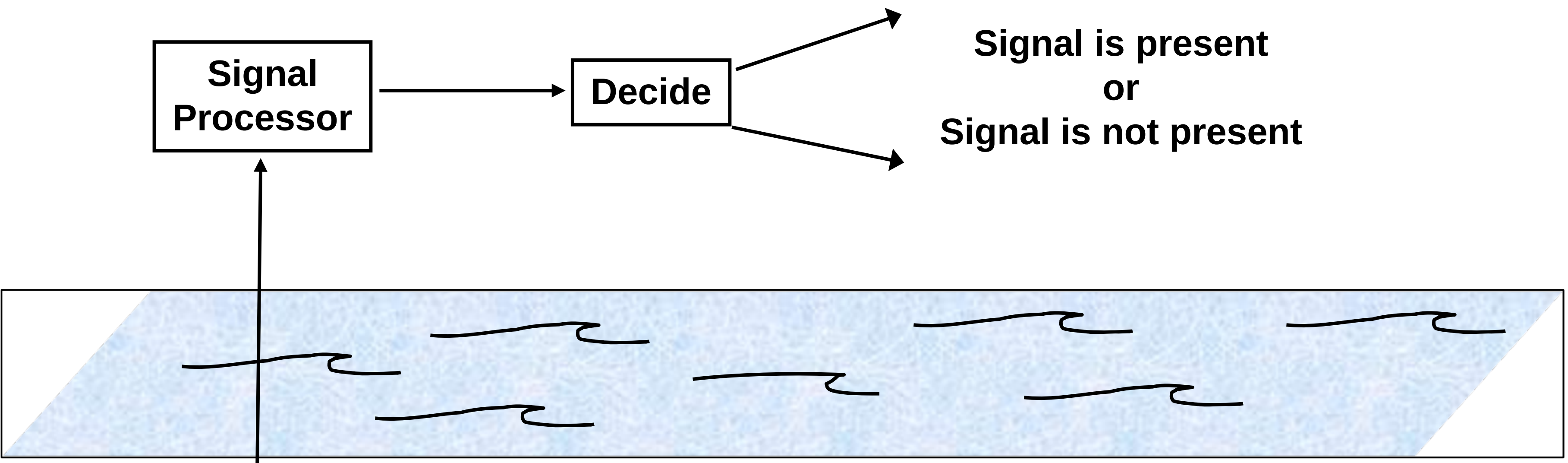
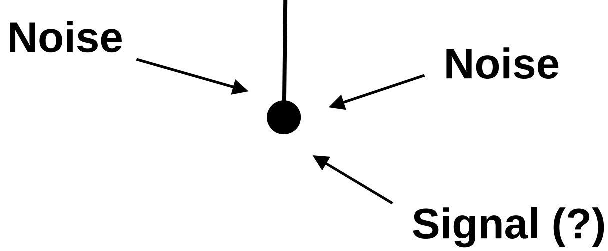
# Decision Theory and Receiver Design

