PREFERENCE FUNCTION AND TRADE-OFFS ASSOCIATED WITH THE AHP
HIERARCHIC COMPOSITION LAW

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Abstract

The Hierarchic Composition Law (HCL) of the Analytic Hierarchy Process (AHP) is examined as a device for aggregation. Preference functions can be similarly described. Hence the question arises as to the relationship between the two constructs. A class of fundamental functional equations is derived which relates priorities derived as in the AHP with those derived directly from a ratio preference function. It is possible to associate trade-off curves to the hierarchic composition law by this technique. These trade-off curves have a number of desirable and interesting features. Our main results indicate that the number of alternatives, m, being compared in the AHP is the critical issue. We show that, in general, different values of m determine trade-off curve systems, and hence different preference structures. By limiting comparisons to the case of m=2, a fixed preference structure is maintained and rank reversal can be eliminated. Two classical cases of rank reversal, on account of alternative addition and alternative deletion respectively, are reexamined. The rank reversal can be resolved in both cases by this methodology provided the decision maker declares one alternative as having unit criteria levels in the relative measurement case. The results obtained here establish a methodological link between the AHP and classical preference function theory.

(AHP; Hierarchic Composition Law; Aggregation; Preference Functions; Rank Reversal; Functional Equations; Trade-off Curve Equivalence; Etalon-Based AHP)
1. Introduction

The Analytic Hierarchy Process (AHP) (Saaty 1980) has been a somewhat controversial topic in the management science literature. In particular, a set of papers which debated the relative merits of AHP and Multiattribute Utility Theory (MAUT) has appeared (Winker 1990; Dyer 1990a, 1990b; Harker and Vargas 1990; Saaty 1990). A brief summary of the most salient issues is as follows. Users of the AHP find that it has good apparent face validity and ease of use in a variety of situations for which MAUT can also apply. Critics of the AHP seem to concede these points but are disconcerted that AHP permits the occurrence of rank reversal of alternatives when new alternatives are added or deleted. This latter phenomenon is not acceptable for many theoreticians and practitioners, but is regarded as an additional strength of the AHP by its proponents.

Other issues, such as normative versus descriptive orientations, and ratio versus interval scales, were also addressed in the above set of papers. However, interest clearly focused on the rank reversal phenomenon of the AHP. Proponents of the AHP suggest the use of the ideal, or absolute measurement mode of AHP if rank preservation is desired. Dyer (1990a, b) appears to regard this as avoiding the issue of rank reversal occurring with the hierarchic composition law itself. Thus we focus on the relationship between rank reversal and this composition law. A brief discussion of the ideal mode is also presented, however.

This paper provides a route for establishing some common theoretical ground between
these two methodologies. Specifically, the focus is on the Hierarchic Composition Law (HCL) of the AHP as a device for aggregating different attribute levels of alternatives into a single scale. 

Criterion priorities are the main parameters for use of the HCL in the AHP. We assume that these criterion priorities are given and do not address the issue of how they should be obtained. Thus our approach is formal and mathematical in order to gain a precise understanding of how the HCL works in terms of preference functions. Also we limit attention to cases of certainty and therefore contact MAUT only in so far as it employs preference function representations.

By assuming that criterion priorities are given, we are able to focus more carefully on the precise mechanical relationships involved in the hierarchic composition law, and their implications for certain related preference functions. This leaves the inconsistency of judgement issues of the eigenvector scaling technique of the AHP out of the present scope. We may assume for our purpose that criterion and alternative priorities have, in fact, been obtained by eigenvector scaling method. Thus our results relate to what may be called inherent rank reversal caused by use of the HCL. We propose modified method which can avoid this inherent rank reversal. However use of this method does not necessarily preclude the possibility of rank reversal due to inconsistencies of pairwise comparisons.

Given the criterion priorities, the HCL, and unidimensional value functions on the individual criteria, we construct a ratio preference function and examine its trade-off curves. Preference functions are real-valued representations of an underlying preference structure. Thus any monotone increasing function of a given preference function will also represent the same preference structure for the purpose of this paper. At the same time such a transformation will have the same trade-off curves, however.

The rest of the paper is organized as follows. Section 2 defines priorities of the positive
real numbers in a manner consistent with the principal eigenvector method of the AHP. Section 3
derives the general and special forms of the fundamental functional equation which result from the
definition of priorities of the positive real numbers. Section 4 discusses a few solutions to these
equations which are applied to the Belton and Gear (1982) example in Section 5 and the example
of Troutt (1988) in Section 6, respectively. Section 7 provides a discussion and Section 8
presents conclusions with some suggestions for further research.

2. Priorities and the HCL

A basic concept in the AHP is that of priorities of alternatives with respect to a criterion.
These priorities are a scaling of the alternative values or levels on a criterion into the interval (0,1)
in such a way that their sum is unity. We assume that criteria are measured on coordinates and
use the following definitions.

**Definition 1:** Let \( x^j \) be \( j = 1, \ldots, m \) vectors in \( \mathbb{R}^n \), with coordinates, \( x^j_i \), \( i = 1, \ldots, n \).
Let \( \mathbb{R}^+_n = \{ x \in \mathbb{R}^n : x_i > 0 \} \). Let \( u_i(x_i) \) be unidimensional value functions on the coordinates, \( x_i \),
such that \( u_i(x_i) \in \mathbb{R}^+_n \). Define the relative priority of vector \( x^j \) with respect to criterion (or
coordinate) \( i \), by

\[
p^j_i = u_i(x^j_i) / \sum_{j=1}^{n} u_i(x^j_i)
\]

If \( u_i(x_i) = x_i \) for all \( x \in \mathbb{R}^+_n \) then (2.1) becomes

\[
p^j_i = x^j_i / \sum_{j=1}^{n} x^j_i
\]

and we call these the *natural priorities*.

