AIAA 2001-0279

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39th Aerospace Sciences Meeting & Exhibit
8-11 January 2001 / Reno, NV
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ABSTRACT

Preconditioned time-marching algorithms are developed for a class of isothermal compressible multi-phase mixture flows, relevant to the modeling of sheet- and super-cavitating flows in hydrodynamic applications. Using the volume fraction and mass fraction forms of the multi-phase governing equations, three closely related but distinct preconditioning forms are derived. The resulting algorithm is incorporated within an existing multi-phase code and several representative solutions are obtained to demonstrate the capabilities of the method. Comparisons with measurement data suggest that the compressible formulation provides an improved description of the cavitation dynamics compared with previous incompressible computations.

INTRODUCTION

Multi-phase flows are encountered in a wide range of applications involving heat exchange, cavitation, sprays, porous media, etc. The computation of multi-phase flows has received growing research attention in recent years, due in part to the evolving maturity of single-phase algorithms. There remain, however, several physical and modeling challenges. A primary issue is the strong coupling of acoustic phenomena [1-5] due to the fact that the speed of sound in two-phase mixtures can be extremely low compared to the sound speeds in the individual component phases. Thus, multi-phase flows are frequently characterized by local regions, wherein the flow may be transonic or even supersonic with the presence of shocks, although the bulk of the flow may remain incompressible. This situation presents a unique challenge to the design of CFD algorithms. The development of appropriate numerical schemes for such multi-phase problems is the subject of the present paper.

There are many levels of modeling that may be utilized in multi-phase computations [6]. In general, one may distinguish between methods that employ an Eulerian framework for both phases and those that employ Eulerian for the gas-phase and Lagrangian for the liquid-phase. In the Eulerian-Eulerian framework, the simplest approach is to employ a single continuity equation for both phases, with the fluid density being described as a continuous function varying between the vapor and liquid phases [7,8]. At a more detailed level of modeling, separate continuity equations for the liquid and vapor phases are employed along with appropriate mass transfer terms to represent the phase-change phenomena [9-12]. The gas-liquid interface is, however, assumed to be in dynamic and thermal equilibrium and, consequently, mixture momentum and energy equations are used. This model is usually referred to as the homogeneous mixture model [6] and is the level of modeling considered in this paper. The model is appropriate for modeling cavitation in naval hydrodynamics applications, the area of primary interest to us. Finally, we note that full multi-fluid modeling, involving separate momentum and energy equations for each of the phases, has also been utilized for certain classes of multi-phase flow [6], but is not addressed here.

The crucial requirement of multiphase algorithms is the ability to accurately and efficiently span both incompressible and compressible flow regimes. For single-phase applications, time-marching techniques have long been established as the method of choice for high-speed compressible flows, while artificial compressibility or preconditioning techniques have enabled the extension of these methods to the incompressible and low-speed compressible regimes [13-16]. Preconditioning methods...
essentially maintain proper conditioning of the controlling time-scales of the time-marching system by introducing appropriate pseudo-time derivatives. Indeed, it is widely recognized that the careful selection of these derivatives is crucial for ensuring efficiency and accuracy over a wide range of Mach numbers. Reynolds numbers and Strouhal numbers \([15,16]\). In this paper, we seek to extend the preconditioning formulation to multi-phase mixture flows.

Several researchers have previously reported preconditioning formulations for multi-phase mixtures. Merkle et al. employed a two-species formulation, using mass fraction as the dependent variable \([9]\). Kunz et al. also employed a multi-species formulation, but used volume fraction as the dependent variable \([10]\). Both these formulations assumed constant densities for both liquid and vapor phases and did not account for compressibility effects in the two-phase mixture region. Ahuja et al. have developed a multi-phase algorithm, including compressibility effects in the component phases \([12]\). In this paper, we review these formulations and, in the case of the first two methods, we show that they can readily be extended to the compressible situation. In particular, we derive all three formulations using a common framework, facilitating comparative study. We find that the three approaches are, in fact, nearly the same with only minor differences between them.

The focus of this paper is on the isothermal multiphase system, in which the densities of the fluids are assumed to be functions of the pressure, but not the temperature. Under this assumption, the energy equation is not solved and only the continuity and momentum equations are considered. It should be noted that the system is still compressible because the pressure dependence of the densities gives rise to finite acoustic speeds. The isothermal compressible model serves as a useful intermediate step between the incompressible model and the fully compressible system, and development of the fully compressible model is currently underway. Nevertheless, we observe that the isothermal assumption is generally valid for the class of hydrodynamic cavitation problems considered here.

Our primary interest lies in sheet- and super-cavitating flows encountered in naval hydrodynamics applications. These flowfields are characterized by large density ratios between the liquid and vapor states. Further, at the Reynolds numbers of the interest, the flows are fully turbulent. Moreover, most problems also exhibit large-scale unsteadiness because of cavity reentrant jets and cavity pinching \([11]\). Accordingly, the preconditioning formulation derived in this paper is incorporated into a time-accurate, Reynolds-averaged, multi-phase Navier-Stokes code (UNCLE-M, see ref. \([10]\) for details). The code is applied to a hierarchy of test cases, both simple one-dimensional and more practical multi-dimensional flows. In particular, we consider cavitating flows over several axisymmetric ogive configurations for which measurement data are available. These computational test cases are used to demonstrate the overall capabilities of the formulation.