# Signal Detection and Performance Estimation



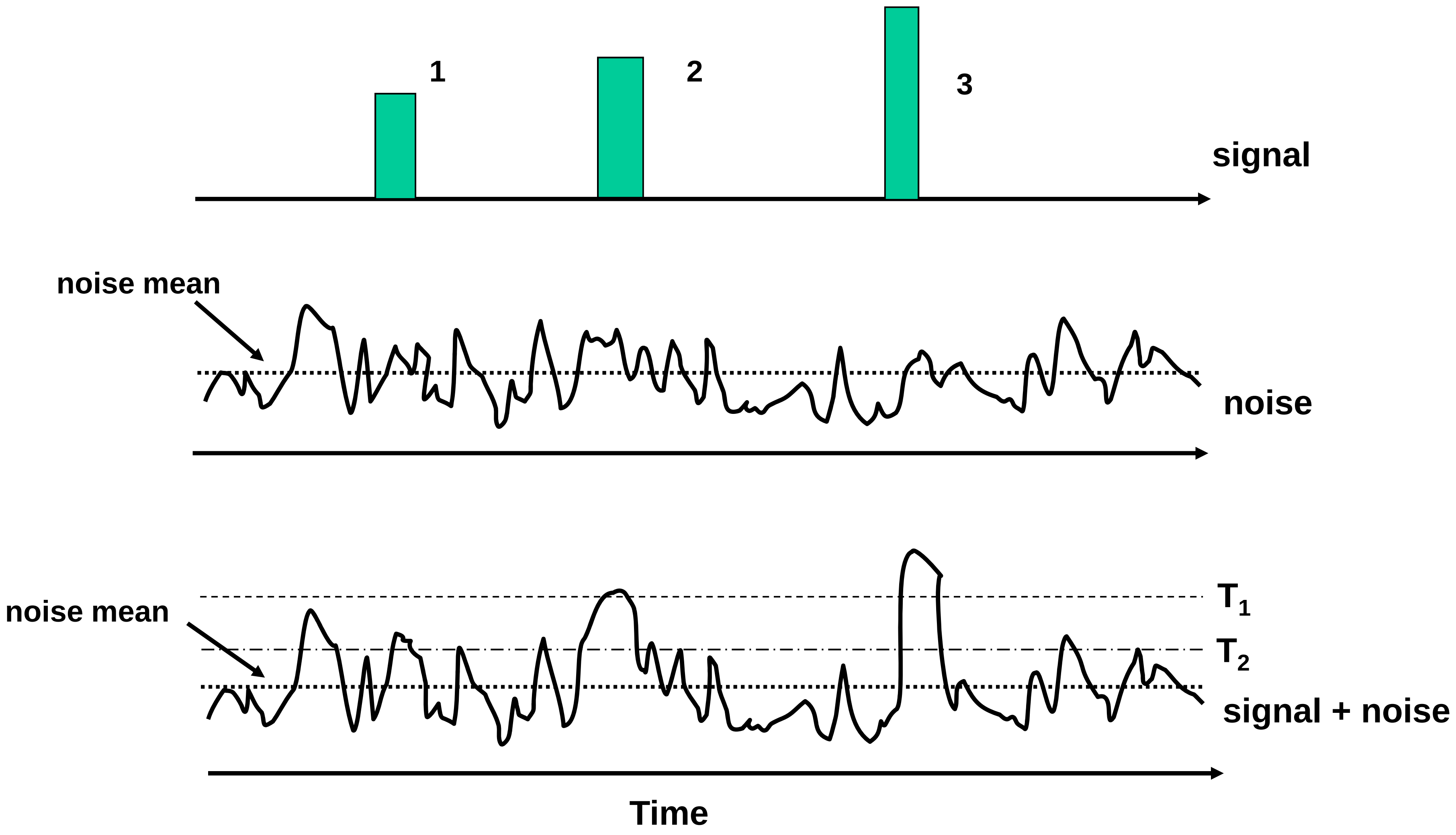




- <u>Problem</u>: How should received signals be processed in order to detect signals in noise? What kind of detection performance can be expected?
- <u>The approach to solution</u>:
  - Must be statistical, since noise is involved
  - Implement hypothesis testing

# Signal Detection

- Input to detector is signal plus noise.
- Requirements expressed in terms of
  - probability of detection
  - probability of false alarm
- Threshold for declaring detection is set based on models for signal and noise
- Noise background estimation can be performed on data to improve model.
- Outputs of detector are threshold crossings
- Performance defined by receiver operating characteristic (ROC) curve – probability of detection vs. probability of false alarm for a particular SNR.

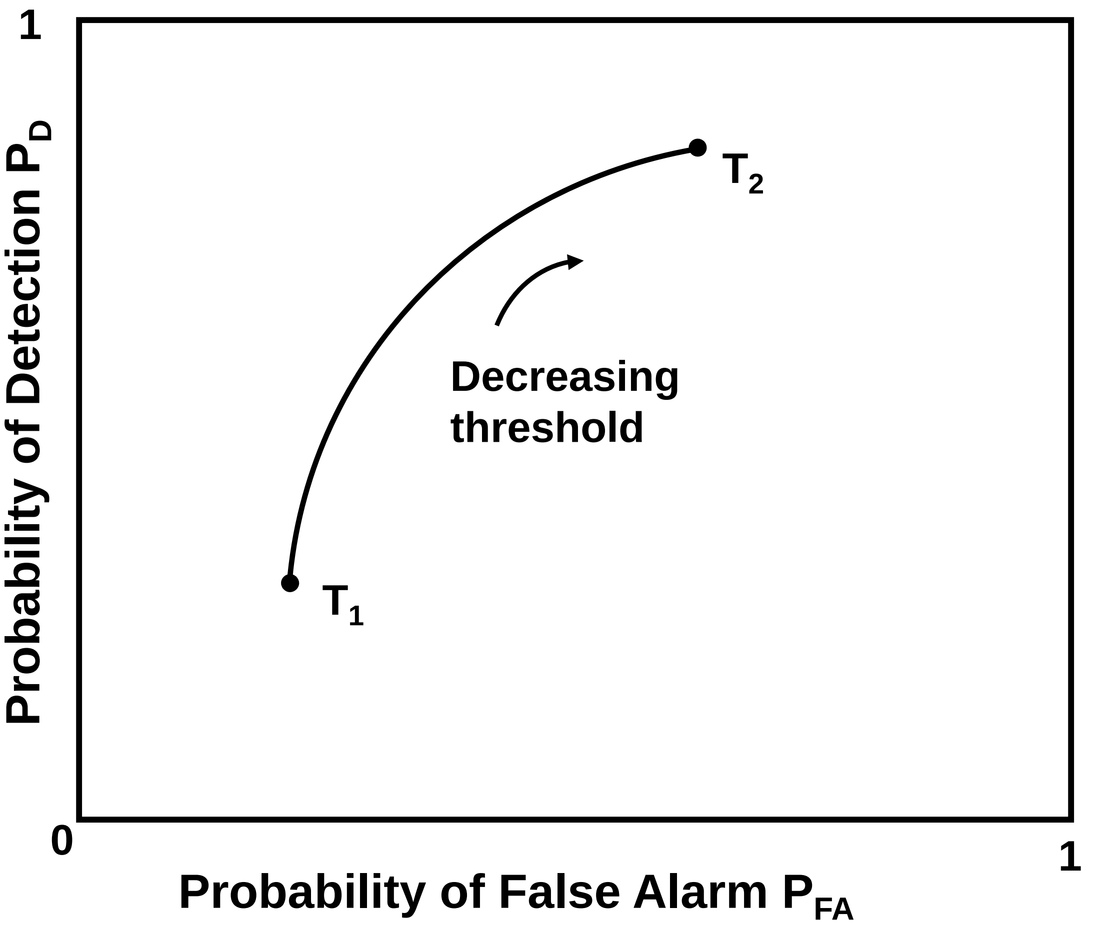
# Detection In Noise



# Detection Threshold

- Performance Criteria:
  - Probability of detection $P_D$
  - Probability of false alarm $P_{FA}$
- These criteria are not independent: a lower threshold increases $P_D$, but also increases $P_{FA}$.
- Theoretical ROC is used to set thresholds.
- True test is performance in water.