Thus, for example, the natural priorities of three values, 1, 6, and 3, on the \( x_1 \) criterion,
say, are 0.1, 0.6, and 0.3 respectively.

**Theorem 1:** Definition 1 is consistent with the principal eigenvalue definition of priorities. Namely, under perfect consistency the matrix of pairwise comparison ratios will have normalized principal eigenvector solution given by (2.2).

**Proof:** The proof is the same for any i. Let $w_j$ be the normalized relative value of alternative $x_i^j$ to the decision maker (DM) for criterion i. By the eigenvalue method of the AHP the priorities are obtained as follows. The matrix $[a_{jk}]$ of pairwise comparison ratios provides estimates of $w_j/w_k$.

Hence, under perfect consistency $a_{jk} = w_j/w_k$, and $\sum_{k=1}^{m} a_{jk} w_k = mw_j$. Thus $w_j (\sum_{j=1}^{m} w_j)^{-1}$ may be recognized as the normalized principal eigenvector of $[a_{jk}]$. Now if $w_j = u_i(x_i^j)$ then the result easily follows by substitution.

Thus, for example, if $u(x_1^1) = 1$, $u(x_2^2) = 6$, and $u(x_3^3) = 3$ for three alternatives in $R$, then we have that $A = [a_{jk}]$ satisfies exactly

$$\begin{bmatrix}
1 & 1/6 & 1/3 \\
6 & 1 & 2 \\
3 & 1/2 & 1
\end{bmatrix} \begin{bmatrix}
0.1 \\
0.6 \\
0.3
\end{bmatrix} = \begin{bmatrix}
0.1 \\
0.6 \\
0.3
\end{bmatrix}.$$

Thus a decision maker with unidimensional value function $u_i(x_i)$ can compute alternate priorities with respect to a criterion by the principal eigenvector method. That is,

$$p^i_j = u_i(x_i^j) (\sum_{j=1}^{m} u_i(x_i^j))^{-1}.$$  

(2.3)
This relation may be used in two ways. First, the unidimensional values $u_i(\cdot)$ might be determined via traditional MAUT methods such as midvalue splitting or lottery equivalents. Then the AHP criterion priorities may be obtained by (2.3). Alternatively the priorities may be obtained by the AHP approach of pairwise comparisons. Then (2.3) might be solved, in principle, for the $u_i(x_i^j)$. For example, with the $p_i^j$ as given constraints, (2.3) may be rewritten as a homogenous linear equation in the $u_i(x_i^j)$. Particular solutions could be found by designating one of these as having unit value.

**Definition 2:** In the AHP, the hierarchic composition law is used to combine the individual priorities into an overall priority of alternatives.\(^2\) Namely, let $p_i$ be given as the priority of criterion $i$, and let $P_j$ be the priority of vector $j$. Then

$$P_j = \sum_i p_i \cdot p_i^j$$

is the priority of alternative $j$ according to the hierarchic composition law.

Suppose the decision maker has a preference structure representable by the ratio preference function $v(x)$, which can be used to weakly order all points on $R^a_+$. Suppose further that $v(0) = 0$ and $v(x) > 0$ for all $x \in R^a_+$. We will call such a $v(x)$ a *ratio preference function*.

If $x^j$, $j = 1,\ldots, m$ are a set of alternatives in $R^a_+$, then their relative priorities can be obtained directly via $v(\cdot)$. This follows since the $v(x^j)$ are positive real numbers and by Definition 1 the direct or overall relative priority of alternative $j$ is given by

$$p_j = p_i \cdot p_i^j$$
In the next section we derive the fundamental functional equation which can connect a specific ratio preference function with the HCL.

3. The Fundamental Functional Equation

From the foregoing discussion, priorities of positive alternatives, $x^j=x_i^j$, may be derived by the three step AHP process given as follows:

i) The DM provides subjective criterion priorities $p_i$, $i=1,n$.

ii) Alternative by criterion priorities, $p_i^j$, may be obtained by Definition 1. That is, 

$$p_i^j = x_i^j \left( \sum_{i=l}^{n} x_i^j \right)^{-1}$$

iii) The HCL may then be used to obtain overall alternative priorities $P_j$. Namely 

$$P_j = \sum_{i=l}^{n} p_i x_i^j \left( \sum_{j=1}^{m} x_i^j \right)^{-1}$$

More generally, if unidimensional utilities, $u_i$, are permitted for the individual dimensions, then

On the other hand, if the DM possesses a ratio preference function $v(x)$ then the $P_j$ may be obtained directly by

$$P_j = \sum_{i=l}^{n} p_i u_i(x_i^j) \left( \sum_{j=1}^{m} u_i(x_i^j) \right)^{-1}.$$  \hspace*{1cm} \hspace*{1cm} \hspace*{1cm} \hspace*{1cm} (3.4)

Comparing (3.3) and (3.4) gives the fundamental functional equation (FFE).

