A Deformation of the Poincaré Lie Algebra and Kinematical Confinement

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Abstract. The notion of kinematical confinement introduced by Flato, Fronsdal, Sternheimer [1] and their coworkers must be considered as one of the most fascinating and powerful ideas of modern theoretical physics. Here we hope to sharpen the meaning of this remarkable mechanism for explaining quark confinement. We recall $so(2,3)$ as a well-known deformation of the Poincaré Lie algebra [2] and we solve the defining equations of the deformation map for the Poincaré translation generators to obtain the desired result, namely: an isomorphic copy of the Poincaré Lie algebra in which the translation generators depend upon the deformation parameter that is essentially the reciprocal of the radius of the anti-de Sitter space [3]. Representations of this isomorphic copy of the Poincaré Lie algebra go over into Segal-Inönü-Wigner contractions [4] of the corresponding representations of the anti-de Sitter algebra as the deformation parameter goes to zero. We apply our results to the singleton representations of the anti-de Sitter algebra, and implications for kinematical confinement are then given.

1. $SO_{0}(2,3)$, the Poincaré Group and their Lie Algebras

Let $\beta_0 = \text{diag}(1, -1, -1, -1, 1)$ and $\dagger$ denote transpose of a matrix, then

$$SO(2,3) = \{g \in SL(5,\mathbb{R}) | g \beta_0 g^\dagger = \beta_0\}.$$  \hfill (1.1)

$G = SO_o(2,3)$ is the component connected to the identity of $SO(2,3)$. Let $\mathcal{G}$ be the Lie algebra of $G$.

A Cartan-Weyl basis for $\mathcal{G}$ is given by

$$-i H_1 = 2 L_{21}, H_2 = (L_{40} - L_{21}), -i X^\pm_k = L_{31} \mp i L_{23},$$  \hfill (1.2a)

$$-i X^\pm_2 = \pm 1/2 (L_{41} + L_{20}) - i/2 (L_{10} - L_{42}).$$  \hfill (1.2b)

$H_1$ and $H_2$ are a basis for a Cartan subalgebra $\mathcal{H}_C$ of $\mathcal{G}$. The commutation relations for the basis vectors of the simple roots are: $[H_j, H_k] = 0$, $[H_j, X^\pm_k] = \pm a_{jk} X^\pm_k$, $[X^+_j, X^-_k] = \delta_{jk} H_k$ (No summation is intended in these equations.)
We introduce the following elements of all Lie structures on such that
\[ \sum \]

The center Lorentz vector operator
\[ j_{\mu} \]
\[ L^{\mu} \]
\[ J_{j}^{(i)} \cdot L_{0j} \]
\[ L_{ij} \]
\[ i j k \]
\[ \mu \]
\[ [x, y] = \mu(x, y) \]
\[ \sum \]
\[ \mu(x, y, z) = 0 \] (Jacobi identity)
\[ \tau_{\lambda} \]
\[ (3.1) \]

2. Lie Algebra Deformations

Given \( V \), a finite dimensional vector space of dimension \( n \) over \( \mathbb{R} \) or \( \mathcal{G} \), let \( \mathcal{M} \) be the space of all Lie structures on \( V \) i.e. the space of all equivalence classes of bilinear mappings

where two such maps are equivalent if they give isomorphic Lie algebras.

**Definition:** Let \( \Lambda \in \mathbb{R}^+ \), then a deformation of a given Lie algebra \( \mathcal{G} \) is a mapping \( \psi : [0, \Lambda] \rightarrow M_n \) with \( \mathcal{G} = (V, \psi(0)) \) i.e. \( \psi(0) \) is the Lie structure of \( \mathcal{G} \). \( \psi \) is a trivial deformation if \( \psi(t) \) for all \( t \in [0, \Lambda] \) is equivalent to \( \psi(0) \).