The paper is organized as follows. We begin with the equations of motion for two-phase homogeneous mixture flows. The equations are presented both in their volume fraction and mass fraction forms. We examine the eigenvalues of these systems and determine their behavior under limiting circumstances. We then use this insight to derive preconditioning forms that maintain well-conditioned behavior in both compressible and incompressible flow regimes. These investigations reveal that the preconditioning formulation is not unique and several distinct (but closely related) forms may be arrived at depending upon the precise version of the governing equations used in the analysis. In the Results section, we perform a series of computations of simplified and practical test problems to assess the performance capabilities of the formulations. Finally, we conclude with a summary and a brief description of current and future work.

**EQUATIONS OF MOTION**

The governing equations for two-phase flow are customarily written in terms of volume fraction variables. For the purposes of the theoretical derivation, we employ only the one-dimensional inviscid flow equations, although, in practical implementation, the full multi-dimensional Reynolds-averaged equations are used.

\[
\frac{\partial \rho \alpha_t}{\partial t} + \frac{\partial \rho \alpha_t u}{\partial x} = 0 \tag{1}
\]

\[
\frac{\partial \rho \alpha_v}{\partial t} + \frac{\partial \rho \alpha_v u}{\partial x} = 0 \tag{2}
\]

\[
\frac{\partial \rho u}{\partial t} + \frac{\partial \rho u^2}{\partial x} + \frac{\partial p}{\partial x} = 0 \tag{3}
\]

with the mixture density being defined as:

\[
\rho = \rho \alpha_t + \rho \alpha_v \tag{4}
\]

Note that the individual phasic densities, \(\rho_t\) and \(\rho_v\) are defined as mass of the phase/volume occupied by that phase.

The above system is closed by the phasic equations of state:

\[
\rho_v = \rho_v(p) \text{ and } \rho_t = \rho_t(p) \tag{5}
\]
where the individual phasic densities are assumed to be functions of pressure only. In our previous work [10,11], these quantities were taken to be constant, representing a mixture of incompressible phases.

The above equations may equivalently be written in terms of mass fraction variables as well. This form is customarily used in the case of gaseous mixtures, but is in fact equally valid for multi-phase mixtures.

\[
\frac{\partial \rho Y_v}{\partial t} + \frac{\partial \rho Y_v u}{\partial x} = 0
\]  

\[
\frac{\partial \rho Y_l}{\partial t} + \frac{\partial \rho Y_l u}{\partial x} = 0
\]

with the momentum equation having the same form as Eqn. 3.

Note that it is sometimes useful to define the individual phasic densities, \( \rho_v = \rho Y_v \) and \( \rho_l = \rho Y_l \), which are given as the mass of phase/volume occupied by the phase. Note that these phasic densities are distinct from those introduced earlier and are related by the following expressions:

\[
\rho_v = \tilde{\rho}_v \alpha_v \quad \rho_l = \tilde{\rho}_l \alpha_l \quad (8)
\]

Further, it is clear that the mixture density, \( \rho = \rho_v + \rho_l \). It is straightforward to see that the mass and volume fraction forms are in fact identical.

In the case of gaseous mixtures, it is customary to replace one of the phasic (or species) continuity equations by an overall continuity equation (obtained by summing up the individual continuity equations):

\[
\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0 \quad (9)
\]

However, for multi-phase flows, where significant density differences may exist between the phases, this procedure is prone to error. It is thus advisable to use all the individual continuity equations as shown above.

Alternately, some researchers prefer to define an overall mixture volume continuity equation, obtained by dividing the phasic continuities by the respective phasic density and then summing them up:

\[
\frac{1}{\rho_v} \frac{\partial \tilde{\rho}_v \alpha_v}{\partial t} + \frac{1}{\rho_l} \frac{\partial \tilde{\rho}_l \alpha_l}{\partial t} + \frac{1}{\rho_v} \frac{\partial \tilde{\rho}_v \alpha_v u}{\partial x} + \frac{1}{\rho_l} \frac{\partial \tilde{\rho}_l \alpha_l u}{\partial x} = 0 \quad (10)
\]

In this paper, we develop the preconditioning formulation for the system with all the phasic continuity equations, but we note that the overall mixture continuity form may be derived from it.