$$v(x^j) \left( \sum_{j=1}^{m} v(x^j) \right)^{-1} = \sum_{i=l}^{n} p_i u_i(x_i^j) \left( \sum_{j=1}^{m} u_i(x_i^j) \right)^{-1}, \text{ for } j=1, \ldots, m.$$ \hspace*{1cm} \hspace*{1cm} \hspace*{1cm} \hspace*{1cm} \hspace*{1cm} (3.5)

If unidimensional utilities, $u_i$, are given by $u_i(x_i^j)=x_i^j$ then the special case, SFFE, results:
There is an important subtlety involved here due to the nonuniqueness of the \( v(x) \) representations of preferences. Given a choice of unidimensional value functions, \( u_i(x_i) \), then (3.5) may be solved, in principle, to find one particular solution, \( v^0(x) \), which agrees with the HCL (AHP) approach. Given a monotone increasing function \{ \cdot \}, then \( \{ v^0(x) \} \) provides the same trade-offs and preference structure as does \( v^0(x) \). However, \( \{ v^0(x) \} \) does not necessarily satisfy (3.5).

The main importance of the above results is that they provide a connection between the priority constructs of the AHP and the more familiar theoretical framework of preference functions. Hence the somewhat philosophical discussion of the differences between these may be replaced or supplemented by specific mathematical questions. Two of these are as follows and are examined further below.

QI: Under what circumstances does there exist a preference function \( v(x) \) on \( \mathbb{R}^n_+ \)

\[
\left\{ x^j \right\} \in \prod_{j=1}^m \mathbb{R}_+^n \\
\] corresponding to given criterion priorities \( p \), such that (3.6) holds for every set

QII: Given a specific \( v(x) \) and set \( \{ x^j \} \in \prod_{j=1}^m \mathbb{R}_+^n \) do there exist criterion priorities for which FFE or SFFE hold(s)?

A second important point for these results is that if a unique (in the sense of trade-off
curves) ratio \( v(x) \) can be associated to the HCL then the problem of rank reversal can be resolved, while maintaining consistency with the HCL. This is because AHP overall alternative priorities \( P_j \) depend both on the criterion priorities, \( p_i \), and the alternative by criterion priorities, \( p_i^j \). Hence in short, the \( P_j \) depend on which particular alternative set is being compared. On the other hand, a preference function value, \( v(x^i) \), is independent of whether other alternatives are, or are not, to be compared. That is, if a ratio \( v(x) \) can be associated with the given priorities, then rank reversal can be avoided while still using the mechanics of the hierarchic composition law; that is, without changing from the HCL to the ideal mode. Two examples of this kind are shown below for a specific kind of solution of functional equation (3.6). The next section discusses solutions of FFE and SFFE.

4. Solutions of the Main Functional Equations

Two practical results can be given here for question QI. The first result discusses solution of SFEE when (3.6) is only required to hold pairwise. That is, only two terms are included in the denominator of the left side of (3.6). Second, a point of reference must be available or specified, to which all other points are compared. In the case of absolute measurement, the actual coordinates of the alternatives are given so that \( e = (1, 1, \ldots, 1) \) is naturally available for that purpose. In the case of relative measurement, the coordinates of the alternatives are not given. Here it is necessary to associate one of the alternatives, or perhaps some other possible alternative, with the unit vector \( e \). Thus consider pairwise comparisons between a generic point \( x \in \mathbb{R}^n \) and \( x^o \). The point \( x^o \) may be thought of as a base comparison point and can be called the *etalon*. Now
clearly has the unique solution

\[
v(x) = v\left( x^o \right) \left( \sum_{i=1}^{n} p_i \frac{x_i}{x_i + x_i^o} \right) \left( 1 - \sum_{i=1}^{n} p_i \frac{x_i}{x_i + x_i^o} \right)^{-1}.
\]

A useful simplification of (4.1) can be given as follows. Suppose \( x^o \) is chosen as \( e = (1,1,...,1) \in \mathbb{R}_+^n \). Also since \( v(x) \) can be multiplied by any positive constant, then we may scale \( v(x) \) so that \( v(e) = 1 \). Then (4.1) becomes

\[
v(x) = \left( \sum_{i=1}^{n} p_i \frac{x_i}{x_i + 1} \right) \left( \sum_{i=1}^{n} p_i \frac{1}{x_i + 1} \right)^{-1}.
\]  \hspace{1cm} (4.2)

Note that \( v(x) = (w(x)) \) where \( (w) = w(1-w)^{-1} \) is monotone increasing on \((0,1)\) and

\[
w(x) = \sum_{i=1}^{n} p_i \frac{x_i}{x_i + 1} \text{ has range } (0,1).
\]

It follows that the level curves of \( w(x) \) and \( v(x) \) are identical. Several of these trade-off curves are illustrated in Figure 1.

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**Figure 1 About Here**

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Figure 1 shows three level curves of

\[
w(x) = 0.3 \frac{x_1}{x_1 + 1} + 0.7 \frac{x_2}{x_2 + 1} = c
\]  \hspace{1cm} (4.3)
for c = 0.1, 0.25, and 0.5 respectively.

The preference function v(x) defined by (4.1) has a number of desirable features. First w(x) can be seen to be an additive separable value function with unit scalings. Next in Hemming (1979) several potentially desirable features of preference functions have been proposed and discussed. We present and check each of these in turn as follows or leave the proof to the reader where the assertion is clear.

P1: v(x) is increasing in each coordinate.

P2: v(x) is concave on $\mathbb{R}^n_+$. 