3. Algebraic Results

Let \( \mathcal{K} \) be the skew field of \( \mathcal{G} \) and \( \mathcal{K}(\mathcal{G}_\lambda) \) be the skew field of \( \mathcal{G}_\lambda \) (see below). Define commutative algebraic extensions of \( \mathcal{K}(\mathcal{G}) \) and \( \mathcal{K}(\mathcal{G}_\lambda) \): \( \mathcal{K} \) \( \mathcal{G} \) satisfies the equation

\[ \hat{Y}^4 + \lambda^2 \ C_2 \ \hat{Y}^2 + \lambda^4 \ C_4 = 0 \] with

\[ C_2 = \left( C_2 + \frac{\lambda}{2} \right) \]
\[ C_4 = \left( C_4 + \frac{\lambda}{4} \right) \]
\[ \frac{\lambda}{16} \]

\( (I \) is the identity in \( \mathcal{K}(\mathcal{G}_\lambda)) \) Now define a mapping \( \tau_{\lambda} \) from \( \mathcal{G}_\lambda \) to \( \mathcal{K}(\mathcal{G}) \) by

\[ \tau_{\lambda}(L_{\mu v}) = \frac{i\lambda}{2Y}[Q_2, P_\mu] + P_\mu \]
The $\tau^{-1}_\lambda(\bar{L}_{4\mu})$ and $\tau_\lambda(\tilde{L}_{\mu\nu})$ satisfy the commutation relations of the generators of $\mathcal{G}$. The $\tau_\lambda(\bar{L}_{4\mu})$ and $\tau_\lambda(\tilde{L}_{\mu\nu})$ are a basis for an isomorphic copy $\mathcal{G}_\lambda$ of $\mathcal{G}$, which differs from $\mathcal{G}$ by a scaling factor $\lambda$ in the $L_{4\mu}$ directions. Let $\tau_{\lambda=1}(\bar{Y}) = Y$, then $\tau = \tau_{\lambda=1}$ can be extended to a homomorphism of $\mathcal{H}(\mathcal{G})^{\text{ext}}$ into $\mathcal{H}(\mathcal{P})^{\text{ext}}$ in an obvious way, which, because of Theorem II, is actually surjective. Denote this extension also by $\tau$. Elements of $\mathcal{H}(\mathcal{G})^{\text{ext}}$ have a tilde to keep them distinct from elements of $\mathcal{H}(\mathcal{P})^{\text{ext}}$, and we introduce $\ast$ structures on $\mathcal{H}(\mathcal{P})^{\text{ext}}$ and $\mathcal{H}(\mathcal{G})^{\text{ext}}$.

**Theorem I:** Let $\tau(\mathcal{G}_\lambda)$ be the isomorphic copy of $\mathcal{G}$ having basis elements $L_{ij} \in \mathcal{P}$ and $L_{4\mu} \in \mathcal{H}(\mathcal{P})^{\text{ext}}$ defined by eqns. (3.1). Then (for $\lambda = 1$) the following holds:

$$C_2 = -Y^2 - \left[ \frac{W}{Y^2} + \frac{9}{4} I \right], \quad C_4 = \left[ Y^2 + \frac{1}{4} \right] \frac{W}{Y^2}. \quad (3.2)$$

**Theorem II:** Solutions $P_\mu(\lambda)$ to eqns. (I) are given by:

$$P_\mu(\lambda) = D^{-1} A^\nu_\mu L_{\nu\mu} \quad (3.3)$$

with

$$A^\nu_\mu = -C_4^{\nu} \delta^\mu_\nu + \frac{i}{2} \left[ \left( Q_2 + \frac{1}{4} \right) \delta^\nu_\mu - \frac{3}{2} L_{\mu\nu} - L_{\mu\rho} L^{\rho\nu} - Q_4 e^\nu_\mu \lambda_{\mu\rho} L^{\rho\tau} \right] \frac{Y}{\lambda}$$

$$- \left[ (Q_2 + \frac{1}{4} - C_4^{\nu}) \delta^\nu_\mu - L_{\mu\nu} - L_{\mu\rho} L^{\rho\nu} \right] \frac{Y^2}{\lambda} + i \left[ (Q_2 - C_4^{\nu}) \delta^\nu_\mu - L_{\mu\nu} - L_{\mu\rho} L^{\rho\nu} \right] \frac{Y^3}{\lambda^3},$$

$$D = Q_4 + \frac{3}{4} Q_2 - C_4^{\nu} + \frac{3}{16} I + i \left[ Q_2 + \frac{1}{2} \right] Y \lambda - \left( Q_2 - C_4^{\nu} - 1/2 \right) \frac{Y^2}{\lambda^2} + 2i \frac{Y^3}{\lambda^3}.$$  

Also $Y^2$ satisfies $Y^2 + \lambda^2 C_4^{2} Y^2 + \lambda^4 C_4^{4} \tau_\lambda = 0$, where $C_4^{2}$ and $C_4^{4}$ are the second and fourth order Casimir operators of $\mathcal{G}$, defined analogously to $C_2$ and $C_4$ which are given above.

