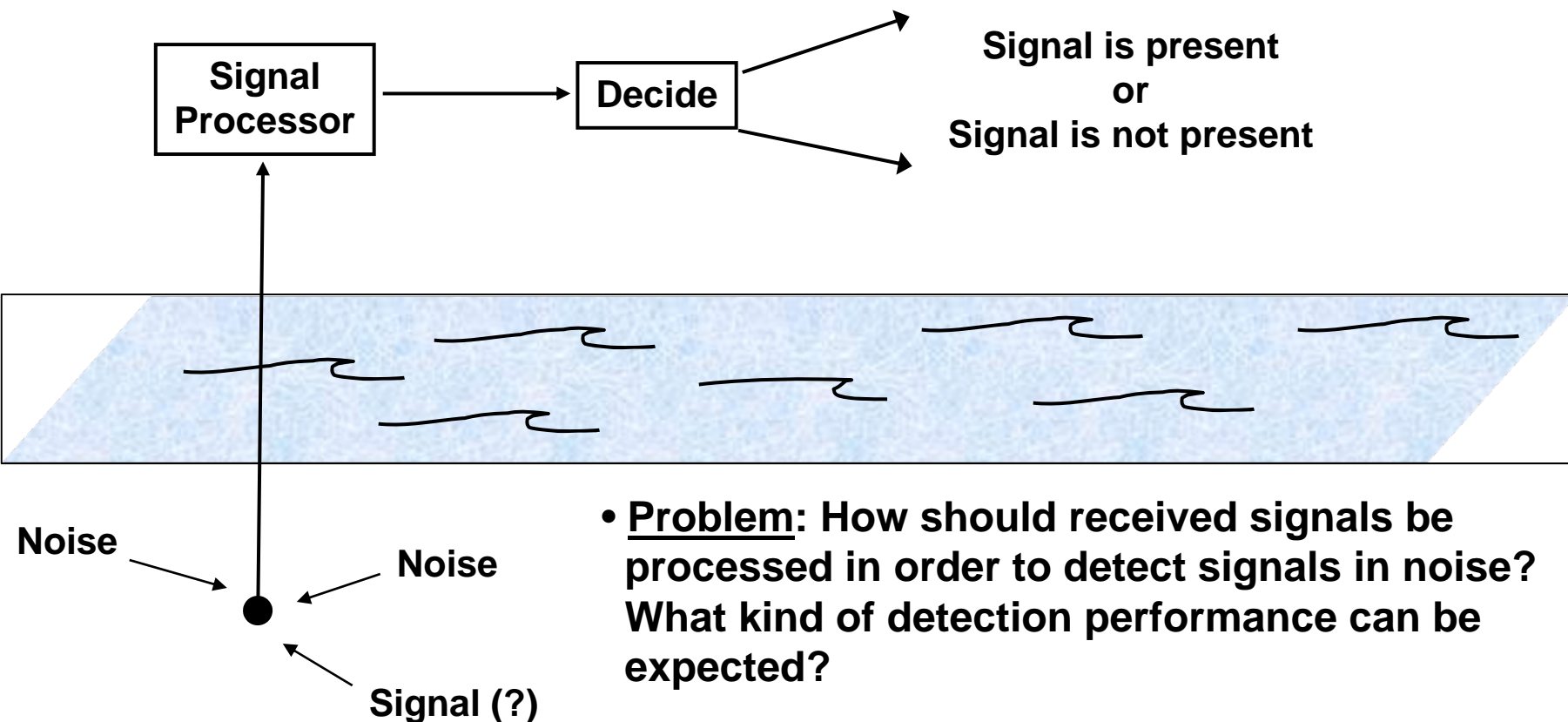




Decision Theory and Receiver Design



Signal Detection and Performance Estimation



- **Problem:** How should received signals be processed in order to detect signals in noise? What kind of detection performance can be expected?

- **The approach to solution:**
 - Must be statistical, since noise is involved
 - Implement hypothesis testing