P3: (Production Function Properties):
   a: $v(0,0,...,0) = 0$,
   b: $v(x) > 0$ for at least one x in $\mathbb{R}^n_+$.
      In fact $v(e) = 1$.
   c: $v(kx) \geq v(x)$ if $k \geq 1$.

Proof: It is sufficient to show this to be true for w(x) due to the monotonicity of ( . ). Thus

$$w(kx) = \sum p_i \frac{k x_i}{k x_i + 1} = \sum p_i \frac{x_i}{x_i + 1/k} \geq \sum p_i \frac{x_i}{x_i + 1} \quad (4.4)$$

by the nonnegativity of the $p_i$.

d: If v (kx) > 0 for some k, then

$$k \to \infty \text{ implies } v(kx) \to \infty .$$

Proof: Suppose $v(kx^0) > 0$. We note that

$$w(kx) = \sum p_i \frac{x_i^0}{x_i + 1/k} \to 1 \quad \text{as} \quad k \to \infty$$
and it follows that $v(x) = (w(x)) \to \infty$ as a result.

e: $v(x)$ is continuous on $\mathbb{R}^n$.

**Remark:** The only other general property proposed in Hemming (1979) is that of homogeneity of degree $h$, namely that

$$v(kx) = k^h v(x), \text{ for some } h > 0.$$ 

Neither $w(x)$ nor $v(x)$ has this property. However, it may be checked that $v(ke) = k$ for all $k > 0$.

The preference function class (4.2) is used further below in sections 5 and 6 to discuss two classical RR examples.

An additional interesting and potentially useful property of such trade-off curve classes can be seen in Figure 1. These curves intersect the axis of the criterion with the higher priority, here $x_2$. For the curve associated with $c = 0.5$, there is a finite amount of $x_2$, namely 2.5, which will compensate for a zero level of the criterion measured by $x_1$. That is, pairs $(x_1, x_2)$ can always be replaced by $(0, x_2^0)$ with $x_2^0$ finite. Also this curve is asymptotic to $x_2 = 0.4$. Thus for $c = 0.5$ no amount of $x_1$ is able to compensate a level of $x_2$ lower than 0.4. This suggests that criterion $x_2$ has what may be called a "quasi-preemptive" priority over $x_1$. A pair can not achieve a value of $c = 0.5$ unless $x_2$ is sufficiently high ($> 0.4$). Also, trade-off curves from linear preference functions cut both axes; while those from the Cobb-Douglas function are asymptotic to both axes. Hence the present model might be considered a bridge between these two classes. A situation of this kind can be imagined in the valuation of collectible coins based on age and condition. A coin of sufficiently old age may have a fixed minimum value regardless of how poor its condition.

We now turn to a second case of question QI. Specifically, suppose $v(x)$ is a specific
linear or additive preference function on $\mathbb{R}^n$. Then given $v(x)$ and an arbitrary set of alternatives in $\mathbb{R}_+^n$, how must criterion priorities be calculated in order to solve SFFE?

**Theorem 2:** Suppose $v(x) = \sum_{i=1}^{n} a_i u_i(x_i)$ where $a_i > 0$ and $u_i(x_i) > 0$ for all $i$ and all $x$. Then given any alternative set $\{x^j, j=1,..., m\}$, the criterion priorities

$$P_i = \left( \sum_{j=1}^{m} \left( \sum_{i=1}^{n} a_i u_i(x_i^j) \right)^{-1} a_i \sum_{j=1}^{m} u_i(x_i^j) \right)$$

solve FFE. Moreover, if $u_i(x_i) = x_i$ for all $i$ then the result corresponding to (4.5) solves SFFE for all sets $\{x^j, j = 1,..., m, x^j \in \mathbb{R}_+^n \}$.

**Proof:** By use of Definition 1 we obtain the overall priorities, $P_i$, as

$$P_j = \left( \sum_{j=1}^{m} \sum_{i=1}^{n} a_i u_i(x_i^j) \right)^{-1} v(x^{supj})$$

$$= \left( \sum_{j=1}^{m} \sum_{i=1}^{n} a_i u_i(x_i^j) \right)^{-1} \sum_{i=1}^{n} a_i u_i(x_i^j)$$

$$= \sum_{i=1}^{n} a_i u_i(x_i^j) \left( \sum_{j=1}^{m} \sum_{i=1}^{n} a_i u_i(x_i^j) \right)^{-1}$$

$$= \sum_{j=1}^{m} \sum_{i=1}^{n} a_i u_i(x_i^j) \left( \sum_{j=1}^{m} \sum_{i=1}^{n} a_i u_i(x_i^j) \right)^{-1}$$

$$\sum_{i=1}^{n} \left( \sum_{i=1}^{n} a_i u_i(x_i^j) \right)^{-1} \left( \sum_{i=1}^{m} a_i \sum_{i=1}^{n} u_i(x_i^j) \right) \left( \sum_{j=1}^{m} u_i(x_i^j) \right)^{-1} \right). \quad (4.6)$$

On the other hand, these priorities are given also by the HCL as

$$\sum_{i=1}^{n} \left( \sum_{i=1}^{n} a_i u_i(x_i^j) \right)^{-1} \left( \sum_{j=1}^{m} u_i(x_i^j) \right)^{-1} \right).$$

Comparison of (4.6) and (4.7) shows that (4.5) solves FFE for all sets of admissible alternatives.