# Receiver Operating Curve (ROC)



# Hypothesis Testing

- **Possible Hypotheses:**
  - $H_0$ **: Only noise is present**
  - $H_1$ **: Signal is present in addition to noise**
- **Steps in forming hypotheses:**
  - **Process array output to obtain a <u>detection statistic</u> x.**
  - **Calculate the a posteriori probabilities $P(H_0|x)$ and $P(H_1|x)$.**
  - **Pick the hypothesis whose probability is the highest: the maximum a posteriori, or MAP estimate.**

$$\frac{P(H_1 \mid x)}{P(H_0 \mid x)} \begin{cases} \geq 1, & \text{Choose } H_1 \\ < 1, & \text{Choose } H_0 \end{cases}$$

# Hypothesis Testing (Cont'd)

- Equivalently, we can use Bayes' rule to write:

$$P(H_1 | x)P(x) = P(x | H_1)P(H_1)$$
$$P(H_0 | x)P(x) = P(x | H_0)P(H_0)$$

- $P(H_1)$ and $P(H_0)$ are called *a priori* probabilities
- Then the test can be written:

$$\frac{P(H_1 | x)}{P(H_0 | x)} = \frac{P(x | H_1)P(H_1)}{P(x | H_0)P(H_0)} \begin{cases} \geq 1, & \text{Choose } H_1 \\ < 1, & \text{Choose } H_0 \end{cases}$$

# Hypothesis Testing (Cont'd)

- **An equivalent test is**

$$\frac{P(x\,|\,H_1)}{P(x\,|\,H_0)}\begin{cases}\le \dfrac{P(H_0)}{P(H_1)}, & \text{Choose } H_1\\[2ex] > \dfrac{P(H_0)}{P(H_1)}, & \text{Choose } H_0\end{cases}$$

- $\dfrac{P(x\,|\,H_1)}{P(x\,|\,H_0)} \equiv \lambda(x)$ **is called the <u>likelihood ratio</u>**

# Aside: Bayes' Rule and Notation

- **Probability density functions are often used to describe continuous random variables:**

$$P(x_0) \equiv \int_{-\infty}^{x_0} p(x)dx$$

- **Bayes' Rule as written for probabilities also holds for probability density functions (pdf).**
- **A compact notation is used in what follows:**

$$p(x \,|\, H_1) \equiv p_1(x) \qquad\qquad p(x \,|\, H_0) \equiv p_0(x)$$

- **Likelihood ration test written in terms of pr**

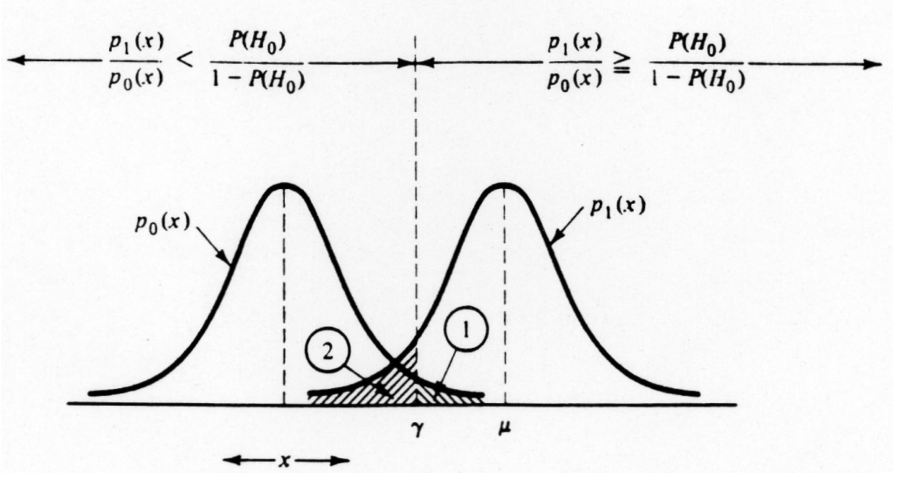
$$\frac{p_1(x)}{p_0(x)} \begin{cases} \le \dfrac{P(H_0)}{P(H_1)}, & \text{Choose } H_1 \\ > \dfrac{P(H_0)}{P(H_1)}, & \text{Choose } H_0 \end{cases}$$

# A First Example: Constant Signal

- **The possible inputs are:**

$$H_0: \quad x(t) = n(t) \qquad \text{(Noise only)}$$
$$H_1: \quad x(t) = \mu + n(t) \qquad \text{(Signal plus noise)}$$

- **If $n(t)$ is Gaussian distributed and $\mu \neq 0$, then $p(x \mid H_1) = \mathrm{p}_1(\mathrm{x})$ and $p(x \mid H_0) = \mathrm{p}_0(\mathrm{x})$ are as shown below:**




$$p_0(x) = (2\pi\sigma^2)^{-1/2} \, exp\left(-\frac{x^2}{2\sigma^2}\right)$$
$$p_1(x) = (2\pi\sigma^2)^{-1/2} \, exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

- **At time t, we receive a signal *x(t)*. Knowing $p_0(x)$ and $p_1(x)$, we can calculate the likelihood ratio**

$$\lambda(x) = \frac{p_1(x)}{p_0(x)}$$

**and compare it to a threshold**

$$\lambda_0 = \frac{P(H_0)}{P(H_1)}$$

**and decide accordingly:**

$$\lambda(x)\begin{cases} \geq \lambda_0, & \text{Choose } H_1 \\ < \lambda_0, & \text{Choose } H_0 \end{cases}$$

