Decision Theory and Receiver Design
Signal Detection and Performance Estimation

- **Problem**: How should received signals be processed in order to detect signals in noise? What kind of detection performance can be expected?
- **The approach to solution**:
  - Must be statistical, since noise is involved
  - Implement hypothesis testing
Signal Detection

- Input to detector is signal plus noise.
- Requirements expressed in terms of
  - probability of detection
  - probability of false alarm
- Threshold for declaring detection is set based on models for signal and noise
- Noise background estimation can be performed on data to improve model.
- Outputs of detector are threshold crossings
- Performance defined by receiver operating characteristic (ROC) curve – probability of detection vs. probability of false alarm for a particular SNR.
Detection In Noise

signal

noise mean

noise

signal + noise

Time

T_1

T_2
Detection Threshold

• Performance Criteria:
  – Probability of detection $P_D$
  – Probability of false alarm $P_{FA}$

• These criteria are not independent: a lower threshold increases $P_D$, but also increases $P_{FA}$.

• Theoretical ROC is used to set thresholds.

• True test is performance in water.
Receiver Operating Curve (ROC)

- Decreasing threshold
- Points T1 and T2
- Probability of Detection $P_D$ vs. Probability of False Alarm $P_{FA}$
Hypothesis Testing

• Possible Hypotheses:
  - $H_0$: Only noise is present
  - $H_1$: Signal is present in addition to noise

• Steps in forming hypotheses:
  - Process array output to obtain a detection statistic $x$.
  - Calculate the a posteriori probabilities $P(H_0|x)$ and $P(H_1|x)$.
  - Pick the hypothesis whose probability is the highest: the maximum a posteriori, or MAP estimate.

$$
\begin{align*}
\frac{P(H_1|x)}{P(H_0|x)} & \begin{cases} 
\leq 1, & \text{Choose } H_0 \\
\geq 1, & \text{Choose } H_1
\end{cases}
\end{align*}
$$
Hypothesis Testing (Cont’d)

• Equivalently, we can use Bayes’ rule to write:
  
  \[ P(H_1 \mid x)P(x) = P(x \mid H_1)P(H_1) \]
  
  \[ P(H_0 \mid x)P(x) = P(x \mid H_0)P(H_0) \]

• \( P(H_1) \) and \( P(H_0) \) are called a priori probabilities

• Then the test can be written:

  \[
  \frac{P(H_1 \mid x)}{P(H_0 \mid x)} = \frac{P(x \mid H_1)P(H_1)}{P(x \mid H_0)P(H_0)} \begin{cases} \geq 1, & \text{Choose } H_1 \\ < 1, & \text{Choose } H_0 \end{cases}
  \]
Hypothesis Testing (Cont’d)

- An equivalent test is

\[
\frac{P(x \mid H_1)}{P(x \mid H_0)} \begin{cases} \leq \frac{P(H_0)}{P(H_1)}, & \text{Choose } H_1 \\ > \frac{P(H_0)}{P(H_1)}, & \text{Choose } H_0 \end{cases}
\]

- \( \frac{P(x \mid H_1)}{P(x \mid H_0)} \equiv \lambda(x) \) is called the likelihood ratio
Aside: Bayes’ Rule and Notation

- Probability density functions are often used to describe continuous random variables:
  \[ P(x_0) \equiv \int_{-\infty}^{x_0} p(x) \, dx \]
- Bayes’ Rule as written for probabilities also holds for probability density functions (pdf).
- A compact notation is used in what follows:
  \[ p(x \mid H_1) \equiv p_1(x) \quad \quad p(x \mid H_0) \equiv p_0(x) \]
- Likelihood ratio test written in terms of pr
  \[
  \frac{p_1(x)}{p_0(x)} \begin{cases} \leq \frac{P(H_0)}{P(H_1)}, & \text{Choose } H_1 \\ > \frac{P(H_0)}{P(H_1)}, & \text{Choose } H_0 \end{cases}
  \]
A First Example: Constant Signal

- The possible inputs are:
  \[
  H_0 : \quad x(t) = n(t) \quad \text{(Noise only)}
  \]
  \[
  H_1 : \quad x(t) = \mu + n(t) \quad \text{(Signal plus noise)}
  \]