The simpler analogous result for SFFE follows easily and was previously obtained in Troutt and Tadisina (1990). This concludes the proof.
Remarks: Theorem 2 does not preclude the possibility that other specific solutions can be obtained for a given \( v(x) \) and set \( \{ x^j \} \). However it does show that criterion priorities can always be adjusted so as to give the same overall priorities as does a specified linear separable or additive preference function\(^3\).

Next, regarding question QII the result is negative except for the foregoing case in which \( v(x) \) is linear or additive separable. Namely, a counter-example (for natural priorities) has been given in Troutt and Tadisina (1990). In that example a specific set of alternatives in \( \mathbb{R}^n \) was exhibited such that no criterion priorities exist which provide the same overall priorities as does a specified Cobb-Douglas preference function. From the results in this paper however, we see that a more important question is whether the equivalence fails to hold after considering all possible monotone increasing functions of the Cobb-Douglas function. While we do not obtain a direct answer to that question in the present paper the foregoing trade-off curve comparisons suggest that the negative result will still hold. Namely, all such monotone transformations will have the same Cobb-Douglas trade-off curves, none of which has the above quasi-preemptive property of the HCL.

Finally, we obtain results on the general case, FFE (3.5) which give insight on RR and further justify attention to solutions of the form (4.2).

**Theorem 3:** Any solution of FFE must be of the form

\[
\text{Proof: } \text{Suppose } m \geq 2. \text{ Let } x^1 \text{ be the generic point } x, \text{ and choose } x^j = e, \text{ for each } j = 2, \ldots, m.
\]

Substitution of these values into FFE yields
\[
\frac{v(x)}{v(x) + \sum_{j=2}^{m} v(e)} = \sum_{i=1}^{ton} P_i \frac{u_i(x)}{u_i(x) + \sum_{j=2}^{m} u_i(e)}.
\]

However, if \( v(x) \) is any solution, we may clearly scale it so that \( v(e) = 1 \). Hence the result follows as in (4.1) - (4.2).

The essential question raised by Theorem 3 is whether the valuations, \( v_m(x) \), \( m \geq 2 \), are trade-off curve equivalent. That is, do these valuations describe the same preference structure?

The next and main result is that the answer is false for a large class of \( u_i(x) \) including \( u_i(x) = x_i \).

It is first useful to simplify notation. We consider the class of \( u_i(\cdot) \) such that \( u_i(0) = 0 \) for all \( i \) and such that \( u(x) \) maps \( \mathbb{R}^n_+ \) onto itself. Then \( v_m(x) = V_m(u(x)) \). Also let \( e_i = u_i(e) \).

\[
V^0_m(u) = (\Sigma P_i \frac{u_i}{u_i + (m-1) e_i})(\Sigma P_i \frac{e_i}{e_i + (m-1) e_i})^\wedge.
\]

(4.10)

(4.8) becomes

We are now ready to state Theorem 4.

**Theorem 4**: Let \( p_i < 1 \) for \( i = 1, \ldots, n \). Then for every \( m \geq 2 \), \( V^0_m(u) \) and \( V^0_{m+1}(u) \) yield distinct trade-off curves.

**Proof**: As in the decomposition, \( v(x) = (w(x)) \), discussed above after (4.2), the trade-off curves for \( V^0_m(u) \) are the same as those of
Hence it is sufficient to show that $W_m^0(u)$ and $W_{m+1}^0(u)$ have distinct trade-off curves. Towards this end we note that $W_m^0(u)$ and $W_{m+1}^0(u)$ are trade-off curve equivalent if and only if one is a monotone function of the other. It follows that the value of $W_{m+1}^0(u)$ must be constant on any level curve of $W_m^0(u)$. The strategy of the proof is to demonstrate that there exist two points on a fixed trade-off curve of $W_m^0(u)$, but with different values of $W_{m+1}^0(u)$. However, there are two cases depending on $p$. Namely, either $p_{i_0} > 1/n$ for some $i_0$ (case 1), or $p = (1/n)e$ (case 2).

Case 1: Let $u^1 = \hat{e}$. Then

Now assume $i_0 = 1$ without loss of generality. Define $u^2 = (r, 0, 0, ..., 0)$ where $r$ is determined so

that $W_m^0(u^2) = 1/m$. This requires

Now evaluating $W_{m+1}^0(u)$ at $u^1$ and $u^2$, we have
If \( p_1 = 1 \) then no trade-offs exist.

Thus we further assume that \( 0 < p_1 < 1 \). With this assumption we have

\[
W_{m+1}^0(u^1) = \sum_{i=1}^{n} \text{BOLDP}_i \frac{e_i}{e_i + m} = \frac{1}{m+1} , \quad (4.14)
\]

and

\[
W_{m+1}^0(u^2) = \frac{2}{p_1^2(m-1)} , \quad (4.15)
\]

If \( p_1 = 1 \) then no trade-offs exist.

Thus we further assume that \( 0 < p_1 < 1 \). With this assumption we have

\[
p_1^2 (m^2 - 1) < m^2 - 1 < m^2 - p_1 \quad (4.16)
\]

which implies

\[
\frac{p_1^2(m^2-1)}{m^2 - p_1} < 1 \quad (4.17)
\]

and

\[
\frac{p_1^2(m-1)}{m^2 - p_1} < \frac{1}{m+1} . \quad (4.18)
\]

Thus \( W_m^0(u^2) \neq W_{m+1}^0(u^1) \).