### EIGENVALUE ANALYSIS

#### Volume Fraction Form

The system given in Eqns. 1-3 may be expressed in the following vector form:

\[
\frac{\partial Q_\alpha}{\partial t} + \frac{\partial E}{\partial x} = 0
\]

where:

\[
Q_\alpha = \begin{bmatrix}
\rho_v \\
\alpha_v \\
u
\end{bmatrix} \quad E = \begin{bmatrix}
\tilde{\rho}_v \alpha_v u \\
\tilde{\rho}_l \alpha_l u \\
u^2 + p
\end{bmatrix}
\]

\[
\Gamma_\alpha = \begin{bmatrix}
\alpha_v \frac{\partial \tilde{\rho}_v}{\partial p} & \tilde{\rho}_v & 0 \\
\alpha_l \frac{\partial \tilde{\rho}_l}{\partial p} & \tilde{\rho}_l & 0 \\
u \frac{\partial \tilde{\rho}_v}{\partial p} & u(\tilde{\rho}_v - \tilde{\rho}_l) & p
\end{bmatrix}
\]

Note that, for the isothermal system under consideration here, the functions \( \frac{\partial \tilde{\rho}_v}{\partial p} \) and \( \frac{\partial \tilde{\rho}_l}{\partial p} \) represent the reciprocal of the squares of the speed of sound in the two individual phases. Also note that:

\[
\frac{\partial \tilde{\rho}_v}{\partial p} \bigg|_{\alpha_v} = \alpha_v \frac{\partial \tilde{\rho}_v}{\partial p} + \alpha_l \frac{\partial \tilde{\rho}_l}{\partial p} \bigg|_{\alpha_v}
\]

To determine the eigenvalues of the above two-phase system, we define the Jacobian:

\[
A_\alpha = \frac{\partial E}{\partial Q_\alpha} = \begin{bmatrix}
u \frac{\partial \tilde{\rho}_v}{\partial p} & \tilde{\rho}_v & 0 \\
u \frac{\partial \tilde{\rho}_l}{\partial p} & \tilde{\rho}_l & 0 \\
1 + u c^2 \frac{\partial \tilde{\rho}_v}{\partial p} & u^2 (\tilde{\rho}_v - \tilde{\rho}_l) & \rho u
\end{bmatrix}
\]

The system eigenvalues are then given by the eigenvalues of \( \Gamma_\alpha^{-1} A_\alpha \):

\[
\Gamma_\alpha^{-1} A_\alpha = \begin{bmatrix}u & 0 & \rho c^2 \\
u & \rho c^2 \alpha_v & \alpha_l \\
1 & 0 & u
\end{bmatrix}
\]
where the sound speed is given by the following expression:

$$\frac{1}{c^2} = \rho \left( \frac{\alpha_1 \hat{\rho}_L}{\rho L \hat{p}} + \frac{\alpha_v \hat{\rho}_V}{\rho V \hat{p}} \right)$$  \hspace{1cm} (16)

which is the standard mixture rule for the sound speed of two-phase mixtures.

It can be readily seen that the eigenvalues of the above system are given as:

$$\lambda (\Gamma^{-1}_\alpha A_\alpha) = u, u \pm c$$  \hspace{1cm} (17)

which has the familiar form of the single-phase compressible system.

**Mass Fraction Form**

The mass-fraction equation system may be expressed in the following vector form:

$$\Gamma_y \frac{\partial Q_y}{\partial t} + \frac{\partial E}{\partial x} = 0$$  \hspace{1cm} (18)

where:

$$Q_\alpha = \begin{bmatrix} \rho \\ Y_v \\ u \end{bmatrix}, \quad E = \begin{bmatrix} \rho Y_v u \\ \rho Y_v u \\ \rho u^2 + p \end{bmatrix}$$

$$\Gamma_\alpha = \begin{bmatrix} \frac{\partial \rho}{\partial Y_v} \\ \frac{\rho Y_v}{\partial Y_v} - \rho + \frac{\partial \rho}{\partial Y_v} \\ \frac{\rho \hat{u}}{\partial Y_v} - \rho u + \frac{\partial \rho}{\partial Y_v} \end{bmatrix}$$  \hspace{1cm} (19)

The Jacobian $A_y$ is given by:

$$A_y = \frac{\partial E}{\partial Q_y} = \begin{bmatrix} u Y_v \frac{\partial \rho}{\partial Y_v} + \rho u + u Y_v \frac{\partial \rho}{\partial Y_v} & \rho Y_v \\ u Y_v \frac{\partial \rho}{\partial Y_v} & -\rho u + u Y_v \frac{\partial \rho}{\partial Y_v} & \rho Y_v \\ 1 + u^2 \frac{\partial \rho}{\partial Y_v} & u^2 \frac{\partial \rho}{\partial Y_v} & \rho u \end{bmatrix}$$  \hspace{1cm} (20)

The system eigenvalues are then given by the eigenvalues of $\Gamma_y^{-1}(A_y)$:

$$\lambda (\Gamma^{-1}_\alpha A_\alpha) = u, u \pm c$$  \hspace{1cm} (25)

**PRECONDITIONING FORMULATION**

Examination of the eigenvalues of the multi-phase system reveals that the system reverts to the standard single-phase eigenvalues when only one phase is present. Under such circumstances, for the liquid phase as well as for the vapor phase at low speeds, the system becomes poorly conditioned for efficient time-marching. Moreover, it is well known that standard discrete formulations (such as Roe’s flux difference scheme) become inaccurate under such stiff conditions. Preconditioning or artificial compressibility methods are the established approach for alleviating such stiffness problems [13-16] within a time-marching algorithmic framework. These methods re-scale the time-derivatives selectively and thereby ensure that the system eigenvalues become well-conditioned.