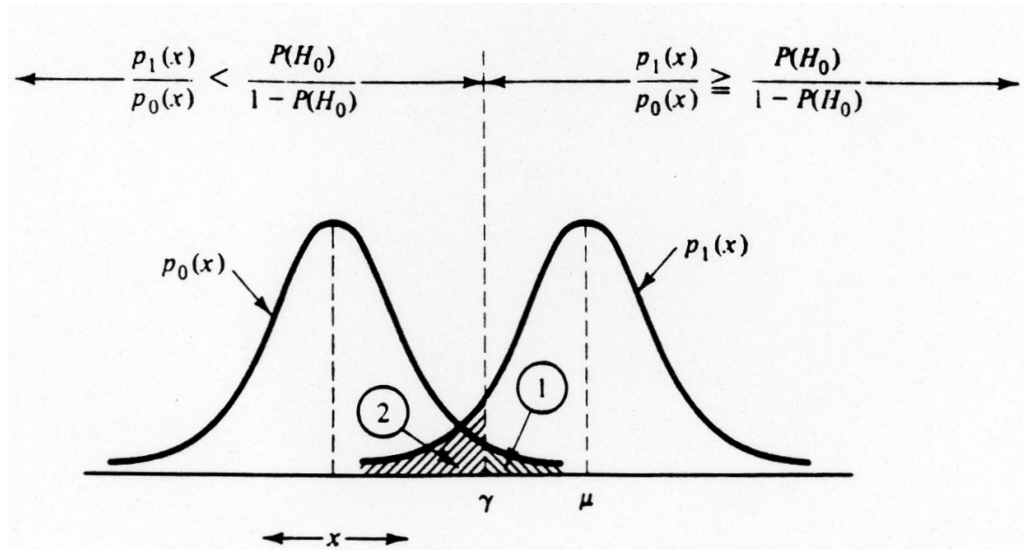
- **Note that $\gamma$ is the value of *x* at which $\lambda(x) = \lambda_0$ in the figure.**

# Errors and Correct Decisions

- **The possible __errors__ are:**
  - **__False Alarm__: We choose $H_1$ when $H_0$ is the right answer.**
  - **__False Dismissal__: We choose $H_0$ when $H_1$ is the right answer.**

- **The possible __correct decisions__ are:**
  - **__Detection__: We choose $H_1$ when it is the right answer.**
  - **__Correct Dismissal__: We choose $H_0$ when it is the right answer.**

# Probabilities of Errors and Correct Decisions




Errors

- $P_{FA} = \int_{\gamma}^{\infty} p_0(x)dx$
- $P_{FD} = \int_{-\infty}^{\gamma} p_1(x)dx$

Correct Decisions

- $P_D = \int_{\gamma}^{\infty} p_1(x)dx$
- $P_{CD} = \int_{-\infty}^{\gamma} p_0(x)dx$

**Note:** $P_{CD} + P_{FA} = 1 = P_{FD} + P_D$ **because** $\int_{-\infty}^{\infty} p(x)dx = 1$

# Neyman-Pearson Criterion

- **Usually we don't know $P(H_1)$ and $P(H_0)$ and thus cannot calculate $\lambda_0$ from their ratio.**
- **Instead, we can specify a desired $P_{FA}$, or false alarm rate, and use it to obtain $\gamma$.**

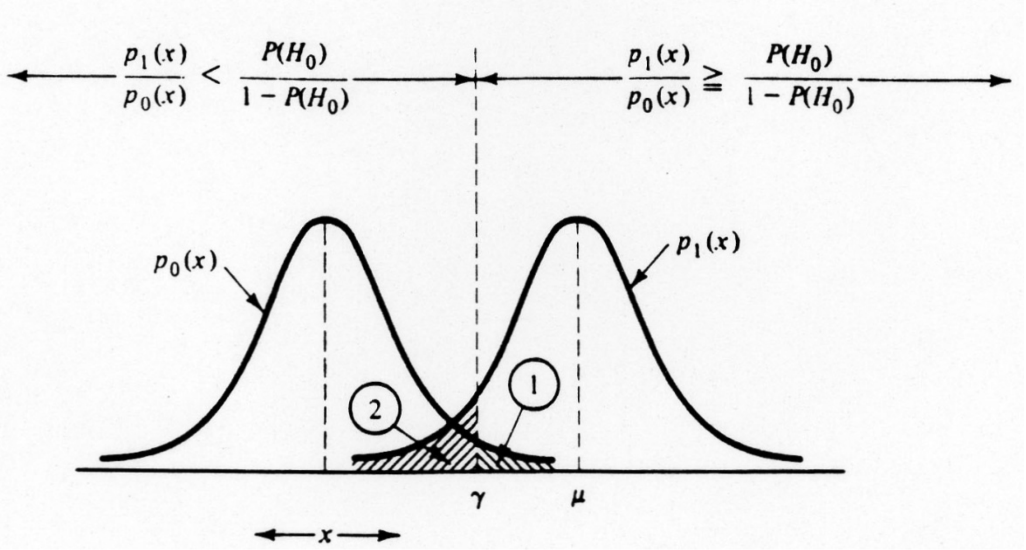
$$P_{FA} = \int_{\gamma}^{\infty} p_0(x)dx \quad = \text{specified false alarm probability}$$

- **Then we can calculate**

$$\lambda_0 = \frac{p_1(\gamma)}{p_0(\gamma)}$$

**or just compare $x$ to $\gamma$ directly.**

# Same Example: Multiple Samples




**For each sample $x_i = x(t_i)$, the probabilities are:**

$$p_0(x_i) = (2\pi\sigma^2)^{-1/2} \exp\left(-\frac{x_i^2}{2\sigma^2}\right), \qquad p_1(x_i) = (2\pi\sigma^2)^{-1/2} \exp\left(-\frac{(x_i - \mu)^2}{2\sigma^2}\right)$$

# Same Example: Multiple Samples (Cont'd)

- If we have a set of M multiple, independent samples,

$$\bar{x} = x_1, x_2, \ldots$$

then their joint probability density functions under $H_1$ and $H_0$ are

$$p_0(\bar{x}) = (2\pi\sigma^2)^{-M/2} \prod_{i=1}^{M} \exp\left(-\frac{x_i^2}{2\sigma^2}\right)$$

$$= (2\pi\sigma^2)^{-M/2} \exp\left(-\sum_{i=1}^{M} \frac{x_i^2}{2\sigma^2}\right)$$

and

$$p_1(\bar{x}) = (2\pi\sigma^2)^{-M/2} \exp\left(-\sum_{i=1}^{M} \frac{(x_i - \mu)^2}{2\sigma^2}\right)$$