Case 2: Again for point $u^1$ we take $\hat{e}$. Point $u^2$ is constructed by perturbing $e$ as follows:

$$u_i^2 = \frac{(m - I)^{\wedge}}{2m - I} e_2.$$  

Let

$$u_2^2 = \frac{3(m - I)^{\wedge}}{2m - 3} e_2.$$  

This value of $u^2_1$ makes the first term of $W^0_m(u)$ have the value $1/2m$. Also let

This value makes the second term of $W^0_m(u)$ have value $3/2m$. For $i \geq 3$ let $u_i^2 = \hat{e}_i$ as in $u^1$. It may then be checked that $u^2$ is another point for which $W^0_m(u) = 1/m$ and $W^0_{m+1}(u^1) = 1/(m+1)$. However, the reader may check that $W^0_{m+1}(u^2) \neq 1/(m+1)$. This concludes the proof.

These difficulties may evidently be avoided by restriction to $m = 2$ and carrying out comparisons to a fixed reference point as in (4.1) - (4.2).

5. The Belton and Gear Example
Here we reconsider the original RR example developed by Belton and Gear (1982) and show how the above results can be used. The key idea is to choose an alternative upon which is endowed unit valuations on the criterion coordinates. We call such an alternative, an *etalon*, or basis of comparison. The resulting operations are similar to the "linking-pins" idea of Schoner et al. (1993).

The Belton and Gear (1982) example considered initially three alternatives A, B, and C and three criteria a, b, and c. Later a fourth alternative, D, was brought into the alternative set. This example stipulated that the criterion priorities were \( p = (1/3, 1/3, 1/3) \). Next three perfectly consistent pairwise ratio comparison matrices were exhibited for which

\[
p_a = (1/11)(1,9,1), \quad p_b = (1/11)(9,1,1), \quad p_c = (1/18)(8,9,1).
\]

Hence by the HCL they obtain \( P = (0.45, 0.47, 0.08) \), so that \( B \_ A \_ C \).

Next a fourth alternative, D, was added to the set of alternatives. The comparison matrices were modified in such a way that the ratios were the same for the D row as for the B row. After these consistent modifications the new alternative priorities become respectively:

\[
p_a = (1/20)(1,9,1,9), \quad p_b = (1/12)(9,1,1,1), \quad p_c = (1/27)(8,9,1,9).
\]

It follows that the new overall priorities became: \( P = (0.37, 0.29, 0.06, 0.29) \), so that the ranking is changed to \( A \_ B = D \_ C \).

Hence by inclusion of an alternative indistinguishable from the original highest priority alternative, the original second highest priority alternative changed to first priority.

We now demonstrate how SFFE solution (4.2) may be applied. First note that this is a case of relative measurement. Hence it is necessary to specify some alternative as the etalon. Let this choice be alternative A. Therefore A is coordinatized as \( e \). Now using proportionality the
coordinates of the other alternatives can be assigned. For example, since \( \frac{p_a^B}{p_a^A} = 9 \) it follows that the first coordinate of alternative B must then be assigned the value, 9. (This approach is also used by Schoner et al., 1993.) Completing this process gives:

\[
A = (1, 1, 1), \quad B = (9, 1/9, 9/8), \quad C = (1, 1/9, 1/8), \quad \text{and} \quad D = B
\]

With \( p = (1/3, 1/3, 1/3) \), and in both the three and four alternative cases we obtain for this decision maker that

\[
\text{Valuation of the alternatives using } v(x) \text{ yields } v(A) = 1, \quad v(B) = 26/25, \quad v(C) = 32/103, \quad \text{and } v(D) = 26/25.
\]

Using Definition 1 we obtain the three and four alternative set overall priority vectors \((0.43, 0.44, 0.13)\), and \((0.30, 0.31, 0.09, 0.31)\), respectively. Hence in either case we see that

\[
B = D \_ A \_ C
\]

is the consistent ordering, hence resolving the RR. Table 1 shows the results to three decimals for other possible choices of the etalon (from among the original alternatives only) in this example.

Table 1 About Here

In this example the same orderings result regardless of the etalon choice. However, we conjecture that instances exist for which the resulting order depends on the etalon choice. The next section deals with a case in which an alternative deletion causes RR.
6. An Alternative Deletion Example

Consider the following example with two criteria a and b, and 3 alternatives A, B, and C originally discussed in Troutt (1988). Suppose the DM data are as given in Table 2. This is an example of RR with absolute measurement. That is, it was assumed that criterion coordinates (measurements) of three alternatives were given by

\[ A = (57, 3), \quad B = (33, 4), \quad \text{and} \quad C = (10, 3) \]

The criterion priorities were stipulated to be 0.3 and 0.7, respectively. The alternative by criterion priorities in Table 2 are derived using Definition 1 applied to the absolute-measured criterion levels. Hence this example rejects the hypothesis that RR occurs due to relative measurement. Second, it is clear that alternative C is strictly dominated by A and B. Hence its inclusion in any further analysis, AHP or otherwise, would apparently need novel justification.

Table 2 About Here

The reader may check using the HCL that \( P = (0.381, 0.379, 0.240) \) for this data, so that the ranking \( A \_ B \_ C \) is obtained. Now however if alternative C is deleted, we may also delete its associated rows and columns from the comparison matrices independently of any other data changes. The reader may verify that the HCL with

\[ p_a = (57/90, 33/90), \quad p_b = (3/7, 4/7), \quad \text{and} \quad p_c = (441/900, 459/900), \] now yields \( B \_ A \).