For the multi-phase system, when both liquid and vapor phases co-exist, the mixture sound-speed is such that the local Mach number may become transonic or even supersonic at relatively low speeds [1-5]. It is therefore important that the preconditioning formulation is capable of handling the low Mach numbers in the pure liquid and vapor phases as well as the transonic/supersonic Mach numbers in the mixture region. In the
present section, we examine several potential approaches for preconditioning the multi-phase system.

**Formulation I**

The volume fraction form given in Eqn. 11 may be preconditioned by the introduction of pseudo properties into the matrix premultiplying the time-derivatives. Thus, we have:

\[
\Gamma^\alpha \frac{\partial Q}{\partial t} + \frac{\partial E}{\partial x} = 0
\] (26)

with the preconditioning matrix being defined as:

\[
\Gamma^\alpha = \begin{bmatrix}
\alpha_v & \frac{\partial \tilde{\rho}_v}{\partial p} & 0 \\
\alpha_l & \frac{\partial \tilde{\rho}_l}{\partial p} & -\tilde{\rho}_l \\
u & \frac{\partial \tilde{\rho}}{\partial p} & \tilde{\rho}_v - \tilde{\rho}_l
\end{bmatrix}
\] (27)

The pseudo-properties are defined so as to render the eigenvalues well-conditioned. One possible choice is suggested by the Kunz et al. formulation for incompressible two-phase mixtures [10]. Accordingly, we define:

\[
\frac{\partial \tilde{\rho}_v}{\partial p} = \frac{\rho_v}{\rho} \frac{1}{V^2} \tilde{\rho}_v, \quad \frac{\partial \tilde{\rho}_l}{\partial p} = \frac{\tilde{\rho}_l}{\rho} \frac{1}{V^2}
\] (28)

\[
\frac{\partial \tilde{\rho}}{\partial p} = \alpha_v \frac{\partial \tilde{\rho}_v}{\partial p} + \alpha_l \frac{\partial \tilde{\rho}_l}{\partial p}
\] (29)

The term “\(V\)” is some characteristic velocity scale, typically the local convective velocity under inviscid conditions.

The system Jacobian, \(\Gamma^\alpha A^\alpha\), is now given as:

\[
\Gamma^\alpha A^\alpha = \begin{bmatrix}
u (c')^2 & 0 & \rho (c')^2 \\
X & u & Y \\
\frac{1}{\rho} & 0 & u
\end{bmatrix}
\] (30)

where

\[
\frac{1}{(c')^2} = \rho \left( \frac{\alpha_v \frac{\partial \tilde{\rho}_v}{\partial p}}{\tilde{\rho}_l} + \frac{\alpha_l \frac{\partial \tilde{\rho}_l}{\partial p}}{\tilde{\rho}_v} \right) = \frac{1}{V^2}
\] (31)

\[
X = \rho u (c')^2 \alpha_v \alpha_v \left[ \frac{\partial \tilde{\rho}_v}{\partial p} \left|_{\alpha_v} \right. - \frac{\partial \tilde{\rho}_l}{\partial p} \left|_{\alpha_v} \right. \right]
\] (32)

\[
Y = \rho (c')^2 \alpha_v \alpha_v \left[ \frac{\partial \tilde{\rho}_v}{\partial p} \left|_{\alpha_v} \right. - \frac{\partial \tilde{\rho}_l}{\partial p} \left|_{\alpha_v} \right. \right]
\] (33)

The eigenvalues of the preconditioned two-phase system are given as:

\[
u \frac{1}{2} \left[ \left(1 + \frac{(c')^2}{c^2}\right) \pm \sqrt{\frac{1}{\left(1 - \frac{(c')^2}{c^2}\right)} + 4(c')^2} \right]
\] (34)

Since \((c')^2 = V^2\), we note that the “acoustic” eigenvalues in the above expression are of the same order as the particle speed, thereby ensuring well-conditioned eigenvalues at all speeds.

The above formulation reduces to the Kunz et al. formulation for incompressible mixtures. It is further evident that it reduces to Chorin’s artificial compressibility for single-phase incompressible flows [13] and to standard preconditioning for single-phase compressible flows [15]. However, unlike the single-phase preconditioning system, this formulation does not automatically switch the preconditioning off for supersonic flows. While this may be viewed as a disadvantage of the method, the effect appears to be small at moderate supersonic Mach numbers.

Finally, we note that the above preconditioning definition may be formally derived using a perturbation procedure applied to the governing equations in the volume fraction form (Eqns. 1-3). The procedure is not presented here for reasons of brevity, but it is similar to that described for single-phase flow in Ref. [15].

