Sonar Signal Processing

Course developed by:
Dr. Martin A. Mazur and Dr. R. Lee Culver

Presented by:
Dr. Martin A. Mazur
Course Outline

Sonar Signal Processing

• Review of the Sonar Equation
• Some Mathematical Preliminaries
• Time/Frequency Signal Processing
• Space/Angle Signal Processing
• Decision Theory and Receiver Design
• Examples (Time Allowing)
What We Will Not Cover (In Any Great Depth)

- Digital signal processing concepts and techniques
- Adaptive signal processing or beamforming
- Post-detection signal processing (e.g. classification, tracking)
- Random variable theory, stochastic processes
- Sonar implementation concepts (covered in a separate course):
  - Detailed transmitter/receiver block diagram
References


(B) – Basic/Overview/Reference (M) – Mid-level/General/Reference (A) – Advanced/Special Topic
Review of the Sonar Equation

The sonar equation, in its simplest form, is:

$$\frac{\text{Signal Power}}{\text{Noise Power}} \geq \frac{\text{Detection Threshold}}{\text{Signal Power}}$$

In decibels, referred to the sonar system input:

$$[S - N]_{\text{in}} - DT > 0$$

"Signal Excess"

Referred to the receiver output, the sonar equation becomes

$$[S - N]_{\text{out}} - OT > 0$$

where

$$[S - N]_{\text{out}} = [S - N]_{\text{in}} + PG$$
Sonar Signal Processing Gain

**Processing Gain**

\[ PG = AG + G \]

**Thresholds:**

\[ DT = \text{Input Detection Threshold} \]
\[ OT = \text{Receiver Output Threshold} \]
\[ OT = DT + PG \]

**Channel Output SNR**

\[ \left[ S - N \right]_{\text{out}} > OT \]

**Input SNR**

\[ \left[ S - N \right]_{\text{in}} < DT \]
Mathematical Tools
Outline

• In this section we touch briefly on some of the mathematical concepts and tools used in signal processing.
• Most of the section is definitions and a quick review of the mathematical notation.
Real and Complex Signals

• A real-valued function of time, f(t), or space, f(x), or both, f(x,t), is often called a “real signal”.

• It is sometimes useful for purposes of analysis to represent a signal as a complex valued function of space, time, or both:

\[ s(t) = u(t) + i \cdot v(t) \]

• More often, such a function is written in polar form:

\[ s(t) = R(t) \cdot e^{i \phi(t)} \]

where

\[ R(t) = \sqrt{u^2(t) + v^2(t)} \quad (\text{magnitude}) \]

\[ \phi(t) = \arctan \left( \frac{v(t)}{u(t)} \right) \quad (\text{phase}) \]

• The real-world signal f(t) represented by s(t) is just the real part of s(t):

\[ f(t) = \Re\{s(t)\} = u(t) = R(t) \cos(\phi(t)) \]
Symbols and Identities

$s(t)$ - a function of time, real or complex valued

$s^*(t)$ - the complex conjugate of $s(t)$, i.e. if

$$s(t) = u(t) + i \cdot v(t) = R(t)e^{i\phi(t)}$$

then

$$s^*(t) = u(t) - i \cdot v(t) = R(t)e^{-i\phi(t)}$$

$|s(t)|^2$ - the squared magnitude of $s(t)$, equal to $s(t) \cdot s^*(t)$

$S(\omega)$ - for deterministic signals, the frequency spectrum of $s(t)$ (the Fourier transform of $s(t)$)

or

spectrogram of $s(t)$ (the Fourier transform of $s(t)$)

$S(f)$ - for random signals, the power spectrum of $s(t)$ (the Fourier transform of the autocorrelation function of $s(t)$)

Note: $\omega = 2\pi f$

$u(x)$ - a function of a spatial coordinate

$U(\theta)$ - the angular spectrum of $u(x)$ (the Fourier transform of $u(x)$)

Euler's Identity: $e^{i\theta} = \cos \theta + i \cdot \sin \theta$
## Special Functions

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Time/Frequency Signal Processing
In this section we touch briefly on some of the concepts and tools used in spectral analysis, one of the most important areas of signal processing.

The section is a barebones outline, but because it is just a sample of a very large topic, it is still a whirlwind tour!

Very little in the vast realm of digital signal processing (DSP) can be covered in this small introduction.
Spectral (Fourier) Analysis

• Any signal, real or complex, that varies with time can be “broken up into its spectrum” in a way similar to that in which light breaks up into its constituent colors by a prism
• The mathematical operation by which this is accomplished is the Fourier transform
• The Fourier transform of a time signal yields the “frequency content” of a signal
• Much signal processing is done in the “frequency domain” by means of mathematical operations (filters) on the Fourier transforms of the signals of interest
• The result of the processing can be converted back to a filtered time signal by means of the inverse Fourier transform
Spectral Analysis of Periodic Signals

- Periodic signals can be represented as a sum of complex exponentials at discrete frequencies. For $s(t) = s(t + nT)$,

$$s(t) = \sum_{k=-\infty}^{\infty} c_k e^{i k \omega_0 t}, \text{ where } \omega_0 = \frac{2\pi}{T}$$

- The $c_k$ are complex numbers representing the spectrum of $s(t)$ and are called the Fourier coefficients of $s(t)$

$$c_k = \frac{1}{T} \int_T s(t) e^{-i k \omega_0 t} dt$$

$$S(\omega) = \sum_{k=-\infty}^{\infty} c_k \delta(\omega - n \omega_0)$$
Spectral Analysis of Continuous Signals

• An arbitrary signal can be represented as an integral of weighted complex exponentials over all frequencies:

\[ s(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega)e^{i\omega t} d\omega \]

• The function \( S(\omega) \) is the frequency spectrum of \( s(t) \) and is called the Fourier transform of \( s(t) \).