# Same Example: Multiple Samples (Cont'd)

- **The likelihood ratio becomes:**

$$\lambda(\bar{x}) = \frac{p_1(\bar{x})}{p_0(\bar{x})} = exp\left(-\sum_{i=1}^{M}\frac{(x_i-\mu)^2 - x_i^2}{2\sigma^2}\right) = exp\left(\frac{\mu M}{\sigma^2}y - \frac{\mu^2 M}{2\sigma^2}\right)$$

**where** $y \equiv \frac{1}{M}\sum_{i=1}^{M} x_i$ **is the mean value of the samples.**

- **Note that each $x_i$ is Gaussian with mean 0 under $H_0$ or $\mu$ under $H_1$. Also, each $x_i$ has variance $\sigma^2$ under both $H_0$ and $H_1$.**

- **Then $y$ is also Gaussian, with the same mean, but with variance** $\frac{\sigma^2}{M}$

# Same Example: Multiple Samples (Cont'd)

- ***y*** **is a detection statistic (i.e. it is a sufficient statistic)**

- **Using the Neyman-Pearson criterion, the probability of a false alarm**

$$P_{FA} = \int_{\gamma}^{\infty} p_0(y)\,dy$$

**can be used to obtain a <u>threshold</u> $\gamma$ for *y*.**

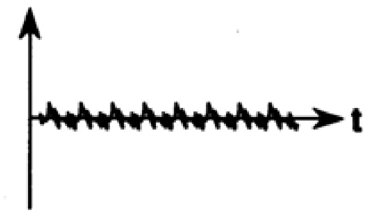
- **Note that using $y \equiv \frac{1}{M}\sum_{i=1}^{M} x_i$ satisfies our intuition that the receiver should counter the effects of noise by averaging the samples.**

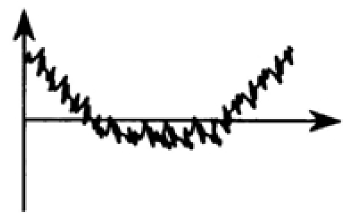
# Second Example: Arbitrary But Known Signal

- **Possible receiver inputs are:**

***$H_0$: x(t) = n(t)*** **(Noise only)**

***$H_1$: x(t) = s(t) + n(t)*** **(Signal plus noise)**



- **If the signal is present, we know its shape exactly.**
- **Assume we have *M* samples $s_i \equiv s(t_i)$ in the interval (0,*T*). The probabilities are:**

$$\text{Under } H_0: \; p_0(\bar{x}) = (2\pi\sigma^2)^{-M/2} \exp\left(-\sum_{i=1}^{M} \frac{x_i^2}{2\sigma^2}\right)$$

$$\text{Under } H_1: \; p_1(\bar{x}) = (2\pi\sigma^2)^{-M/2} \exp\left(-\sum_{i=1}^{M} \frac{(x_i - s_i)^2}{2\sigma^2}\right)$$

- **The likelihood ratio is:**

$$\lambda(\overline{x}) = \frac{p_1(\overline{x})}{p_0(\overline{x})} = exp\left(-\sum_{i=1}^{M}\frac{(x_i - s_i)^2 - x_i^2}{2\sigma^2}\right) = exp\left(\frac{1}{\sigma^2}\sum_{i=1}^{M} x_i s_i - \frac{1}{2\sigma^2}\sum_{i=1}^{M} s_i^2\right)$$

- **The second term can be calculated before receiving the samples.**

- **As we sample more finely in the interval (0,*T*), the summation becomes the integral:**

$$\sum_{i=1}^{M} s_i^2 \rightarrow \int_0^T s^2(t)dt \equiv \mathrm{E}$$

**where E is the energy in the signal.**

- The test statistic in this case is:

$$y(\overline{x}) = \sum_{i=1}^{M} x_i s_i \rightarrow \int_0^T x(t)s(t)dt$$

- Note that the received signal x(t) is being correlated with the signal we are trying to detect *s(t)*.

- Equivalently, we can filter x(t) using a filter with impulse response function

$$h(t)=s(T-\tau)$$

as can be seen from this equation:

$$\int_0^T h(\tau)x(T-\tau)d\tau = \int_0^T s(T-\tau)x(T-\tau)d\tau = \int_0^T s(t)x(t)dt = y$$

- A filter whose impulse response function is matched to the signal in this way is called a matched filter.

# Test Statistic SNR

- **We can define the SNR of *y* to be:**

$$\mathrm{SNR}_y = \frac{\left[E(y_1) - E(y_0)\right]^2}{var(y_0)}$$

- **The expected values of the test statistic *y* under $H_0$ and $H_1$ are**

$$E(y \mid H_0) \equiv \bar{y}_0 = E\left[\int_0^T x(t)s(t)dt\right] = 0$$

$$E(y \mid H_1) \equiv \bar{y}_1 = E\left[\int_0^T (x(t) + n(t))s(t)dt\right] = \mathrm{E}$$

- **The variance of *y* under $H_0$ is (using the shorthand $y_0 \equiv y \mid H_0$ ):**

$$var(y_0) = E\left[(y_0 - \bar{y}_0)^2\right] = E\left[\int_0^T \int_0^T s(t)s(\tau)n(t)n(\tau)dtd\tau\right]$$

- **Let *n(t)* be Gaussian white noise with spectral level $\frac{N_0}{2}$, i.e.:**

$$R_{nn}(\tau) = \frac{N_0}{2}\delta(\tau)$$

- **Then**

$$var(y_0) = \frac{N_0}{2}\int_0^T\int_0^T s(t)s(\tau)\delta(t-\tau)dtd\tau$$

$$= \frac{N_0 \mathrm{E}}{2}$$

- **And so:** $$SNR_y = \frac{2\mathrm{E}}{N_0}$$

- **As long as n(t) is white Gaussian noise (WGN), there is no other receiver, i.e. no other test statistic y, which has a higher SNR. For many other types of noise, the matched filter is optimal or near optimal as well. This is why the matched filter is used.**