**Formulation II**

In their model for isothermal compressible mixtures, Ahuja et al. [12] use a similar form of the preconditioning matrix as in Eqn. 27, but they define the pseudo-properties in a different fashion:

\[
\frac{\partial \tilde{\rho}_v}{\partial p} \left|_{\alpha_v} \right. = \frac{1}{M^2} \frac{\partial \rho_v}{\partial p} \left|_{\alpha_v} \right. , \quad \frac{\partial \tilde{\rho}_l}{\partial p} \left|_{\alpha_v} \right. = \frac{1}{M^2} \frac{\partial \tilde{\rho}_l}{\partial p} \left|_{\alpha_v} \right.
\] (35)

where \(M^2 = V^2/c^2\) and \(V\) is some characteristic velocity scale.

Interestingly, this preconditioning formulation yields the same eigenvalues as those in formulation I. This is a consequence of the following relationship being true:

\[
\frac{1}{(c')^2} = \rho \left( \frac{\alpha_l \frac{\partial \tilde{\rho}_l}{\partial p}}{\tilde{\rho}_l} + \frac{\alpha_v \frac{\partial \tilde{\rho}_v}{\partial p}}{\tilde{\rho}_v} \right) = \frac{1}{c'V^2}
\] (36)

For single-phase compressible flows, formulation II also reduces to the standard preconditioning formulation. For two-phase mixtures, it has the further advantage of automatically turning preconditioning off at supersonic speeds (unlike formulation I). However,
implementation of the method in the incompressible limit is a little clumsy because the Mach number tends to zero in this limit. In practice, this is not a major issue since the incompressible sound speed may arbitrarily be set to a large number.

It is interesting to note that formulation II may also be derived from a perturbation analysis when one of the phasic continuity equations is replaced by the overall mixture continuity equation. Again, for reasons of brevity, we do not provide the details here.

**Formulation III**

A third formulation may be derived by starting with the mass fraction form of the governing equations. We may write the preconditioning system as:

\[
\Gamma^p \frac{\partial Q^p}{\partial t} + \frac{\partial E}{\partial x} = 0
\]  

with the preconditioning matrix being defined as:

\[
\Gamma^p = \begin{bmatrix}
Y_v \frac{\partial \rho'}{\partial p} - \rho + Y_v \frac{\partial \rho}{\partial Y_v} & 0 \\
Y_v \frac{\partial \rho'}{\partial p} - \rho + Y_v \frac{\partial \rho}{\partial Y_v} & 0 \\
u \frac{\partial \rho'}{\partial p} - \rho + \frac{\partial \rho}{\partial Y_v} & 0
\end{bmatrix}
\]

and, as usual, \(V\) is some characteristic velocity scale.

The above formulation is identical to the preconditioning used for gaseous mixtures [15] and is similar to that proposed by Merkle et al. for two-phase flows [9]. It reduces to Chorin’s artificial compressibility for incompressible single-phase flows and to the standard preconditioning form for compressible single-phase flows. For multiple phases, the system is well-conditioned at low Mach numbers and automatically switches off the preconditioning at supersonic Mach numbers. In fact, it can be shown that the eigenvalues of the above formulation are the same as the eigenvalues of formulations I and II (see Ref. [15]).

Although the three formulations are not identical, they possess the same eigenvalues and are closely related. Our experience suggests that the three approaches perform equally well in practice. Thus, the decision of which one to use may well depend upon the particular form of the equations or the solution variables used in the candidate code.

**NUMERICAL METHODOLOGY**

The preconditioning system for isothermal compressible multi-phase flows has been incorporated into the UNCLE-M code. This code was originally developed for incompressible flows at Mississippi State University [17] and later extended to two-phase mixtures by Kunz et al. [10]. The code is structured, multi-block, implicit and parallel with Roe-upwind flux-difference splitting for the spatial discretization and Gauss-Seidel relaxation for the inversion of the implicit operator.

Physical modeling of the phase transformation from liquid to vapor is modeled by a source term that is proportional to the product of the liquid volume fraction and the difference between the local pressure and the vapor pressure. For transformation of vapor to liquid, a simplified form of the Ginzburg-Landau potential is used [10]. A high Reynolds number two-equation turbulence model with wall functions is used for turbulence closure.

For time-accurate unsteady solutions, a dual-time procedure is used. The physical time terms are discretized using second-order one-sided differencing. The discrete algebraic equations at each physical time level are driven to convergence by a set of pseudo-time derivatives, which are preconditioned for maximizing convergence and accuracy.

The UNCLE-M code represents the phasic continuity equations in volume fraction form and, for that reason, only formulations I and II were incorporated into the code. In both cases, the characteristic velocity parameter was supplied as user-specified value, usually expressed as some multiple of the free-stream velocity. More refined definitions of this parameter based upon local variables will be the subject of future study. As mentioned earlier, both formulations performed comparably for the test cases studied here.