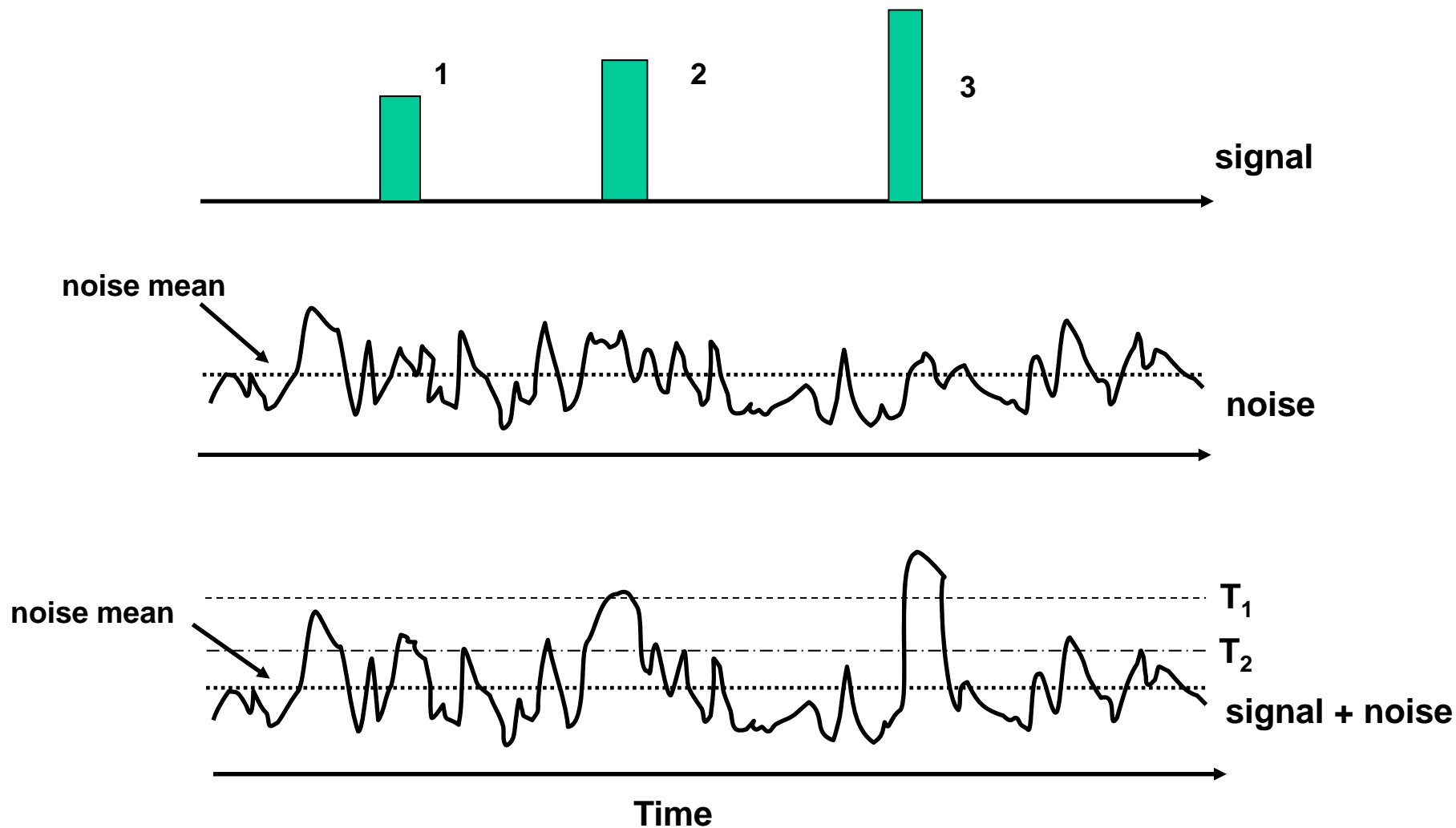


Signal Detection

- **Input to detector is signal plus noise.**
- **Requirements expressed in terms of**
 - probability of detection
 - probability of false alarm
- **Threshold for declaring detection is set based on models for signal and noise**
- **Noise background estimation can be performed on data to improve model.**
- **Outputs of detector are threshold crossings**
- **Performance defined by receiver operating characteristic (ROC) curve – probability of detection vs. probability of false alarm for a particular SNR.**



Detection In Noise



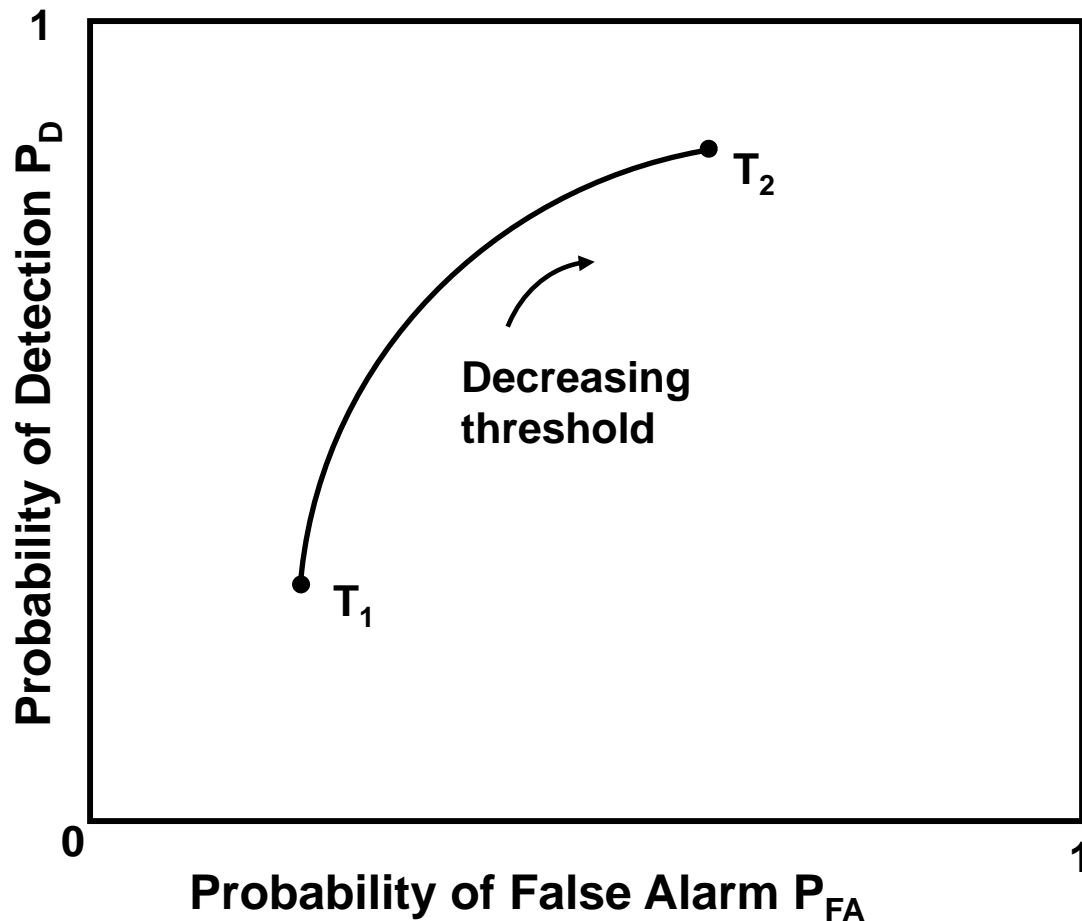


Detection Threshold

- **Performance Criteria:**
 - Probability of detection P_D
 - Probability of false alarm P_{FA}
- **These criteria are not independent: a lower threshold increases P_D , but also increases P_{FA} .**
- **Theoretical ROC is used to set thresholds.**
- **True test is performance in water.**