- If \( n(t) \) is Gaussian distributed and \( \mu \neq 0 \), then \( p(x \mid H_1) = p_1(x) \) and \( p(x \mid H_0) = p_0(x) \) are as shown below:

\[
p_0(x) = (2\pi\sigma^2)^{-1/2} \exp\left(-\frac{x^2}{2\sigma^2}\right)
\]

\[
p_1(x) = (2\pi\sigma^2)^{-1/2} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)
\]
• At time \( t \), we receive a signal \( x(t) \). Knowing \( p_0(x) \) and \( p_1(x) \), we can calculate the likelihood ratio

\[
\lambda(x) = \frac{p_1(x)}{p_0(x)}
\]

and compare it to a threshold

\[
\lambda_0 = \frac{P(H_0)}{P(H_1)}
\]

and decide accordingly:

\[
\lambda(x) \begin{cases} 
\geq \lambda_0, & \text{Choose } H_1 \\
< \lambda_0, & \text{Choose } H_0
\end{cases}
\]

• Note that \( \gamma \) is the value of \( x \) at which \( \lambda(x) = \lambda_0 \) in the figure.
Errors and Correct Decisions

• The possible errors are:
  – False Alarm: We choose $H_1$ when $H_0$ is the right answer.
  – False Dismissal: We choose $H_0$ when $H_1$ is the right answer.

• The possible correct decisions are:
  – Detection: We choose $H_1$ when it is the right answer.
  – Correct Dismissal: We choose $H_0$ when it is the right answer.
Probabilities of Errors and Correct Decisions

- $P_{FA} = \int_{\gamma}^{\infty} p_0(x) \, dx$
- $P_{FD} = \int_{-\infty}^{\gamma} p_1(x) \, dx$
- $P_D = \int_{\gamma}^{\infty} p_1(x) \, dx$
- $P_{CD} = \int_{-\infty}^{\gamma} p_0(x) \, dx$

Note: $P_{CD} + P_{FA} = 1 = P_{FD} + P_D$ because $\int_{-\infty}^{\infty} p(x) \, dx = 1$
Neyman-Pearson Criterion

• Usually we don’t know $P(H_1)$ and $P(H_0)$ and thus cannot calculate $\lambda_0$ from their ratio.

• Instead, we can specify a desired $P_{FA}$, or false alarm rate, and use it to obtain $\gamma$.

\[
P_{FA} = \int_{\gamma}^{\infty} p_0(x)dx = \text{specified false alarm probability}
\]

• Then we can calculate

\[
\lambda_0 = \frac{p_1(\gamma)}{p_0(\gamma)}
\]

or just compare $x$ to $\gamma$ directly.
Same Example: Multiple Samples

For each sample $x_i = x(t_i)$, the probabilities are:

$$p_0(x_i) = (2\pi\sigma^2)^{-1/2} \exp\left(-\frac{x_i^2}{2\sigma^2}\right), \quad p_1(x_i) = (2\pi\sigma^2)^{-1/2} \exp\left(-\frac{(x_i - \mu)^2}{2\sigma^2}\right)$$
Same Example: Multiple Samples (Cont’d)

• If we have a set of M multiple, independent samples,

\[ \bar{x} = x_1, x_2, \ldots \]

then their joint probability density functions under \( H_1 \) and \( H_0 \) are

\[
p_0(\bar{x}) = (2\pi\sigma^2)^{-M/2} \prod_{i=1}^{M} \exp\left(-\frac{x_i^2}{2\sigma^2}\right)
\]

\[
= (2\pi\sigma^2)^{-M/2} \exp\left(-\sum_{i=1}^{M} \frac{x_i^2}{2\sigma^2}\right)
\]

and

\[
p_1(\bar{x}) = (2\pi\sigma^2)^{-M/2} \exp\left(-\sum_{i=1}^{M} \frac{(x_i - \mu)^2}{2\sigma^2}\right)
\]
Same Example: Multiple Samples (Cont’d)

- The likelihood ratio becomes:

\[
\lambda(\bar{x}) = \frac{p_1(\bar{x})}{p_0(\bar{x})} = \exp\left(-\sum_{i=1}^{M} \frac{(x_i - \mu)^2 - x_i^2}{2\sigma^2}\right) = \exp\left(\frac{\mu M}{\sigma^2} y - \frac{\mu^2 M}{2\sigma^2}\right)
\]

where \( y \equiv \frac{1}{M} \sum_{i=1}^{M} x_i \) is the mean value of the samples.

- Note that each \( x_i \) is Gaussian with mean 0 under \( H_0 \) or \( \mu \) under \( H_1 \). Also, each \( x_i \) has variance \( \sigma^2 \) under both \( H_0 \) and \( H_1 \).