**RESULTS**

*Inviscid Flow in Straight Duct*

As a first example, we consider the simple case of inviscid two-phase mixture flow in a two-dimensional straight duct. Figure 1 shows convergence results for three different Mach numbers of 0.01, 0.55 and 2. Results are given for the standard equations without any preconditioning and for two forms of preconditioning (I and II). The results reveal that, at the low Mach number of 0.01, convergence is not obtained with the non-preconditioned equations because of the inherent stiffness of the original time-marching system (at low speeds). On the other hand, efficient convergence is realized with both preconditioning forms tested. Even at the higher
subsonic Mach number of 0.55, the two preconditioning forms are observed to slightly out-perform the non-preconditioned system. Finally, at the supersonic Mach number, all three systems converge at virtually the same rate. We recall that at supersonic Mach numbers, formulation I does not turn off the preconditioning, while formulation II does turn off the preconditioning. The latter case hence becomes identical to the non-preconditioned case for supersonic flows. We note that the results of formulation I are also very nearly the same, indicating that the preconditioning effect is very slight for moderately supersonic flows.

**Shock-Tube Problem**

As a second example, we consider the unsteady two-phase shock tube problem, investigated both experimentally and theoretically by Campbell and Pitscher [5]. The problem can be modeled as a shock wave moving into a stationary and non-condensable, non-vaporizable gas/liquid mixture. Assuming that the liquid is incompressible and the gas is perfect, an exact expression may be obtained for the shock speed [5]:

\[
\frac{u_1^2}{c_1^2} = \frac{p_2}{p_1} \quad (40)
\]

where the subscript 1 denotes conditions in front of the shock, the subscript 2 denotes conditions behind it and \(u_1\) is the shock speed.

Figure 2 shows the results for two different pressure ratios of 2 and 6. In each case, the predicted results after a given period of time are compared with the theoretical shock location and very good agreement is obtained. We note here that the time-accurate results are obtained using dual time-stepping. The characteristic velocity parameter in the definition of the preconditioning is specified based upon the time-scales associated with the shock motion. The results demonstrate that the two-phase preconditioning formulation is capable of resolving acoustic/compressibility effects very well.

**Natural Cavitation on Axisymmetric Bodies**

The application of principal interest to us involves the modeling of sheet cavitation around axisymmetric bodies at high Reynolds numbers. Because of their importance to naval hydrodynamics applications, numerous experimental and computational studies of cavity flowfields around bodies of different shapes have been carried out. Rouse and McNown have documented steady and time-averaged measurements of relevant cavitation parameters for various forebody shapes [18]. May has assembled cavity shape and size parameters for a wide range of cavitation numbers [19]. Stinebring et al. have documented the unsteady cycling behavior of several axisymmetric cavitators [20]. Recently, Kunz et al. and Lindau et al. have modeled sheet cavity flowfields in a variety of configurations using an incompressible mixture model [10,11]. They have made extensive comparisons with experimental data and have determined the strengths and shortcomings of their model. In this section, we perform similar computations with the current isothermal compressible model.

Figures 3 and 4 show flowfield snapshots of the unsteady cavitating flow over a 0-caliber and a 1/4-caliber ogive. The Reynolds numbers are approximately \(1.4 \times 10^5\) and the cavitation number is 0.3. In both sets of figures, corresponding results using the incompressible mixture model have also been included to facilitate direct comparison.

It is evident that the flowfields are rich in complexity. We first consider the 0-caliber results in Fig. 3. Both incompressible and compressible results show the cavity forming at or very near the leading edge of the ogive body. Further, because of the sharp turning of the incoming flow, a recirculating bubble is also formed in this cavity region. The incoming liquid then flows over the bubble and rejoins the surface of the body downstream of the cavity and the recirculating bubble. The pressure recovery in this downstream region then sets up a liquid reentrant jet which shoots into the cavity. As we follow the snapshots through a complete period, we observe that this reentrant jet appears to progress deeper into the cavity until some form of cavity pinching occurs that causes the jet to retract until the cycle can commence again. The unsteady cycle thus seems to be closely co-ordinated with the re-entrant jet and the cavity pinching process.

Figure 3 shows a marked difference between the compressible and incompressible results in the extent to which the liquid reentrant jet penetrates the cavity. In the compressible case, the liquid jet traverses only about one-half of the cavity length before the jet is pinched off and begins to retract. The incompressible case also shows the reentrant jet getting pinched off; however, the liquid bubble appears to remain intact within the cavity until it reaches the leading edge. In fact, at t=0.5, the reentrant liquid appears to have pushed the cavity downstream of the leading edge.

This difference between the compressible and incompressible results is also observed in Fig. 4 for the 1/4-caliber case. In fact, the difference appears even more marked with the compressible case showing the liquid reentrant jet barely reaching one-half length into the cavity, while in the incompressible result, it again appears to traverse most of the cavity length.
These trends in the behavior of the reentrant jet appear to control the unsteady dynamics of the flowfields. Figure 5 shows the instantaneous (pressure) drag coefficient histories for all four cases. The drag coefficient essentially is a measure of the total pressure on the face of the ogive body. The fluctuations in this quantity then result from the upstream propagation of pressure disturbances generated by the cavity pulsations.