Receiver Operating Curve (ROC)





Hypothesis Testing

- **Possible Hypotheses:**
 - H_0 : Only noise is present
 - H_1 : Signal is present in addition to noise
- **Steps in forming hypotheses:**
 - Process array output to obtain a detection statistic x .
 - Calculate the a posteriori probabilities $P(H_0/x)$ and $P(H_1/x)$.
 - Pick the hypothesis whose probability is the highest: the maximum a posteriori, or MAP estimate.

$$\frac{P(H_1/x)}{P(H_0/x)} \begin{cases} \geq 1, & \text{Choose } H_1 \\ < 1, & \text{Choose } H_0 \end{cases}$$



Hypothesis Testing (Cont'd)

- Equivalently, we can use Bayes' rule to write:

$$P(H_1 | x)P(x) = P(x | H_1)P(H_1)$$

$$P(H_0 | x)P(x) = P(x | H_0)P(H_0)$$

- $P(H_1)$ and $P(H_0)$ are called *a priori* probabilities
- Then the test can be written:

$$\frac{P(H_1 | x)}{P(H_0 | x)} = \frac{P(x | H_1)P(H_1)}{P(x | H_0)P(H_0)} \begin{cases} \geq 1, & \text{Choose } H_1 \\ < 1, & \text{Choose } H_0 \end{cases}$$



Hypothesis Testing (Cont'd)

- An equivalent test is

$$\frac{P(x | H_1)}{P(x | H_0)} \begin{cases} \leq \frac{P(H_0)}{P(H_1)}, & \text{Choose } H_1 \\ > \frac{P(H_0)}{P(H_1)}, & \text{Choose } H_0 \end{cases}$$

- $\frac{P(x | H_1)}{P(x | H_0)} \equiv \lambda(x)$ is called the likelihood ratio



Aside: Bayes' Rule and Notation

- Probability density functions are often used to describe continuous random variables:

$$P(x_0) \equiv \int_{-\infty}^{x_0} p(x) dx$$

- Bayes' Rule as written for probabilities also holds for probability density functions (pdf).
- A compact notation is used in what follows:

$$p(x | H_1) \equiv p_1(x) \qquad p(x | H_0) \equiv p_0(x)$$

- Likelihood ratio test written in terms of pr

$$\frac{p_1(x)}{p_0(x)} \begin{cases} \leq \frac{P(H_0)}{P(H_1)}, & \text{Choose } H_1 \\ > \frac{P(H_0)}{P(H_1)}, & \text{Choose } H_0 \end{cases}$$



A First Example: Constant Signal

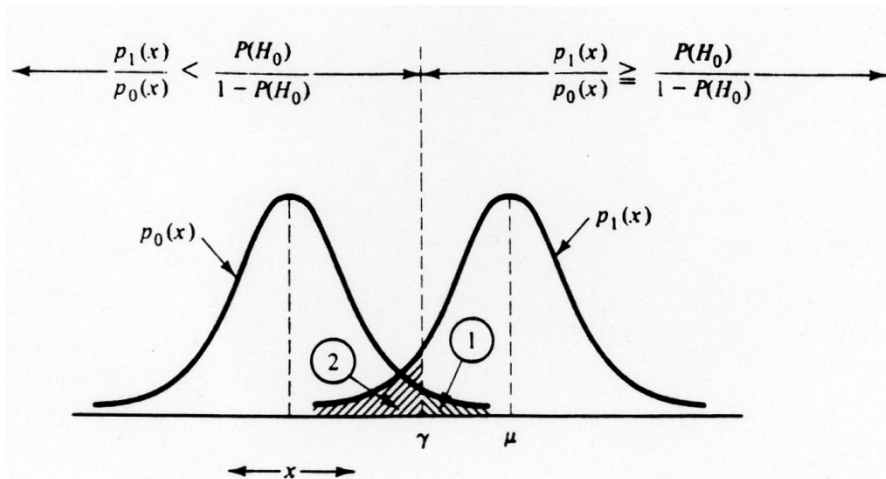
- The possible inputs are:

$$H_0 : x(t) = n(t) \quad \text{(Noise only)}$$

$$H_1 : x(t) = \mu + n(t) \quad \text{(Signal plus noise)}$$

- If $n(t)$ is Gaussian distributed and $\mu \neq 0$, then

$p(x | H_1) = p_1(x)$ and $p(x | H_0) = p_0(x)$ are as shown below :



$$p_0(x) = (2\pi\sigma^2)^{-1/2} \exp\left(-\frac{x^2}{2\sigma^2}\right)$$

$$p_1(x) = (2\pi\sigma^2)^{-1/2} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$



- At time t , we receive a signal $x(t)$. Knowing $p_0(x)$ and $p_1(x)$, we can calculate the likelihood ratio

$$\lambda(x) = \frac{p_1(x)}{p_0(x)}$$

and compare it to a threshold

$$\lambda_0 = \frac{P(H_0)}{P(H_1)}$$

and decide accordingly:

$$\lambda(x) \begin{cases} \geq \lambda_0, & \text{Choose } H_1 \\ < \lambda_0, & \text{Choose } H_0 \end{cases}$$

- Note that γ is the value of x at which $\lambda(x) = \lambda_0$ in the figure.



Errors and Correct Decisions

- The possible errors are:
 - False Alarm: We choose H_1 when H_0 is the right answer.
 - False Dismissal: We choose H_0 when H_1 is the right answer.

- The possible correct decisions are:
 - Detection: We choose H_1 when it is the right answer.
 - Correct Dismissal: We choose H_0 when it is the right answer.