**Lemma I:** $\text{ker}(\tau)|_{\mathcal{G}(\mathcal{G})} = 0$.

**Theorem III:** If $\bar{L}_{\mu\nu} = -L_{\mu\nu}$, $\bar{L}_{4\mu} = -L_{4\mu}$ and if $\bar{Y} = Y$, then $P_\mu = \left( D^{-1} \bar{A}^\rho_\mu L_{4\rho} \right)$ and $\bar{P}_\mu = \left( D^{-1} \bar{A}^\rho_\mu L_{4\rho} \right)^\ast = \bar{L}_{4\rho} \bar{A}^\rho_\mu (\bar{D})^{-1} = \bar{A}^\rho_\mu$. Further $[\bar{P}_\mu, \bar{P}_\nu] = 0$.

Proofs of the results in this section can be found in ref. [3]. More details about the necessarily lengthy and technical proof of Theorem II can be found in ref. [9].

### 4. A Deformation of $\mathcal{P}$

Let $V$ be the underlying vector space of the Lie algebra of $\mathcal{P}$ and define a map $\phi_\lambda : \mathcal{P} \to \mathcal{H}(\mathcal{P})^{\text{ext}}$ by

$$\phi_\lambda(L_{\mu\nu}) = \tau_\lambda(\bar{L}_{\mu\nu}) \quad \phi_\lambda(P_\mu) = \tau_\lambda(\bar{L}_{4\mu}). \quad (4.1)$$

Due to Lemma I, $\phi_\lambda$ is a vector space isomorphism onto its image and thus we can define a new Lie bracket on $\mathcal{P}$ by:

$$[a, b]_\lambda = \phi_\lambda^{-1}([\phi_\lambda(a), \phi_\lambda(b)]) \quad \forall \ a, b \in \mathcal{P}. \quad (4.2)$$

**Corollary:** Let $\psi : I \subset \mathcal{R} \to \mathcal{M}_n$ be defined as

$$\psi(\lambda) = \left[ \bullet, \bullet \right]_\lambda \quad (4.3)$$

with $\lambda \in I$. Then $\psi$ is a (nontrivial) deformation of $\mathcal{P}$, with $\psi(1)$ being the Lie structure of $\mathcal{G}$. 

3
5. Representations

Although a representation of \( \mathcal{G} \) always gives a representation of the enveloping algebra \( \mathcal{E}(\mathcal{G}) \), it does not necessarily give a representation of the skew field. However, from Theorem III we obtain:

**Theorem IV:** Let \( (d\pi, \mathcal{H}) \) be an infinitesmally unitarizable representation of \( \mathcal{G} \) on an Hilbert space \( \mathcal{H} \), and let \( \tilde{Y} \) be a self-adjoint operator on \( \mathcal{H} \) which satisfies \( \tilde{Y}^4 + d\pi(\tilde{C}_1^4)\tilde{Y}^2 + d\pi(\tilde{C}_0^4) = 0 \). Then, if both \( d\pi(\tilde{D})^{-1} \) and \( d\pi(\tilde{D})^i_{-1} \) exist on a suitable, dense domain in \( \mathcal{H} \), there exists a skew symmetric representation \( d\tilde{\pi} \) of \( \mathcal{P} \) on \( \mathcal{H} \) which is defined by:

\[
d\tilde{\pi}(\tilde{L}_{ij}) = d\pi(\bar{L}_{ij}), \quad d\tilde{\pi}(\bar{P}_0) = d\pi(\bar{D})^{-1} d\pi(\sum_{0}^{1} \bar{A}_i \bar{L}_n, i) \quad \text{and} \quad d\tilde{\pi}(\bar{P}_i) = [d\pi(\bar{L}_{i0}), d\pi(\bar{P}_0)](i = 1, 2, 3).
\]

Note that both operators \( d\pi(\bar{D})^{-1} \) and \( d\pi(\bar{D})^i_{-1} \) must exist on a suitable dense domain in \( \mathcal{H} \) in order to be assured of the existence of the \( d\pi(\bar{P}_\mu) \), and, even if \( \tilde{Y} \) is not self adjoint, we still get a representation of \( \mathcal{P} \) except that it is not skew symmetric.