Examination of the 0-caliber drag coefficient histories first reveals that the incompressible amplitudes are significantly higher, a fact that is reflected by the more extreme cavity distortions that are evident in the incompressible snapshots in Fig. 3. Further, the dominant mode of the compressible result shows a slightly higher frequency than the incompressible result. Again, this result appears to correlate with the distance traversed by the reentrant jet in Fig. 3. For instance, the compressible case shows shorter reentrant jet penetration and, hence, a higher frequency.

Drag coefficient results for the 1/4-caliber ogive are also given in Fig. 5. Here, both incompressible and compressible amplitudes are observed to be small, in accordance with the more stream-lined cavity shapes evident in the snapshots in Fig. 4. However, there is now a more significant difference in the cycling frequency, with the compressible case showing a much higher frequency. Again, this result correlates well with the jet penetration distance in Fig. 4. The incompressible case shows the reentrant jet reaching much deeper into the cavity and, hence, has a smaller cycling frequency.

The frequency data for the above cases are compared with measurement data obtained by Stinebring et al. [20] in Fig. 6. While all the computed data lie within the bounds of the experimental data, it is interesting that the experimental data for the hemispherical forebody indicates a higher Strouhal frequency than the corresponding result for the 0-caliber ogive. We point out that this trend is in agreement with the higher frequency obtained for the 1/4-caliber ogive using the compressible model. On the other hand, the incompressible model predicts roughly the same Strouhal frequency for the two shapes.

The above comparison certainly suggests that the compressible model may be capturing the dynamics of the cavity more correctly than the incompressible model. In particular, it would appear that the penetration of the liquid jet (in Fig. 4) is over-predicted by the incompressible model. It is not immediately apparent why this may be the case. We speculate that the compressible nature of the vapor phase may be causing the pressure within the cavity to increase somewhat during the jetting process, thereby weakening the penetration of the jet. In the incompressible case, on the other hand, the vapor phase is treated as an incompressible fluid, which may readily accommodate the incoming jet by expanding the size of the cavity as a whole. More detailed investigations are clearly necessary to reach definitive conclusions regarding these effects.

Extensive time-averaged data are also available to characterize various cavity parameters. Figure 7 shows a comparison of arithmetically averaged data for the 0-caliber and 1/4-caliber ogive cases with average surface pressure measurements from Rouse and McNown [18]. The incompressible results are also included for comparison. In all cases, the computed data generally agree with the experiments. Although there are some differences between the compressible and incompressible results, both sets appear to compare well with the data.

Other parameters of relevance include the cavitator diameter, $d$, cavity length, $L$, and cavity diameter ($d_m$). Some ambiguity is inherent in both the experimental and computational definition of these parameters. Cavitation closure location is difficult to define due to unsteadiness and its dependence upon the afterbody diameter. Accordingly, the cavity length is customarily defined as twice the distance from the cavity leading edge to the location of the maximum cavity diameter. Also, the cavity diameter (in the computations) is determined by examining the $\alpha_j = 0.5$ contour and determining its maximum radial location.

Figure 8 compares computed results with experimental data from May [19]. Figure 8a shows the fineness ratio, $L/d_m$, plotted versus the cavitation number, while Fig. 8b plots the quantity, $L/(d_m \sqrt{C_D})$. At the present time, we have computed data using the isothermal compressible mixture model only for a cavitation number of 0.3. We observe that the results agree well with the data at this condition and are again in good agreement with the incompressible results as well.

In summary, we note that the compressible results for the ogive configurations are in general agreement with previous incompressible results in the time-averaged quantities. Both sets of results also agree well with averaged experimental data. The instantaneous results do show some significant differences in the amplitudes and periods of the fluctuations. The compressible results correctly predict the higher frequency in the 1/4-caliber case compared to the 0-caliber case. Additional results and investigations are needed to confirm the validity of these trends.

Other Applications

In this section, we briefly consider two other applications of interest that involve the modeling of two-phase compressibility effects. Figure 9 shows an underwater supersonic projectile. Both computational results and a corresponding photograph of an actual test are included.
in the figure. The flow Mach number for the case shown is 1.03 and the liquid to gas density ratio is nominally 1000. The experiments and the computations show the presence of a bow shock upstream of the nose. In addition, because of the high velocity, the cavitation number is about $10^{-4}$. Consequently, most of the flow immediately adjacent to the body is completely vaporized as is the downstream wake portion.

The second example shown in Fig. 10 is the plume flowfield of an underwater rocket exhaust. The plume exhaust is supersonic and is slightly under-expanded. It is surrounded by a co-flowing secondary subsonic gas stream, which in turn is surrounded by a liquid water free-stream flow. Again, the nominal liquid to gas density ratio is 1000. Figure 10 shows the shock function field, which exhibits the classic expansion pattern. In particular, the interaction of the compressible gas stream with the incompressible liquid is demonstrated first by the contraction and then by the expansion of the gas stream. In addition, the interface between the liquid and gas phases is comprised of a two-phase mixture, which is also fully supersonic due to the low magnitude of the mixture sound speed.