Thus RR has occurred in a case in which a strictly dominated and lowest priority
alternative was deleted. In this case the relationship between the alternatives and the unit vector $e$

$$v(x) = (0.3 \frac{x_1}{x_1 + 1} + 0.7 \frac{x_2}{x_2 + 1}) (0.3 \frac{1}{x_1 + 1} + 0.7 \frac{1}{x_2 + 1})^t$$

is not arbitrary. The associated $v(x)$ is therefore

The reader may wish to see Figure 1 again for the associated trade-off curves.

Valuation of the alternatives using this $v(x)$ yields $v(A) = 4.549$, $v(B) = 5.720$, and $v(C) = 3.943$, and hence $B \succ A \succ C$ is the ordering which is invariant under the number of comparison points considered.

7. Discussion

The priority concepts of the AHP can be related to a ratio preference function and associated system of trade-off curves via Definition 1 and the Fundamental Functional equations FFE or SFEE. A class of solutions to SFFE can be obtained whenever alternatives are compared, one at a time, to a base comparison point, or etalon. For the absolute measurement case the unit vector $e$ is readily available as the etalon. In the relative measurement case, a similar solution can be obtained if the DM is willing to designate some alternative as the etalon. The procedure can be depicted as in Figures 2 and 3 for criteria (a, b, c, d) and three alternatives (A, B, C). Figure 2 depicts the usual hierarchical arrangement. Figure 3 depicts the modification required for solution (4.2) of SFFE. Thus the alternatives are ranked according to their AHP priorities when compared to a common standard.
This revised AHP procedure, which can be called *Etalon-Based AHP*, is asserted to be a reasonable one. It follows the formal operations of the AHP and the results are consistent with a class of desirable preference functions. A disadvantage in the relative measurement case is that the decision maker must designate an etalon. This raises the question as to whether criterion priorities are anchored in some way to the etalon so chosen.

If the usual hierarchical format, Figure 2, is employed, a linear separable preference function solution might, in principle, be obtained, provided criterion priorities change with the specific sets of alternatives being compared. This possible approach raises the question of whether the DM can and does tailor criterion priorities according to the alternatives under consideration. In these cases it is not clear whether stated criterion priorities relate to the alternative set before or after addition or deletion of alternatives.

8. The Ideal Mode Composition Law

The ideal mode is the currently accepted approach to avoiding rank reversal in AHP theory and is available as an option in ExpertChoice (???????) software. Thus it is an alternative to the HCL or relative measurement mode. Since it also preserves ranks, and is widely used by AHP practitioners, it is useful to discuss it in the context of the present results.

The ideal mode operates as follows. For each criterion, criterion priorities, $p_i$, and alternative by criterion priorities, $p_i^j$, are given or obtained in the usual manner (pairwise
comparison and principal eigenvector method, say). Next the \( p_i^j \) are renormalized, for each \( i \), so that the largest is unity. For this purpose, define
\[
m_i = \max p_i^j
\]
Then the adjusted priorities, called weights, are
\[
m_i^{-1} p_i^j
\]
These weights are first multiplied by the criterion priorities \( p_i \). This has the effect of awarding unity criterion \( i \) weight to the alternative which has the largest priority for that criterion. These weights are then added for all the criteria to form a preliminary total weight for an alternative,
\[
\sum_{i=1}^{n} p_i m_i^{-1} p_i^j
\]
namely
These preliminary total weights are then normalized over all the alternatives to produce the final IMCL priorities, which we denote here by \( P_j \). Thus we have
\[
P_i = (\sum_{i=1}^{n} p_i m_i^{-1} p_i^j) (\sum_{j=1}^{m} \sum_{i=1}^{n} p_i m_i^{-1} p_i^j)^{-1}
\]
IMCL priorities, which we denote here by \( P_j \). Thus we have

This contrasts with (2.4) for the HCL.

First we note that this computation here (8.4) is considerably more complicated than the HCL (2.4). Results of the form derived earlier in the paper might as well be sought for the IMCL as well. That is, using (8.4) as a starting point, functional equations could be developed for it as in section 3. We conjecture that solutions of these functional equations will be difficult to obtain due to the maximum calculations which define the \( m_i \) values. We leave an attempt at solving these as beyond the scope of this paper.
As noted by a referee, the IMCL has the feature of anchoring preferences on a criterion by criterion basis. That is, one or more alternatives receive unit criterion levels for each criterion. In the etalon HCL approach developed above, preferences are anchored on one alternative. However, the etalon approach was developed by solving the functional equations for only a pair of points at one time. The same general strategy might be employed with the IMCL. Such a solution, if possible, would appear to have both criterion and alternative preference anchors and is worthy of future research. In any case, the techniques developed here should aid in understanding between preference functions and IMCL or other priority composition laws which may become of interest in the future.

9. Conclusions and Further Research

This paper has established a connection between the Hierarchic Composition Law of the AHP and the more familiar construct of preference functions. A fundamental functional equation can be used to connect the two preference representations. Some solutions of the main equations were obtained. We have proposed a new modification of the AHP, Etalon-Based AHP, which must always avoid rank reversal while still using the hierarchic composition law. A disadvantage of this modified method in the relative measurement case is that the decision maker must also designate an etalon.