# Third Example: Signal Known Except Amplitude and Start Time

- **This is the most common case, in which we are**
  - **- Looking for a target echo**
  - **- Listening for a radiated signal**

- **Exact arrival time and signal amplitude are unknown. The hypotheses are:**

$$H_0: \quad x(t) = n(t) \qquad \text{(Noise only)}$$

$$H_1: \quad x(t) = a \cdot s(t - t_0) + n(t) \qquad (a, t_0 \text{ unknown})$$

- **As before, *T* is the duration of *s(t)***

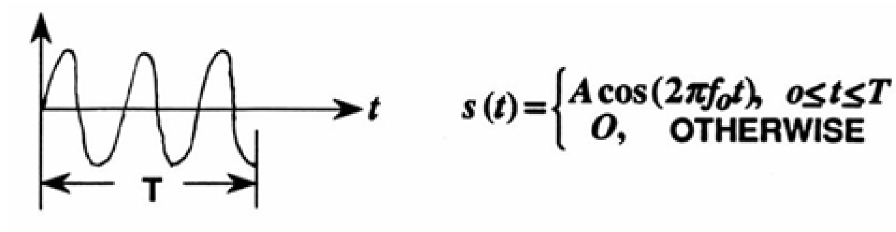
- We apply the signal to a matched filter. under $H_1$, the output is

$$y(t) = \int_0^T h(\tau)x(t-\tau)d\tau$$
$$= \int_0^T s(T-\tau)\left[a \cdot s(t-\tau-t_0) + n(t-\tau)\right] \cdot d\tau$$
$$= \mathrm{a} \cdot \mathrm{R}_s(t-T-t_0) + \int_0^T s(T-\tau)n(t-\tau)d\tau$$

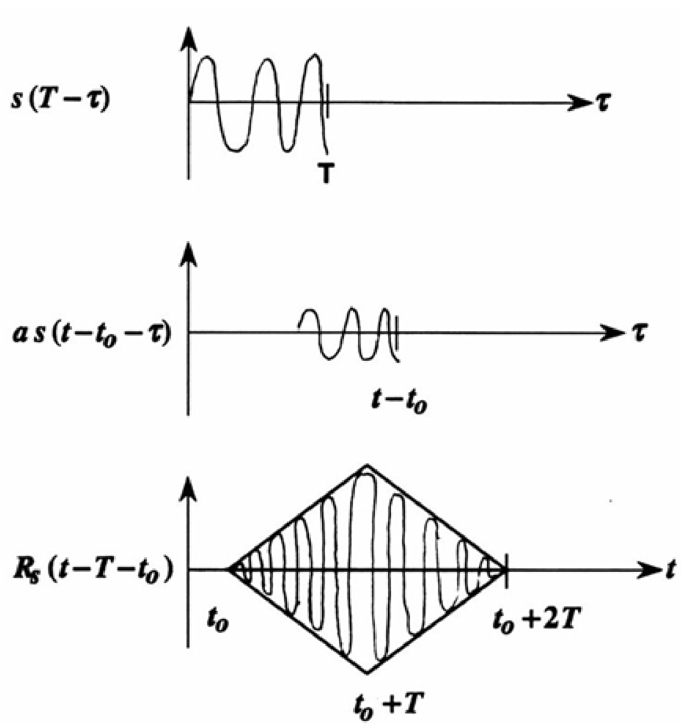
- The first term is the autocorrelation function as *s* at a lag of $t - T - t_0$. It is maximum when $t = T + t_0$, the time corresponding to the end of the pulse arrival

- The second term is random due to the noise.

- **Assume *s*(*t*) is a tone burst:**




- **The autocorrelation function is:**



- **Autocorrelation function is written:**

$$R_s(p) = \begin{cases} \frac{A \cdot a}{2}(T - |p|)\cos(2\pi f_0 t), & -T \le p \le T \\ 0, & \text{otherwise} \end{cases}$$

- **Can get the envelope of Rs(p) by squaring and low-pass filtering**

$$\left[R_s^2(p)\right]_{lpf} = \frac{A^2 a^2}{8}(T - |p|)^2$$

- **This is maximum when** $p = (t-T-t_0) = 0$ **or** $t=T+t_0$.
- **Thus the peak in** $[y^2(t)]_{lpf}$ **occurs at** $t= T+t_0$ **, and since we know** $T$**, can get** $t_0$

- **Therefore, we define a new test statistic *Z*(*t*):**

$$Z(t) \equiv \left[y^2(t)\right]_{lpf}$$

- **The probability density functions of *Z(t)* under $H_0$ and $H_1$ are shown by Burdic to be:**

$$p_0(z) = \frac{1}{\sigma_y^2} exp\left(-\frac{z}{\sigma_y^2}\right), \quad \sigma_y^2 = \frac{\mathrm{E}N_0}{2}$$

$$p_1(z) = \frac{1}{\sigma_y^2} exp\left(-S-\frac{z}{\sigma_y^2}\right) I_0\left[z\left(\frac{zS}{\sigma_y^2}\right)^{1/2}\right]$$

- **Where $S \equiv \mathrm{SNR}_y$ and $I_0(\circ)$ is the zero-order modified Bessel function.**