Probabilities of Errors and Correct Decisions

- $P_{FA} = \int_{\gamma}^{\infty} p_0(x) dx$

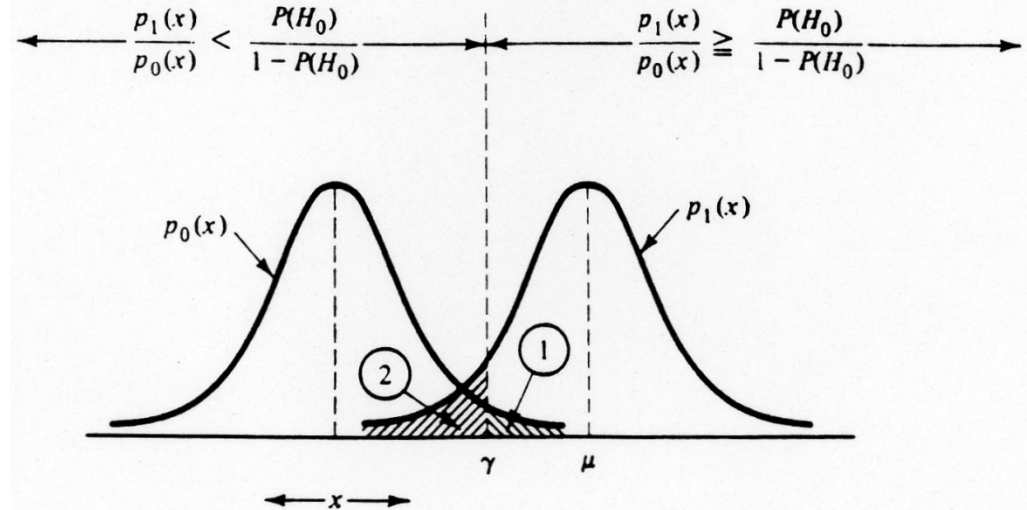
- $P_{FD} = \int_{-\infty}^{\gamma} p_1(x) dx$

Errors

- $P_D = \int_{\gamma}^{\infty} p_1(x) dx$

- $P_{CD} = \int_{-\infty}^{\gamma} p_0(x) dx$

Correct Decisions



Note: $P_{CD} + P_{FA} = 1 = P_{FD} + P_D$ because $\int_{-\infty}^{\infty} p(x) dx = 1$



Neyman-Pearson Criterion

- Usually we don't know $P(H_1)$ and $P(H_0)$ and thus cannot calculate λ_0 from their ratio.
- Instead, we can specify a desired P_{FA} , or false alarm rate, and use it to obtain γ .

$$P_{FA} = \int_{\gamma}^{\infty} p_0(x) dx = \text{specified false alarm probability}$$

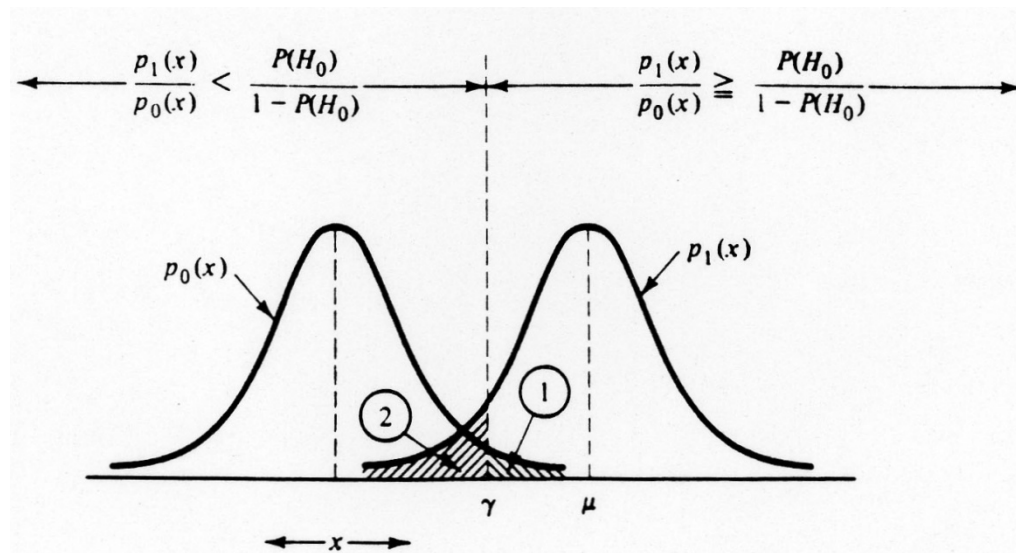
- Then we can calculate

$$\lambda_0 = \frac{p_1(\gamma)}{p_0(\gamma)}$$

or just compare x to γ directly.



Same Example: Multiple Samples



For each sample $x_i = x(t_i)$, the probabilities are:

$$p_0(x_i) = (2\pi\sigma^2)^{-1/2} \exp\left(-\frac{x_i^2}{2\sigma^2}\right), \quad p_1(x_i) = (2\pi\sigma^2)^{-1/2} \exp\left(-\frac{(x_i - \mu)^2}{2\sigma^2}\right)$$



Same Example: Multiple Samples (Cont'd)

- If we have a set of M multiple, independent samples,

$$\bar{\mathbf{X}} = \mathbf{x}_1, \mathbf{x}_2, \dots$$

then their joint probability density functions under H_1 and H_0 are

$$\begin{aligned} p_0(\bar{\mathbf{x}}) &= (2\pi\sigma^2)^{-M/2} \prod_{i=1}^M \exp\left(-\frac{\mathbf{x}_i^2}{2\sigma^2}\right) \\ &= (2\pi\sigma^2)^{-M/2} \exp\left(-\sum_{i=1}^M \frac{\mathbf{x}_i^2}{2\sigma^2}\right) \end{aligned}$$

and

$$p_1(\bar{\mathbf{x}}) = (2\pi\sigma^2)^{-M/2} \exp\left(-\sum_{i=1}^M \frac{(\mathbf{x}_i - \mu)^2}{2\sigma^2}\right)$$