- Then \( y \) is also Gaussian, with the same mean, but with variance \( \frac{\sigma^2}{M} \).
Same Example: Multiple Samples (Cont’d)

- $y$ is a detection statistic (i.e. it is a sufficient statistic)

- Using the Neyman-Pearson criterion, the probability of a false alarm
  
  $$P_{FA} = \int_{\gamma}^{\infty} p_0(y) dy$$

  can be used to obtain a threshold $\gamma$ for $y$.

- Note that using $y = \frac{1}{M} \sum_{i=1}^{M} x_i$ satisfies our intuition that the receiver should counter the effects of noise by averaging the samples.
Second Example: Arbitrary But Known Signal

- Possible receiver inputs are:

  \[ H_0: x(t) = n(t) \]
  (Noise only)

  \[ H_1: x(t) = s(t) + n(t) \]
  (Signal plus noise)

- If the signal is present, we know its shape exactly.

- Assume we have \( M \) samples \( s_i \equiv s(t_i) \) in the interval \((0, T)\). The probabilities are:

  Under \( H_0 \): \[ p_0(\bar{x}) = (2\pi\sigma^2)^{-M/2} \exp\left(-\sum_{i=1}^{M} \frac{x_i^2}{2\sigma^2}\right) \]

  Under \( H_1 \): \[ p_1(\bar{x}) = (2\pi\sigma^2)^{-M/2} \exp\left(-\sum_{i=1}^{M} \frac{(x_i - s_i)^2}{2\sigma^2}\right) \]
• The likelihood ratio is:

\[ \lambda(\bar{x}) = \frac{p_1(\bar{x})}{p_0(\bar{x})} = \exp\left(-\sum_{i=1}^{M} \frac{(x_i - s_i)^2 - x_i^2}{2\sigma^2}\right) = \exp\left(\frac{1}{\sigma^2} \sum_{i=1}^{M} x_i s_i - \frac{1}{2\sigma^2} \sum_{i=1}^{M} s_i^2\right) \]

• The second term can be calculated before receiving the samples.

• As we sample more finely in the interval \((0, T)\), the summation becomes the integral:

\[ \sum_{i=1}^{M} s_i^2 \rightarrow \int_{0}^{T} s^2(t) dt \equiv E \]

where E is the energy in the signal.
• The test statistic in this case is:

\[ y(\bar{x}) = \sum_{i=1}^{M} x_i s_i \rightarrow \int_{0}^{T} x(t)s(t)dt \]

• Note that the received signal \( x(t) \) is being correlated with the signal we are trying to detect \( s(t) \).

• Equivalently, we can filter \( x(t) \) using a filter with impulse response function

\[ h(t) = s(T-\tau) \]

as can be seen from this equation:

\[
\int_{0}^{T} h(\tau)x(T-\tau)d\tau = \int_{0}^{T} s(T-\tau)x(T-\tau)d\tau = \int_{0}^{T} s(t)x(t)dt = y
\]

• A filter whose impulse response function is matched to the signal in this way is called a matched filter.
Test Statistic SNR

• We can define the SNR of \( y \) to be:

\[
\text{SNR}_y = \frac{[E(y_1) - E(y_0)]^2}{\text{var}(y_0)}
\]

• The expected values of the test statistic \( y \) under \( H_0 \) and \( H_1 \) are

\[
E(y \mid H_0) \equiv \bar{y}_0 = E\left[ \int_0^T x(t)s(t)dt \right] = 0
\]

\[
E(y \mid H_1) \equiv \bar{y}_1 = E\left[ \int_0^T (x(t) + n(t))s(t)dt \right] = E
\]

• The variance of \( y \) under \( H_0 \) is (using the shorthand \( y_0 \equiv y \mid H_0 \)):

\[
\text{var}(y_0) = E[(y_0 - \bar{y}_0)^2] = E\left[ \int_0^T \int_0^T s(t)s(\tau)n(t)n(\tau)dtd\tau \right]
\]
Let $n(t)$ be Gaussian white noise with spectral level $\frac{N_0}{2}$, i.e.:

$$R_{nn}(\tau) = \frac{N_0}{2} \delta(\tau)$$

Then

$$\text{var}(y_0) = \frac{N_0}{2} \int_0^T \int_0^T s(t)s(\tau)\delta(t-\tau)dt\,d\tau$$

$$= \frac{N_0 E}{2}$$

And so:

$$\text{SNR}_y = \frac{2E}{N_0}$$

As long as $n(t)$ is white Gaussian noise (WGN), there is no other receiver, i.e. no other test statistic $y$, which has a higher SNR. For many other types of noise, the matched filter is optimal or near optimal as well. This is why the matched filter is used.
Third Example: Signal Known Except Amplitude and Start Time