- **The probability density functions are plotted below**

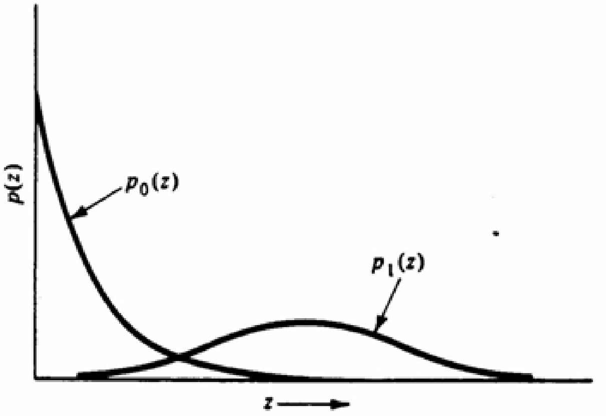


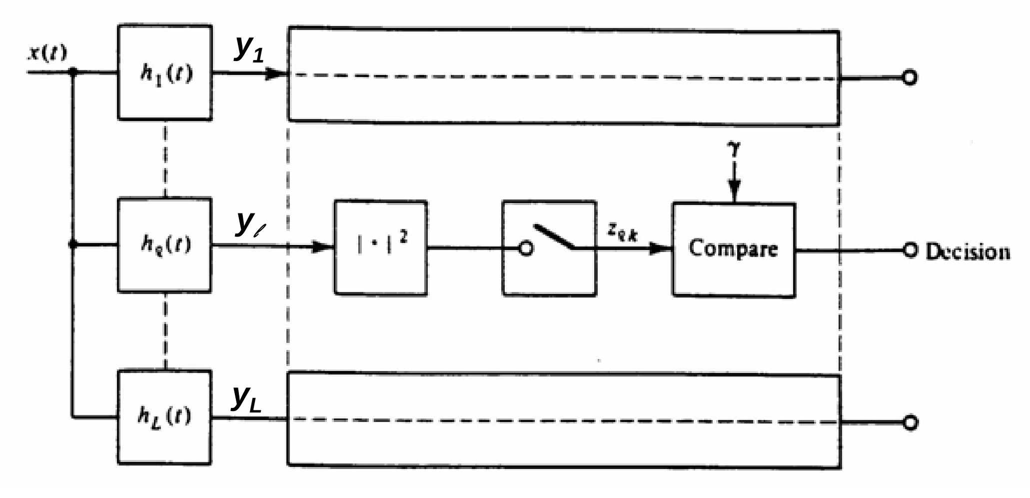

**Figure 13-4** Probability density functions at output of a squared-magnitude envelope detector for noise only and signal plus noise.

- **Can use the Neyman-Pearson criterion to get $\gamma$, then calculate $P_D$**

$$P_{FA} = \int_{\gamma}^{\infty} p_0(z)\,dz$$

# Fourth Example: Possible Doppler Shift

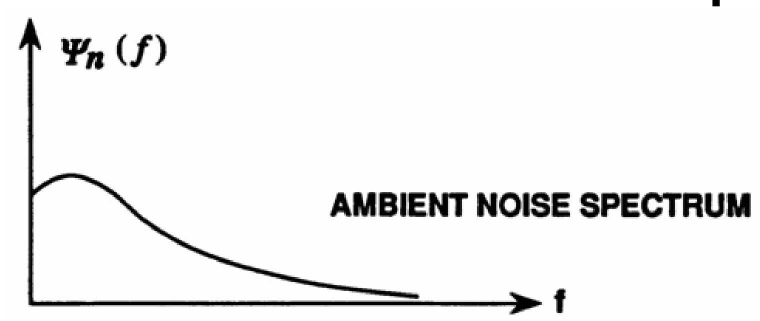
- **Non-zero radial motion between a transmitter (or reflector) and receiver causes the frequency of the received signal to be shifted relative to the transmitted signal. This is called <u>Doppler Shift</u>.**
- **This complication is usually met by implementing a parallel bank of filters (or FFT), each matched to a different frequency.**



# Passive Broadband Detection

**Want to detect targets with broadband signatures:**


**Assume we know the ambient noise power spectrum**



# Passive Broadband Detection (Cont'd)

- Use the receiver shown below, where $h_1(t)$ and $h_2(t)$ are filters whose impulse functions need to be determined.

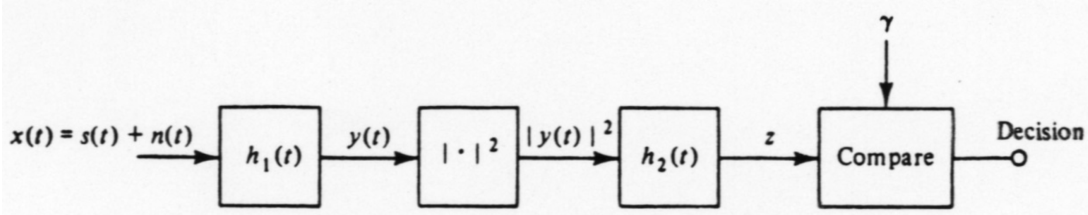



**Figure 13-8** Passive broadband receiver mechanization.

# Passive Broadband Detection (cont.)

- **It has been shown that the Eckart Filter is optimal for $h_1(t)$:**

$$\left|H_1(f)\right|^2 = \frac{\psi_s(f)}{\psi_n^2(f)} \qquad \textbf{Eckart Filter}$$

- **Note: when the noise is white, $H_1(f)$ looks like $\Psi_s(f)$. Otherwise, $H_1(f)$ is minimized when $\Psi_n(f)$ is large**

- **The power spectrum of $y$ under $H_0$ and $H_1$ is then:**

$$\Psi_{y_0}(f) = \Psi_n(f)\left|H_1(f)\right|^2 = \frac{\Psi_s(f)}{\Psi_n(f)}$$

$$\Psi_{y_1}(f) = \left(\Psi_s(f) + \Psi_n(f)\right)\left|H_1(f)\right|^2 = \frac{\Psi_s^2(f)}{\Psi_n^2(f)} + \frac{\Psi_s(f)}{\Psi_n(f)}$$