Same Example: Multiple Samples (Cont'd)

- The likelihood ratio becomes:

$$\lambda(\bar{x}) = \frac{p_1(\bar{x})}{p_0(\bar{x})} = \exp\left(-\sum_{i=1}^M \frac{(x_i - \mu)^2 - x_i^2}{2\sigma^2}\right) = \exp\left(\frac{\mu M}{\sigma^2} y - \frac{\mu^2 M}{2\sigma^2}\right)$$

where $y \equiv \frac{1}{M} \sum_{i=1}^M x_i$ is the mean value of the samples.

- Note that each x_i is Gaussian with mean 0 under H_0 or μ under H_1 . Also, each x_i has variance σ^2 under both H_0 and H_1 .
- Then y is also Gaussian, with the same mean, but with variance $\frac{\sigma^2}{M}$



Same Example: Multiple Samples (Cont'd)

- y is a detection statistic (i.e. it is a sufficient statistic)
- Using the Neyman-Pearson criterion, the probability of a false alarm

$$P_{FA} = \int_{\gamma}^{\infty} p_0(y) dy$$

can be used to obtain a threshold γ for y .

- Note that using $y \equiv \frac{1}{M} \sum_{i=1}^M x_i$ satisfies our intuition that the receiver should counter the effects of noise by averaging the samples.

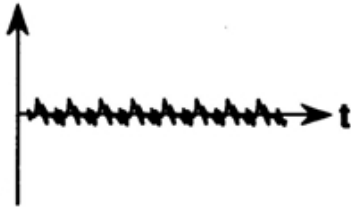


Second Example: Arbitrary But Known Signal

- Possible receiver inputs are:

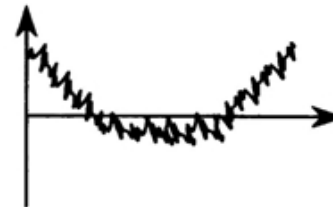
$$H_0: x(t) = n(t)$$

(Noise only)



$$H_1: x(t) = s(t) + n(t)$$

(Signal plus noise)



- If the signal is present, we know its shape exactly.
- Assume we have M samples $s_i \equiv s(t_i)$ in the interval $(0, T)$. The probabilities are:

$$\text{Under } H_0: p_0(\bar{x}) = (2\pi\sigma^2)^{-M/2} \exp\left(-\sum_{i=1}^M \frac{x_i^2}{2\sigma^2}\right)$$

$$\text{Under } H_1: p_1(\bar{x}) = (2\pi\sigma^2)^{-M/2} \exp\left(-\sum_{i=1}^M \frac{(x_i - s_i)^2}{2\sigma^2}\right)$$



- The likelihood ratio is:

$$\lambda(\bar{x}) = \frac{p_1(\bar{x})}{p_0(\bar{x})} = \exp\left(-\sum_{i=1}^M \frac{(x_i - s_i)^2 - x_i^2}{2\sigma^2}\right) = \exp\left(\frac{1}{\sigma^2} \sum_{i=1}^M x_i s_i - \frac{1}{2\sigma^2} \sum_{i=1}^M s_i^2\right)$$

- The second term can be calculated before receiving the samples.
- As we sample more finely in the interval $(0, T)$, the summation becomes the integral:

$$\sum_{i=1}^M s_i^2 \rightarrow \int_0^T s^2(t) dt \equiv E$$

where E is the energy in the signal.



- The test statistic in this case is:

$$y(\bar{x}) = \sum_{i=1}^M x_i s_i \rightarrow \int_0^T x(t) s(t) dt$$

- Note that the received signal $x(t)$ is being correlated with the signal we are trying to detect $s(t)$.
- Equivalently, we can filter $x(t)$ using a filter with impulse response function

$$h(t) = s(T - t)$$

as can be seen from this equation:

$$\int_0^T h(\tau) x(T - \tau) d\tau = \int_0^T s(T - \tau) x(T - \tau) d\tau = \int_0^T s(t) x(t) dt = y$$

- A filter whose impulse response function is matched to the signal in this way is called a matched filter.



Test Statistic SNR

- We can define the SNR of y to be:

$$\text{SNR}_y = \frac{[E(y_1) - E(y_0)]^2}{\text{var}(y_0)}$$

- The expected values of the test statistic y under H_0 and H_1 are

$$E(y | H_0) \equiv \bar{y}_0 = E \left[\int_0^T x(t)s(t)dt \right] = 0$$

$$E(y | H_1) \equiv \bar{y}_1 = E \left[\int_0^T (x(t) + n(t))s(t)dt \right] = E$$

- The variance of y under H_0 is (using the shorthand $y_0 \equiv y | H_0$):

$$\text{var}(y_0) = E[(y_0 - \bar{y}_0)^2] = E \left[\int_0^T \int_0^T s(t)s(\tau)n(t)n(\tau)dtd\tau \right]$$



- Let $n(t)$ be Gaussian white noise with spectral level $\frac{N_0}{2}$, i.e.:

$$R_{nn}(\tau) = \frac{N_0}{2} \delta(\tau)$$

- Then

$$\begin{aligned} \text{var}(y_0) &= \frac{N_0}{2} \int_0^T \int_0^T s(t)s(\tau)\delta(t-\tau)dt d\tau \\ &= \frac{N_0 E}{2} \end{aligned}$$

- And so: $SNR_y = \frac{2E}{N_0}$

- As long as $n(t)$ is white Gaussian noise (WGN), there is no other receiver, i.e. no other test statistic y , which has a higher SNR. For many other types of noise, the matched filter is optimal or near optimal as well. This is why the matched filter is used.