Both of the above examples involve supersonic Mach numbers in the bulk flow for at least one of the phases. The current isothermal assumption is clearly inadequate to fully represent the dynamics of these flowfields. Nevertheless, we have presented preliminary results using the isothermal formulation. Detailed modeling using a fully compressible two-phase formulation (including the energy equation) will be the subject of future research.

**CONCLUSIONS**

Multi-phase mixture flows present a unique challenge to CFD algorithms because of the simultaneous existence of incompressible flow in the liquid phase, low-speed compressible flow in the vapor phase and transonic and supersonic flows in the two-phase mixture region. The CFD algorithm has to be efficient and accurate over all these Mach number regimes.

We have developed a preconditioned time-marching algorithm for the computation of multi-phase mixture flows. The preconditioning formulation introduces pseudo-time derivatives, which automatically adapt to keep the system well-conditioned in the incompressible as well as compressible regimes and, thereby, ensure that proper accuracy and optimal efficiency are maintained.

Three closely related but distinct preconditioning formulations have been derived for isothermal compressible flows. The differences arise from the precise form of the governing equations---volume or mass fraction---used in the derivation. However, we have shown that all three systems possess identical eigenvalues and, therefore, should perform comparably in practice.

The preconditioning algorithm has been incorporated into an existing multi-phase code (UNCLE-M). Simple test cases such as an inviscid straight duct and shock tube are used to verify the formulation. The code is then applied to more practical application problems.

The principal focus of the paper is the modeling of sheet- and super-cavitating flowfields in hydrodynamics applications. Computational results are obtained for flow over 0-caliber and 1/4-caliber ogives and compared with experimental data as well as previously obtained incompressible results. The computed compressible and incompressible results agree well with each other and with the experiments for time-averaged data.

The unsteady results, however, show some marked differences between the compressible and incompressible cases. The flowfields are characterized by unsteady effects caused by cavity reentrant liquid jets and cavity pinching. In the compressible case, the cavity reentrant jets appear to be relatively short-lived compared with the incompressible case. In the latter case, the reentrant jet persists almost until it reaches the leading edge of the cavity. In turn, these effects lead to higher frequencies for the compressible model, which is in agreement with measured Strouhal frequency data. Thus, it appears that compressibility effects may need to be accounted for correctly describe the cavity dynamics. Additional investigations are necessary to confirm these findings.

Additional computations have also been performed for underwater supersonic projectile and underwater rocket plume flowfields. In these examples, the bulk flow of one of the phases is supersonic, suggesting the need for a fully compressible formulation (including the energy equation). The development of preconditioning algorithms for the fully compressible two-phase system will be the subject of future research.

**ACKNOWLEDGMENTS**

This work is supported by the Office of Naval Research, contract #N00014-98-0143, with Dr. Kam Ng as contract monitor.
REFERENCES


FIGURES

Figure 1: Convergence history for inviscid flow in a straight duct. History shown with two types of preconditioning and no preconditioning
a) Mach 0.01
b) Mach 0.55
c) Mach 2.0.

Figure 2: Mixture shock tube computations and comparison with theory of Campbell and Pitscher [5]. Shock moves from left to right with initial location at 0. Model solution shown at even time intervals beginning after initial state. Fluid to right of shock is at rest with liquid to gas density ratio of 1000, and $\alpha_l=0.95$.
a) Pressure ratio 2.
b) Pressure ratio 6.
Figure 3: Volume fraction contours and streamlines. Snapshots span an approximate modeled cycle. 0-caliber ogive at $Re_D=1.46 \times 10^5$ and $\sigma=0.3$.
   a) Isothermal form
   b) Incompressible form

Figure 4: Volume fraction contours and streamlines. Snapshots span an approximate modeled cycle. 0.25-caliber ogive at $Re_D=1.36 \times 10^5$ and $\sigma=0.3$.
   a) Isothermal form
   b) Incompressible form
Figure 5: Drag coefficient histories for 0-caliber and 1/4-caliber ogive shapes modeled with incompressible and isothermal forms.

a) 0-caliber ogive based on isothermal model.
b) 0-caliber ogive based on incompressible model.
c) 1/4-caliber ogive based on incompressible and isothermal models

Figure 6: Strouhal frequency comparison modeling results and data from Stinebring [20].

Figure 7: Comparison of time averaged results with data of Rouse and McNown [18].

a) 0-caliber
b) 1/4-caliber
Figure 8: Comparison of isothermal and incompressible model results to the data of May [19].

Figure 9: Isothermal model result and photograph of supersonic underwater projectile. Model result contains flow field density contours (red high blue low) and projectile surface colored by pressure (blue high red low).

Figure 10: Cartoon vehicle and 3-stream, axisymmetric aft flow region. Center jet (diameter=1) surrounded by gas at free stream velocity (outer diameter=2). Liquid to gas density ratio 1000. Shock function: $V \cdot \nabla p_m$