6. Singleton Representations of \( SO(2,3) \)

There exists geometrical descriptions of the singleton representations of \( SO_0(2,3) \) which have generalizations to higher dimensions; in particular, the \( SO(4,4) \) case has been treated in great detail by Bertram Kostant and his student Witold Biedrzycki[6]. The \( SO_0(2,3) \) singleton representations are called the \( Di \) and the \( Rac \). Kobayashi and Orsted [7] have described generalizations of the \( Rac \) for arbitrary \( SO(p,q) \) groups. Angelopoulos and Laoues [8] describe the singleton representations of \( SO_0(2,p) \) for arbitrary \( p \). The \( Di \) is technically somewhat more complicated than the \( Rac \), and, for simplicity we treat only the case of the \( Rac \) here.

For the description of the \( Rac \) we consider a three dimensional version of a compactification of the Einstein universe, namely:

\[
M \cong S^1 \times S^2 = \left\{ (u_0, u_1, u_2, u_3, u_4) \mid u_0^2 + u_1^2 + u_2^2 + u_3^2 = R^2 \right\}.
\]

(6.1)

It turns out that \( R \) is the radius of anti-de Sitter space and \( R = \frac{1}{\lambda} \) where \( \lambda \) is the contraction parameter introduced above. We may introduce spherical coordinates on \( M \) as follows:

\[
u_0 = R \sin \tau, \quad u_1 = R \sin \theta \cos \phi, \quad u_2 = R \sin \theta \sin \phi, \quad u_3 = R \cos \theta, \quad u_4 = R \cos \tau.
\]

(6.2)

with ranges of the angular parameters being \( 0 < \tau < 2 \pi, 0 < \theta < \pi, 0 < \phi < 2 \pi \). An \( SO(2) \times SO(3) \) invariant measure on \( M \) is

\[
du = d\tau \sin \theta \, d\theta d\phi.
\]

(6.3)

\( SO_0(2,3) \) acts as the group of conformal transformations on \( M \), and the conformally invariant wave equation on \( M \) is

\[
\frac{\partial^2}{\partial \tau^2} \psi - \Delta_{S^2} \psi = \frac{1}{4 R^2} \psi,
\]

(6.4)

where \( \Delta_{S^2} \) is the Laplacian on \( S^2 \) which in spherical coordinates is

\[
\Delta_{S^2} = -\frac{1}{\sin \theta} \left( \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin \theta} \frac{\partial^2}{\partial \phi^2} \right),
\]

(6.5)
Solutions of the wave equation (6.4) are:
\[
\psi_{E \ell m}(\tau, \theta, \phi) = \frac{(-1)^m (2m)!}{m!2^m} \left[ \frac{(2\ell + 1)(\ell - m)!}{(\ell + m)!} \right]^{1/2} e^{iE \tau} \sin^m(\theta) C^{m+1/2}_{\ell-m}(\cos \theta) e^{i m \phi}
\]
with \( \ell = 0, 1, 2, 3, \ldots, 0 \leq m \leq \ell, \ m \in \mathbb{Z}^+ \). The \( C^{m+1/2}_{\ell-m}(\cos \theta) \) are Gegenbauer polynomials. We easily check that \( \int_M |\psi_{E \ell m}|^2 \, du = \) constant, so that the \( \psi_{E \ell m} \) are normalized up to an inessential constant which we do not bother to determine.

Substituting the solutions (eqn. (6.6)) into the wave equation gives the following spectral equation for the \( \mathcal{R}ac \):
\[
E^2 - \frac{1}{R^2} \ell(\ell + 1) = \frac{1}{4 R^2}.
\]  
(6.7)

We let
\[
\mathcal{H}^+_\ell = 1.s.\{ \psi_{E \ell m} \mid E = + \frac{1}{R} (\ell + \frac{1}{2}), \ell \text{ fixed} \},
\]  
(6.8a)

and
\[
\mathcal{H}^-_\ell = 1.s.\{ \psi_{E \ell m} \mid E = - \frac{1}{R} (\ell + \frac{1}{2}), \ell \text{ fixed} \},
\]  
(6.8b)