- This is the most common case, in which we are
  - Looking for a target echo
  - Listening for a radiated signal

- Exact arrival time and signal amplitude are unknown. The hypotheses are:

$$H_0 : \quad x(t) = n(t) \quad \text{(Noise only)}$$

$$H_1 : \quad x(t) = a \cdot s(t - t_0) + n(t) \quad (a, t_0 \text{ unknown})$$

- As before, $T$ is the duration of $s(t)$
• We apply the signal to a matched filter. Under $H_1$, the output is

$$y(t) = \int_0^T h(\tau)x(t-\tau)d\tau$$

$$= \int_0^T s(T-\tau)[a \cdot s(t-\tau-t_0) + n(t-\tau)] \cdot d\tau$$

$$= a \cdot R_s(t-T-t_0) + \int_0^T s(T-\tau)n(t-\tau)d\tau$$

• The first term is the autocorrelation function as $s$ at a lag of $t-T-t_0$. It is maximum when $t = T + t_0$, the time corresponding to the end of the pulse arrival

• The second term is random due to the noise.
• Assume $s(t)$ is a tone burst:

$$s(t) = \begin{cases} 
A \cos(2\pi f_0 t), & 0 \leq t \leq T \\
0, & \text{otherwise} 
\end{cases}$$

• The autocorrelation function is:
• Autocorrelation function is written:

\[
R_s(p) = \begin{cases} 
\frac{A \cdot a}{2} (T - |p|) \cos(2\pi f_0 t), & -T \leq p \leq T \\
0, & \text{otherwise}
\end{cases}
\]

• Can get the envelope of Rs(p) by squaring and low-pass filtering

\[
[R_s^2(p)]_{lpf} = \frac{A^2 a^2}{8} (T - |p|)^2
\]

• This is maximum when \( p = (t - T - t_0) = 0 \) or \( t = T + t_0 \).
• Thus the peak in \([y^2(t)]_{lpf}\) occurs at \( t = T + t_0 \), and since we know \( T \), can get \( t_0 \).
• Therefore, we define a new test statistic $Z(t)$:

$$Z(t) \equiv [y^2(t)]_{pf}$$

• The probability density functions of $Z(t)$ under $H_0$ and $H_1$ are shown by Burdic to be:

$$p_0(z) = \frac{1}{\sigma_y^2} \exp \left( - \frac{z}{\sigma_y^2} \right), \quad \sigma_y^2 = \frac{E N_0}{2}$$

$$p_1(z) = \frac{1}{\sigma_y^2} \exp \left( - S - \frac{z}{\sigma_y^2} \right) I_0 \left[ z \left( \frac{zS}{\sigma_y^2} \right)^{1/2} \right]$$

• Where $S \equiv \text{SNR}_y$ and $I_0(\cdot)$ is the zero-order modified Bessel function.
The probability density functions are plotted below.

Can use the Neyman-Pearson criterion to get $\gamma$, then calculate $P_D$

$$P_{FA} = \int_{\gamma}^{\infty} p_0(z) \, dz$$
Fourth Example: Possible Doppler Shift

- Non-zero radial motion between a transmitter (or reflector) and receiver causes the frequency of the received signal to be shifted relative to the transmitted signal. This is called Doppler Shift.
- This complication is usually met by implementing a parallel bank of filters (or FFT), each matched to a different frequency.
Passive Broadband Detection

Want to detect targets with broadband signatures:

Assume we know the ambient noise power spectrum
Passive Broadband Detection (Cont’d)

• Use the receiver shown below, where $h_1(t)$ and $h_2(t)$ are filters whose impulse functions need to be determined.

Figure 13-8  Passive broadband receiver mechanization.
Passive Broadband Detection (cont.)