***and***

$$\mathrm{SNR}_y = \frac{\int_{\infty}^{\infty}\left[\Psi_{y_1}(f) - \Psi_{y_0}(f)\right]df}{\int_{\infty}^{\infty}\Psi_{y_0}(f)\,df} = \frac{\int_{-\infty}^{\infty}\frac{\Psi_s^2(f)}{\Psi_n^2(f)}df}{\int_{-\infty}^{\infty}\frac{\Psi_s(f)}{\Psi_n(f)}df}$$

# Passive Broadband Detection (cont.)

- **Burdic shows that the SNR of the output of the envelope detector is**

$$\mathrm{SNR}_{|y|^2} = \mathrm{SNR}_y^2$$

- **The commonly-used post detection filter is an averager whose duration is as long as possible,**

$$h_2(t) = \begin{cases} \frac{1}{T}, & -\frac{T}{2} \le \tau \le \frac{T}{2} \\ 0, & \text{otherwise} \end{cases}$$

- **The product of $T\beta_\varepsilon$ is typically large, where $\beta_\varepsilon$ is the effective noise bandwidth at the output of the pre-detection filter $h_1(\tau)$, i.e. $\beta_\varepsilon$ is the width of a rectangular filter which admits the same noise power. The frequency domain expression for $\beta_\varepsilon$ is derived by Burdic in section 8-4 to be**

$$\beta_\varepsilon = \frac{\left[\int \Psi_n(f)\left|H_1(f)\right|^2 df\right]^2}{\int \Psi_n^2(f)\left|H_1(f)\right|^4 df}$$

# Passive Broadband Detection (cont.)

- **Using the Eckert Filter**

$$\beta_\varepsilon = \frac{\left[\int \frac{\Psi_s(f)}{\Psi_n(f)} df\right]^2}{\int \frac{\Psi_s^2(f)}{\Psi_n^2(f)} df}$$

- **Given large $T\beta_\varepsilon$, Burdic shows that the SNR at the averager output is**

$$\text{SNR}_z = T\beta_\varepsilon \text{SNR}_{|y|^2} = T\beta_\varepsilon \text{SNR}_y^2$$

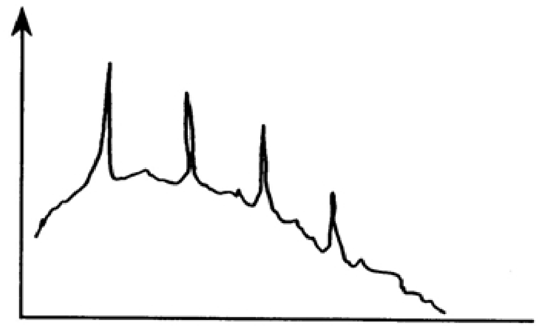
- **Using the expressions for $\text{SNR}_y$ and $\beta_\varepsilon$**

$$\text{SNR}_z = T \frac{\left[\int \frac{\Psi_s(f)}{\Psi_n(f)} df\right]^2}{\int \frac{\Psi_s^2(f)}{\Psi_n^2(f)} df} \frac{\left[\int \frac{\Psi_s^2(f)}{\Psi_n^2(f)} df\right]^2}{\left[\int \frac{\Psi_s(f)}{\Psi_n(f)} df\right]^2} = T \int \frac{\Psi_s^2(f)}{\Psi_n^2(f)} df$$

- **Note the effect on $\text{SNR}_z$ of increasing T.**

# Passive Narrowband Detection

- **Want to detect targets that emit pure tone signatures:**



- **Receiver is shown below (essentially a spectrum analyzer)**

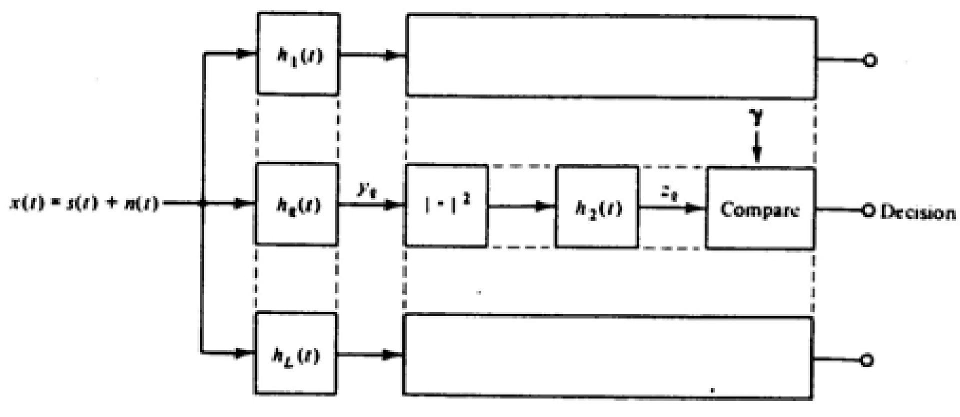



**Figure 13-9** Passive narrowband receiver mechanization.

# Passive Narrowband Detection (Cont'd)

- **Typically implemented by Fournier transforming the input signal. Second filter is an integrator ("averager"). Long averages are usually employed, so that $T\beta$ >>1.**

***If :***

**Signal Spectrum :** $\Psi_s(f) = a^2\delta(f - f_0)$

**Filter :**

$$|H_1(f)|^2 = \begin{cases} 1, & -\frac{\beta}{2} \le f - f_0 \le \frac{\beta}{2} \\ 0, & \text{otherwise} \end{cases}$$

**Noise Spectrum :** $\Psi_n(f) \approx$ **Constant around** $f_0$

# Passive Narrowband Detection (Cont'd)

- **Then**

$$\mathrm{SNR}_y = \frac{a^2}{\psi_n(f_0)\cdot\beta}$$

- **As before, the SNR of the test statistic *Z* is**

$$\mathrm{SNR}_z = T\beta\,\mathrm{SNR}_y^2$$

- **Putting these together**

$$\mathrm{SNR}_z = \frac{T}{\beta}\left(\frac{a^2}{\psi_n(f_0)}\right)^2$$