Then
\[
\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-.
\]  
(6.9)

with
\[
\mathcal{H}^+ = \sum_{\ell=0}^{\infty} \mathcal{H}^+_\ell \quad \text{and} \quad \mathcal{H}^- = \sum_{\ell=0}^{\infty} \mathcal{H}^-_\ell.
\]  
(6.10)

The spaces \( \mathcal{H}^+ \) and \( \mathcal{H}^- \) are carrier spaces for positive and negative energy irreducible and infinitesimally unitarizable representations of \( so(2, 3) \). We denote the \( so(2, 3) \) representations on \( \mathcal{H}^\pm \) by \( d_{\pi_{1/2}}^\pm \), and the representation on \( \mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^- \) by \( d_{\pi_{1/2}} \). We call the positive energy one the Rac representation. We have for \( R = 1 (\partial_x = \frac{\partial}{\partial x}) \):
\[
d_{\pi_{1/2}}(L_{ij}) = u_i \partial_{u_j} - u_j \partial_{u_i} \quad (i = 1, 2, 3)
\]  
(6.11a)

\[
d_{\pi_{1/2}}(L_{40}) = - \partial_\tau
\]  
(6.11b)

\[
d_{\pi_{1/2}}(L_{03}) = \cos \tau \sin \theta \partial_\theta + \cos \theta \sin \tau \partial_\tau + \frac{1}{2} \cos \theta \cos \tau
\]  
(6.11c)

\[
d_{\pi_{1/2}}(L_{43}) = \sin \tau \sin \theta \partial_\theta + \cos \theta \cos \tau \partial_\tau + \frac{1}{2} \cos \theta \cos \tau
\]  
(6.11d)

All other commutation relations follow from
\[
L_{0i} = [L_{03}, L_{3i}], \quad L_{4i} = [L_{43}, L_{3i}] \quad (i = 1, 2, 3)
\]  
(6.12)

Let the ket corresponding to \( \psi_{E, \ell, m}(\tau, \theta, \phi) \) be denoted by \( |\ell m> \), then the actions of the \( H_1, \ H_2, \ X_1^\pm, \ X_2^\pm \) and \( X_4^\pm \) in the Rac representation on the basis \( |\ell m> \) are as follows:
\[
H_1|\ell m> = 2m|\ell m>, \quad H_2|\ell m> = (1/2 + \ell - m)|\ell m>,
\]  
(6.13a)

\[
X_1^\pm|\ell m> = ([\ell \mp m] [(\ell \pm m + 1)])^{1/2} |\ell m \pm 1>,
\]  
(6.13b)

\[
X_2^\pm|\ell m> = \pm ([1/2(\ell - m \pm 1)] [1/2(\ell - (m \mp 1) + 1)])^{1/2} \times |\ell \pm 1 m \pm 1>,
\]  
(6.13c)

\[
X_4^\pm|\ell m> = \pm ([1/2(\ell + m \pm 1)] [1/2(\ell + (m + 1) \pm 1)])^{1/2} \times |\ell \pm 1 m \pm 1>.
\]  
(6.13d)
7. Kinematical Confinement: Why the Di and Rac Cannot Be Realized on Minkowski Space

According to Theorem IV it is necessary that both $d\pi_{1/2}^+(\tilde{D})^{-1}$ and $d\pi_{1/2}^+(\tilde{D})^{+1}$ exist on a suitable, dense domain in $\mathcal{H}^+$, in order that there exists a representation $d\tilde{\pi}_{1/2}$ of $\mathcal{P}$ on $\mathcal{H}^+$. For both the Di and Rac is $\tilde{Y}^2 = -1, -\frac{1}{4}$ which implies not only that $d\pi_{1/2}(\tilde{P}_\mu)$ are not skew-symmetric operators, but, e.g. for the Rac: $d\pi_{1/2}(\tilde{D}) = 0$ (with $\tilde{Y} = -i$). Thus the $d\tilde{\pi}_{1/2}(\tilde{P}_\mu)$ do not exist in any sense as operators on the Hilbert space of the representation; and, since,

$$\lim_{\lambda \to 0} \tilde{P}_\mu(\lambda) = \lim_{\lambda \to 0} \tilde{L}_{4\mu} = P_\mu$$

(7.1)

where $P_\mu$ is the contracted of $L_{4\mu}$, the Di and the Rac do not have contractions to representations of the Poincaré group.

References