Third Example: Signal Known Except Amplitude and Start Time

- This is the most common case, in which we are
 - Looking for a target echo
 - Listening for a radiated signal
- Exact arrival time and signal amplitude are unknown. The hypotheses are:

$$H_0 : x(t) = n(t) \quad (\text{Noise only})$$

$$H_1 : x(t) = a \cdot s(t - t_0) + n(t) \quad (a, t_0 \text{ unknown})$$

- As before, T is the duration of $s(t)$



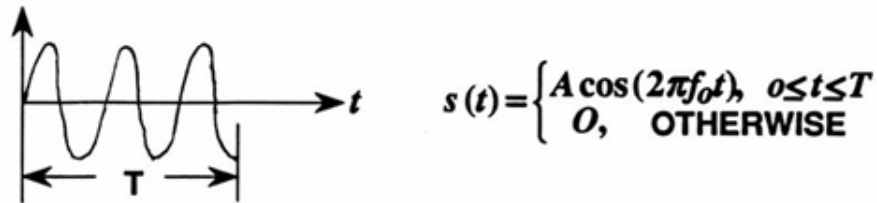
- We apply the signal to a matched filter. under H_1 , the output is

$$\begin{aligned}y(t) &= \int_0^T h(\tau) x(t - \tau) d\tau \\ &= \int_0^T s(T - \tau) [a \cdot s(t - \tau - t_0) + n(t - \tau)] \cdot d\tau \\ &= a \cdot R_s(t - T - t_0) + \int_0^T s(T - \tau) n(t - \tau) d\tau\end{aligned}$$

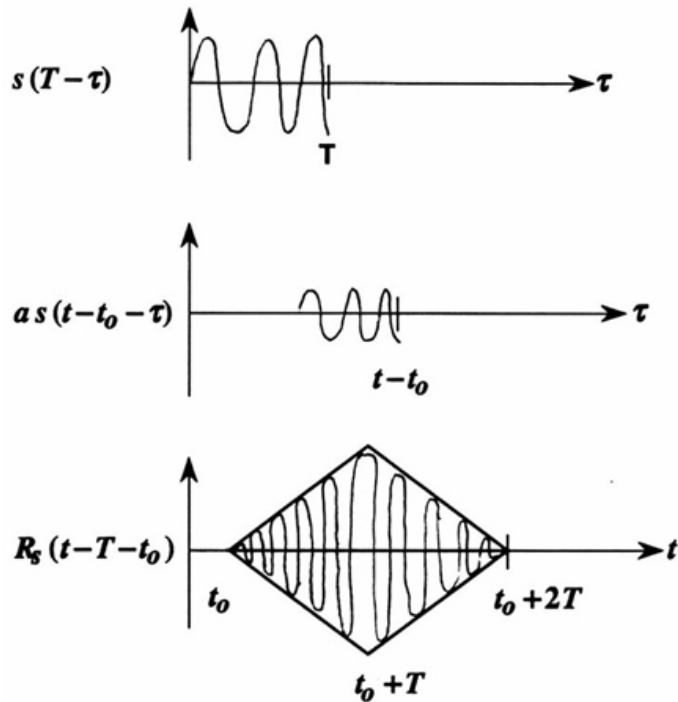
- The first term is the autocorrelation function as s at a lag of $t - T - t_0$. It is maximum when $t = T + t_0$, the time corresponding to the end of the pulse arrival
- The second term is random due to the noise.



- Assume $s(t)$ is a tone burst:



- The autocorrelation function is:





- Autocorrelation function is written:

$$R_s(p) = \begin{cases} \frac{A \cdot a}{2} (T - |p|) \cos(2\pi f_0 t), & -T \leq p \leq T \\ 0, & \text{otherwise} \end{cases}$$

- Can get the envelope of $R_s(p)$ by squaring and low-pass filtering

$$[R_s^2(p)]_{lpf} = \frac{A^2 a^2}{8} (T - |p|)^2$$

- This is maximum when $p = (t - T - t_0) = 0$ or $t = T + t_0$.
- Thus the peak in $[y^2(t)]_{lpf}$ occurs at $t = T + t_0$, and since we know T , can get t_0



- Therefore, we define a new test statistic $Z(t)$:

$$Z(t) \equiv [y^2(t)]_{lpf}$$

- The probability density functions of $Z(t)$ under H_0 and H_1 are shown by Burdick to be:

$$p_0(z) = \frac{1}{\sigma_y^2} \exp\left(-\frac{z}{\sigma_y^2}\right), \quad \sigma_y^2 = \frac{EN_0}{2}$$

$$p_1(z) = \frac{1}{\sigma_y^2} \exp\left(-S - \frac{z}{\sigma_y^2}\right) I_0\left[z\left(\frac{zS}{\sigma_y^2}\right)^{1/2}\right]$$

- Where $S \equiv \text{SNR}_y$ and $I_0(\circ)$ is the zero-order modified Bessel function.



- The probability density functions are plotted below

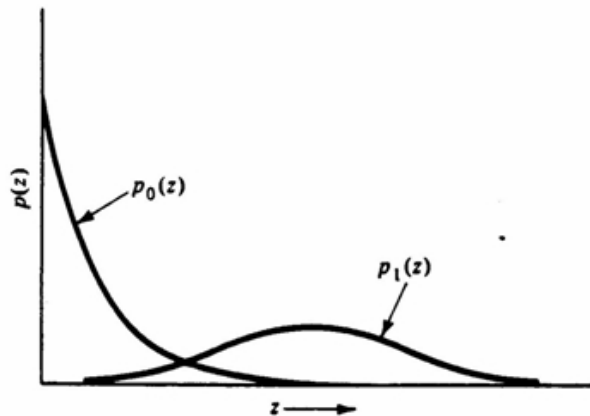


Figure 13-4 Probability density functions at output of a squared-magnitude envelope detector for noise only and signal plus noise.