Our results suggest that RR in the AHP is not due to relative measurement. It was observed that RR cannot occur with the preference function approach. We believe that RR in the AHP occurs because of either of the following conditions:

(i) Use of the standard hierarchy, Figure 2, prevents the possibility of associating a
specific preference function to the data when two or more alternatives are under consideration due to Theorem 4, or

(ii) In the linear preference function case, the DM is unable to consistently adjust criterion priorities, at least in an unaided mode, to the alternative set or sets being considered.

Furthermore, we conjecture that if the standard hierarchy, Figure 2, is used, then the following result is likely. Namely, starting with a specific set of alternatives, then with the addition of appropriate other alternatives, any specified reordering of the original alternatives may be obtained.

An interesting possibility for further research is to exploit the connection between priorities and preference functions on the one hand, and the eigensystem estimation method on the other hand. In some simple, perhaps expressly designed test cases, it should be relatively straightforward for decision makers to provide overall pairwise comparisons on the alternatives directly, as well as for the alternatives and criteria in the usual manner. Then an associated \( v(x) \) might be ascertained in both ways to empirically check behavioral consistency with the present theoretical results. Such a study might also compare the relative performance of the hierarchic composition law and the ideal mode composition law.

In this paper we have attempted to isolate the mechanics of the AHP from inconsistency issues. This was done by assuming that criterion priorities were given. Thus we viewed inconsistency as related to the correct estimation of criterion priorities which we regard to be adequately handled by the principal eigenvector method. Similarly, we assumed absolute measurements of alternatives on the criteria were given without inconsistency. A different approach might be developed for future research by studying the effect of the AHP composition
laws on the combined inconsistencies in both the criteria, and alternatives.
Footnotes

1. The authors do not dispute that rank reversal can, does and should appropriately occur in one or more real decision situations. However whether the AHP or some other methodology accurately models such real and appropriate rank reversal has not yet been convincingly demonstrated in the literature.

2. In the AHP methodology much attention is given to estimation of these priorities, chiefly from pairwise comparison matrices. In this paper we are not concerned with such estimation or related consistency issues since these priorities are assumed to be given.

3. Interestingly, a version of result (4.5) has also been obtained independently in Schoner and Wedley (1989). Interested readers may contact the first author for a correspondence of the differing notations.
REFERENCES


**Table 1**: Belton and Gear Example Results

<table>
<thead>
<tr>
<th>Etalon Chosen:</th>
<th>A</th>
<th>B or D</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>P Vector - 3 Alternatives:</td>
<td>[0.43 0.44 0.13]</td>
<td>[0.43 0.44 0.13]</td>
<td>[0.43 0.44 0.013]</td>
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<tr>
<td>P Vector - 4 Alternatives:</td>
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<td>[0.29 0.31 0.09 0.31]</td>
<td>[0.30 0.305 0.09 0.305]</td>
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Table 2: Pairwise comparison matrices and their priorities.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>( p_a )</th>
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<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>57/33</td>
<td>57/10</td>
<td>0.57</td>
</tr>
<tr>
<td>B</td>
<td>33/57</td>
<td>1</td>
<td>33/10</td>
<td>0.33</td>
</tr>
<tr>
<td>C</td>
<td>10/57</td>
<td>10/33</td>
<td>1</td>
<td>0.10</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>( p_b )</th>
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<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>3/4</td>
<td>1</td>
<td>0.3</td>
</tr>
<tr>
<td>B</td>
<td>4/3</td>
<td>1</td>
<td>4/3</td>
<td>0.4</td>
</tr>
<tr>
<td>C</td>
<td>1</td>
<td>3/4</td>
<td>1</td>
<td>0.3</td>
</tr>
</tbody>
</table>
Figure 1: Some level curves for $w(x)=c$ with $c=0.1, 0.25, 0.50$
Figure 2: Standard AHP Hierarchy
Figure 3: Etalon-Based AHP Hierarchy
## Correspondence of Key Notations and Definitions


<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
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</thead>
<tbody>
<tr>
<td>$x^i_j$</td>
<td>level of criterion $i$ for alternative $x^j$</td>
<td>$T_{i,k}$</td>
<td>absolute measurement of criterion $k$ for option $i$</td>
</tr>
<tr>
<td>$v(x^j)$</td>
<td>value or worth or overall importance of alternative $x^j$</td>
<td>$v_i$</td>
<td>value of option $i$</td>
</tr>
<tr>
<td>$p_i$</td>
<td>criterion $i$ priority</td>
<td>$x_k$</td>
<td>criterion $k$ importance (priority)</td>
</tr>
<tr>
<td>no symbol used</td>
<td>(natural) priority of alternative $j$ on criterion $i$, $\sum_{j=1}^{k} x^j_i / \sum_{j=1}^{k} x^j_i$</td>
<td>$w_{i,k}$</td>
<td>criterion $k$ importance (priority)</td>
</tr>
<tr>
<td>Troutt (1988) none, Troutt and Tadisina (1991) $a_i$</td>
<td>coefficient of criterion $i$ in a linear preference function</td>
<td>$q_k$</td>
<td>a scale factor which converts measurement on attribute $k$ to units of the objective</td>
</tr>
<tr>
<td>Troutt (1988) $v^o$, Troutt and Tadisina (1991) no symbol used</td>
<td>composite priority of alternative $j$</td>
<td>$w_i$</td>
<td>composite priority of option $i$</td>
</tr>
</tbody>
</table>