- It has been shown that the Eckart Filter is optimal for $h_1(t)$:

$$\left| H_1(f) \right|^2 = \frac{\psi_s(f)}{\psi_n^2(f)}$$

Eckart Filter

- Note: when the noise is white, $H_1(f)$ looks like $\Psi_s(f)$. Otherwise, $H_1(f)$ is minimized when $\Psi_n(f)$ is large

- The power spectrum of $y$ under $H_0$ and $H_1$ is then:

$$\Psi_{y_0}(f) = \Psi_n(f)\left| H_1(f) \right|^2 = \frac{\psi_s(f)}{\psi_n(f)}$$

$$\Psi_{y_1}(f) = (\psi_s(f) + \psi_n(f))\left| H_1(f) \right|^2 = \frac{\psi_s^2(f)}{\psi_n^2(f)} + \frac{\psi_s(f)}{\psi_n(f)}$$

and

$$\text{SNR}_y = \frac{\int_{-\infty}^{\infty} [\Psi_{y_1}(f) - \Psi_{y_0}(f)] df}{\int_{-\infty}^{\infty} \Psi_{y_0}(f) df} = \frac{\int_{-\infty}^{\infty} \frac{\psi_s^2(f)}{\psi_n^2(f)} df}{\int_{-\infty}^{\infty} \frac{\psi_s(f)}{\psi_n(f)} df}$$
Passive Broadband Detection (cont.)

• Burdic shows that the SNR of the output of the envelope detector is
  \[ \text{SNR}_{y} = \text{SNR}_{y}^{2} \]

• The commonly-used post detection filter is an averager whose duration is as long as possible,
  \[ h_{2}(t) = \begin{cases} 
  \frac{1}{T}, & -\frac{T}{2} \leq \tau \leq \frac{T}{2} \\
  0, & \text{otherwise} 
\end{cases} \]

• The product of \( T\beta_{\epsilon} \) is typically large, where \( \beta_{\epsilon} \) is the effective noise bandwidth at the output of the pre-detection filter \( h_{1}(\tau) \), i.e. \( \beta_{\epsilon} \) is the width of a rectangular filter which admits the same noise power. The frequency domain expression for \( \beta_{\epsilon} \) is derived by Burdic in section 8-4 to be
  \[
  \beta_{\epsilon} = \left[ \int \Psi_{n}(f)|H_{1}(f)|^{2} df \right]^{2} / \int \Psi_{n}^{2}(f)|H_{1}(f)|^{4} df
  \]
Passive Broadband Detection (cont.)

- Using the Eckert Filter
  \[ \beta_\varepsilon = \left( \int \frac{\Psi_s(f)}{\Psi_n(f)} \, df \right)^2 \left( \int \frac{\Psi_s^2(f)}{\Psi_n^2(f)} \, df \right) \]

- Given large \( T \beta_\varepsilon \), Burdic shows that the SNR at the averager output is
  \[ \text{SNR}_z = T \beta_\varepsilon \text{SNR}_y \]

- Using the expressions for \( \text{SNR}_y \) and \( \beta_\varepsilon \)
  \[ \text{SNR}_z = T \left( \int \frac{\Psi_s(f)}{\Psi_n(f)} \, df \right)^2 \left( \int \frac{\Psi_s^2(f)}{\Psi_n^2(f)} \, df \right)^2 = T \int \frac{\Psi_s^2(f)}{\Psi_n^2(f)} \, df \]

- Note the effect on \( \text{SNR}_z \) of increasing \( T \).
Passive Narrowband Detection

- Want to detect targets that emit pure tone signatures:

- Receiver is shown below (essentially a spectrum analyzer)
Passive Narrowband Detection (Cont’d)

- Typically implemented by Fournier transforming the input signal. Second filter is an integrator ("averager"). Long averages are usually employed, so that \( T\beta >> 1 \).

**If:**

Signal Spectrum: \( \Psi_s(f) = a^2 \delta(f - f_0) \)

Filter:

\[
|H_1(f)|^2 = \begin{cases} 
1, & -\frac{\beta}{2} \leq f - f_0 \leq \frac{\beta}{2} \\
0, & \text{otherwise}
\end{cases}
\]

Noise Spectrum: \( \Psi_n(f) \approx \text{Constant around } f_0 \)
Passive Narrowband Detection (Cont’d)

• Then

\[ \text{SNR}_y = \frac{a^2}{\psi_n(f_0) \cdot \beta} \]

• As before, the SNR of the test statistic \( Z \) is

\[ \text{SNR}_z = T \beta \text{SNR}_y^2 \]

• Putting these together

\[ \text{SNR}_z = \frac{T}{\beta} \left( \frac{a^2}{\psi_n(f_0)} \right)^2 \]