [54; p. 80]:

"The study of Ramanujan's work and of the problems to which it gives rise inevitably recalls to mind Lamé's remark that, when reading Hermite's papers on modular functions, "on a la chair de poule." I would express my own attitude with more prolixity by saying that such a formula as

$$\int_0^\infty e^{-3\pi x^2}\frac{\sinh \pi x}{\sinh 3\pi x}dx = \frac{1}{e^{i\pi}\sqrt{3}}\sum_{n=0}^{\infty}\frac{e^{-2n(n+1)\pi}}{(1+e^{-\pi})^2(1+e^{-3\pi})^2\dots\left(1+e^{-(2n+1)\pi}\right)^2}$$

gives me a thrill which is indistinguishable from the thrill which I feel when I enter the Sagrestia Nuova of the Capelle Medicee and see before me the austere beauty of the four statues representing Day, Night, Evening, and Dawn which Michelangelo has set over the tombs of Guiliano de Medici and Lorenzo de Medici.

Ramanujan's discovery of the mock theta functions makes it obvious that his skill and ingenuity did not desert him at the oncoming of his untimely end. As much as any of his earlier work, the mock theta functions are an achievement sufficient to cause his name to be held in lasting remembrance. To his students such discoveries will be a source of delight and wonder until the time shall come when we too shall make our journey to that Garden of Proserpine where

Pale, beyond porch and portal,
   Crowned with calm leaves, she stands
Who gathers all things mortal
   With cold immortal hands."

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