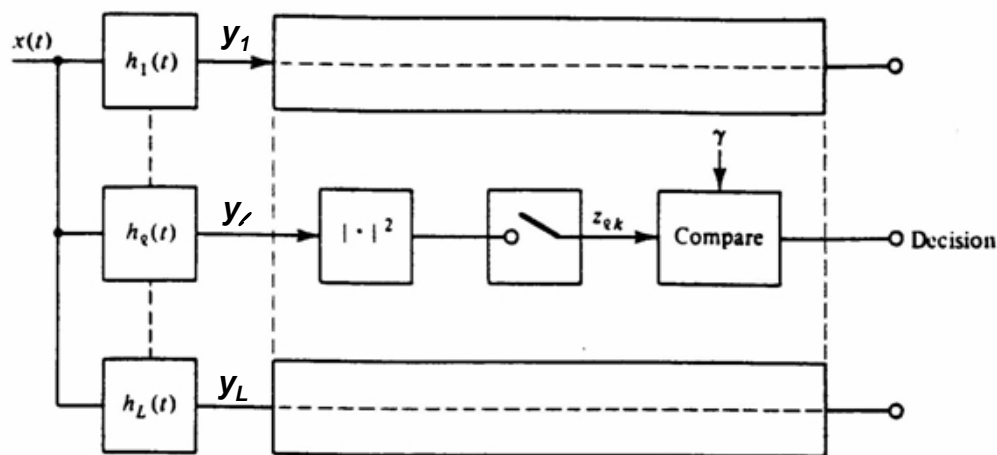
- Can use the Neyman-Pearson criterion to get γ , then calculate P_D

$$P_{FA} = \int_{\gamma}^{\infty} p_0(z) dz$$



Fourth Example: Possible Doppler Shift

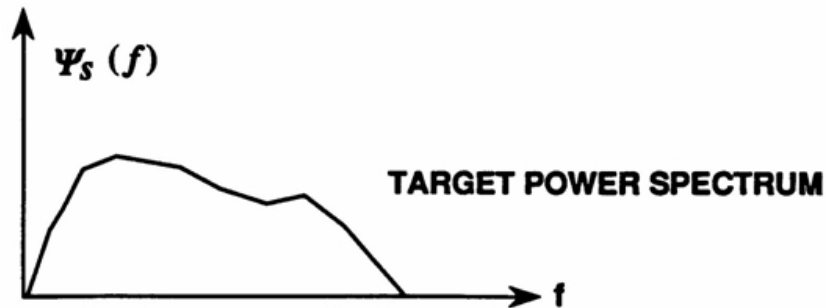
- Non-zero radial motion between a transmitter (or reflector) and receiver causes the frequency of the received signal to be shifted relative to the transmitted signal. This is called Doppler Shift.
- This complication is usually met by implementing a parallel bank of filters (or FFT), each matched to a different frequency.



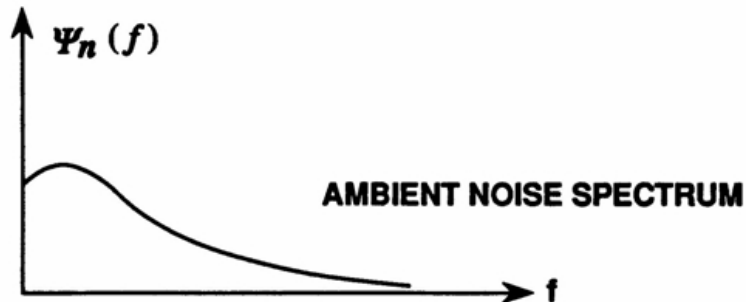


Passive Broadband Detection

Want to detect targets with broadband signatures:



Assume we know the ambient noise power spectrum





Passive Broadband Detection (Cont'd)

- Use the receiver shown below, where $h_1(t)$ and $h_2(t)$ are filters whose impulse functions need to be determined.

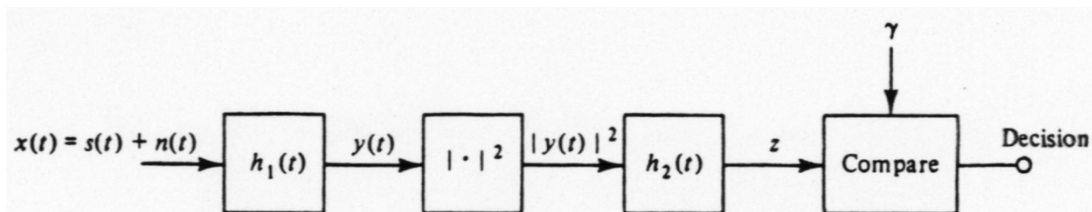


Figure 13-8 Passive broadband receiver mechanization.



Passive Broadband Detection (cont.)

- It has been shown that the Eckart Filter is optimal for $h_1(f)$:

$$|H_1(f)|^2 = \frac{\psi_s(f)}{\psi_n^2(f)} \quad \text{Eckart Filter}$$

- Note: when the noise is white, $H_1(f)$ looks like $\Psi_s(f)$. Otherwise, $H_1(f)$ is minimized when $\Psi_n(f)$ is large
- The power spectrum of y under H_0 and H_1 is then:

$$\Psi_{y_0}(f) = \Psi_n(f) |H_1(f)|^2 = \frac{\Psi_s(f)}{\Psi_n(f)}$$

$$\Psi_{y_1}(f) = (\Psi_s(f) + \Psi_n(f)) |H_1(f)|^2 = \frac{\Psi_s^2(f)}{\Psi_n^2(f)} + \frac{\Psi_s(f)}{\Psi_n(f)}$$

and

$$\text{SNR}_y = \frac{\int_{-\infty}^{\infty} [\Psi_{y_1}(f) - \Psi_{y_0}(f)] df}{\int_{-\infty}^{\infty} \Psi_{y_0}(f) df} = \frac{\int_{-\infty}^{\infty} \frac{\Psi_s^2(f)}{\Psi_n^2(f)} df}{\int_{-\infty}^{\infty} \frac{\Psi_s(f)}{\Psi_n(f)} df}$$



Passive Broadband Detection (cont.)

- Burdick shows that the SNR of the output of the envelope detector is

$$\text{SNR}_{|y|^2} = \text{SNR}_y^2$$

- The commonly-used post detection filter is an averager whose duration is as long as possible,

$$h_2(t) = \begin{cases} \frac{1}{T}, & -\frac{T}{2} \leq \tau \leq \frac{T}{2} \\ 0, & \text{otherwise} \end{cases}$$

- The product of $T\beta_\epsilon$ is typically large, where β_ϵ is the effective noise bandwidth at the output of the pre-detection filter $h_1(\tau)$, i.e. β_ϵ is the width of a rectangular filter which admits the same noise power. The frequency domain expression for β_ϵ is derived by Burdick in section 8-4 to be

$$\beta_\epsilon = \frac{\left[\int \Psi_n(f) |H_1(f)|^2 df \right]^2}{\int \Psi_n^2(f) |H_1(f)|^4 df}$$



Passive Broadband Detection (cont.)

- Using the Eckert Filter

$$\beta_\epsilon = \frac{\left[\int \frac{\Psi_s(f)}{\Psi_n(f)} df \right]^2}{\int \frac{\Psi_s^2(f)}{\Psi_n^2(f)} df}$$

- Given large $T\beta_\epsilon$, Burdick shows that the SNR at the averager output is

$$\text{SNR}_z = T\beta_\epsilon \text{SNR}_{|y|^2} = T\beta_\epsilon \text{SNR}_y^2$$

- Using the expressions for SNR_y and β_ϵ

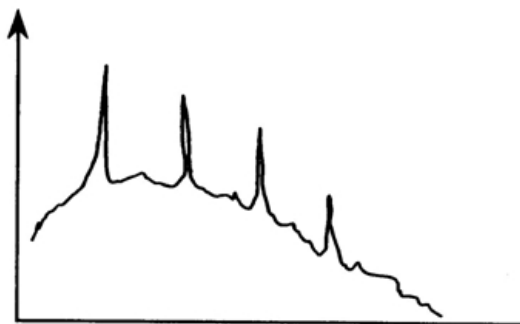
$$\text{SNR}_z = T \frac{\left[\int \frac{\Psi_s(f)}{\Psi_n(f)} df \right]^2}{\int \frac{\Psi_s^2(f)}{\Psi_n^2(f)} df} \frac{\left[\int \frac{\Psi_s^2(f)}{\Psi_n^2(f)} df \right]^2}{\left[\int \frac{\Psi_s(f)}{\Psi_n(f)} df \right]^2} = T \int \frac{\Psi_s^2(f)}{\Psi_n^2(f)} df$$

- Note the effect on SNR_z of increasing T .



Passive Narrowband Detection

- Want to detect targets that emit pure tone signatures:



- Receiver is shown below (essentially a spectrum analyzer)

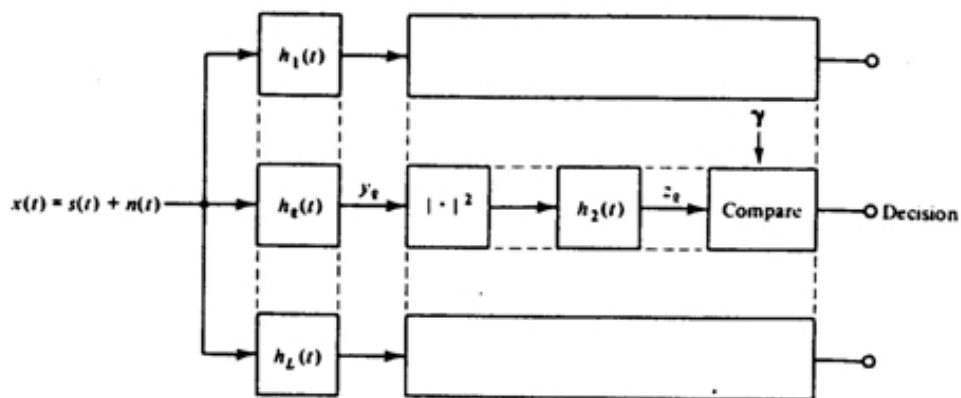


Figure 13-9 Passive narrowband receiver mechanization.



Passive Narrowband Detection (Cont'd)

- Typically implemented by Fourier transforming the input signal. Second filter is an integrator (“averager”). Long averages are usually employed, so that $T\beta \gg 1$.

If :

Signal Spectrum : $\Psi_s(f) = a^2 \delta(f - f_0)$

Filter : $|H_1(f)|^2 = \begin{cases} 1, & -\frac{\beta}{2} \leq f - f_0 \leq \frac{\beta}{2} \\ 0, & \text{otherwise} \end{cases}$

Noise Spectrum : $\Psi_n(f) \approx \text{Constant around } f_0$



Passive Narrowband Detection (Cont'd)

- Then

$$\text{SNR}_y = \frac{a^2}{\psi_n(f_0) \cdot \beta}$$

- As before, the SNR of the test statistic Z is

$$\text{SNR}_z = T\beta \text{SNR}_y^2$$

- Putting these together

$$\text{SNR}_z = \frac{T}{\beta} \left(\frac{a^2}{\psi_n(f_0)} \right)^2$$