• \( S(\omega) \) is calculated from \( s(t) \) as follows:

\[ S(\omega) = \mathcal{F}\{s(t)\} = \int_{-\infty}^{\infty} s(t)e^{-i\omega t} dt \]
## A Few Fourier Transform Relationships

<table>
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<tr>
<th>Description</th>
<th>$f(t)$</th>
<th>$F(\omega)$</th>
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<td>$\int_{-\infty}^{\infty} f(t) e^{j\omega t} dt$</td>
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<td>Time shift</td>
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<td>Modulation</td>
<td>$e^{j\omega_0 t} f(t)$</td>
<td>$F(\omega - \omega_0)$</td>
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$\int_{-\infty}^{\infty} f(t) h(\tau - t) dt = f(t) \otimes h(t)$

$F(\omega) \cdot H(\omega)$

<table>
<thead>
<tr>
<th>And Vice - Versa</th>
<th>$f(t) \cdot h(t)$</th>
<th>$F(\omega) \otimes H(\omega)$</th>
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<td>Fourier transform of a time pulse</td>
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<td>$A \cdot T \cdot \text{rect}\left(\frac{T}{2\pi} \omega\right)$</td>
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- It generally takes a month or two of hard homework assignments for an engineering student to become facile with the mathematics, concepts, and implications of the previous table!
Frequency Domain Signal Processing

- Frequency domain signal processing, or “filtering” alters the frequency spectrum of a time signal to achieve a desired result.
- Examples of filters: band pass, band stop, low pass, high pass, “coloring”,
- Analog filters are electrical devices that work directly on the time signal and shape its spectrum electronically.
- Most filtering nowadays is done digitally. The spectrum of a sampled, digitized time signal is calculated using the Fast Fourier Transform (FFT). The FFT is a sampled version of the signal’s spectrum. Mathematical operations are performed on the sampled spectrum.
- Samples of the filtered signal are recovered using the inverse FFT.
Response of a Linear Time-Invariant System

- A linear system is one described by an equation \( y(t) = L[x(t)] \) that obeys the superposition property:
  - If \( y_1(t) \) is the response, or output, of the system to the input \( x_1(t) \), and \( y_2(t) \) is the response to \( x_2(t) \), and if \( a \) and \( b \) are arbitrary constants, then
    \[
    ay_1(t) + by_2(t) = L[a x_1(t) + b x_2(t)]
    \]
- A system is time-invariant if when the input is shifted in time, the output is correspondingly shifted.
- The response of a linear time-invariant system can be deduced by its response to an impulse:
  \[
  h(t) \equiv \text{Impulse response function} = L[\delta(t)]
  \]
- For any input \( x(t) \), the output is the convolution integral of \( x(t) \) with the impulse response function \( h(t) \):
  \[
  y(t) = L[x(t)] = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau \equiv x(t) \otimes h(t)
  \]
Suppose a linear system \( L \) has impulse response \( h(t) \).

- The Fourier transform of \( h(t) \), \( H(\omega) \), determines the “frequency response” of \( L \). How?

- The “convolution property” of the Fourier transform allows us to easily calculate the spectrum of the output of the system for any input \( x(t) \):

\[
y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau \iff Y(\omega) = H(\omega)X(\omega)
\]

  convolution property: convolution in time domain \( \iff \) multiplication in frequency domain
  (AND VICE VERSA)

- The output time function \( y(t) \) can be obtained by calculating the inverse Fourier transform of \( Y(\omega) \):

\[
y(t) = \mathcal{F}^{-1}\{Y(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(\omega)e^{i\omega t}\,d\omega
\]
Suppose we have two inputs to a receiver, only one of which is desirable. Suppose the inputs are separated in frequency:

**Example of Frequency Domain Processing:**

**Low Pass Filter**

\[ x(t) = A \cos(\omega_1 t) + B \cos(\omega_2 t) \]

\( X(\omega) = \frac{A}{2} \left[ \delta(2\pi(\omega - \omega_1)) + \delta(2\pi(\omega + \omega_1)) \right] + \frac{B}{2} \left[ \delta(2\pi(\omega - \omega_2)) + \delta(2\pi(\omega + \omega_2)) \right] \)

Let \( H(\omega) = \text{rect} \left( \frac{\omega}{2\omega_3} \right) \)

Then the output of this filter is \( Y(\omega) = H(\omega) X(\omega) = \frac{A}{2} \left[ \delta(2\pi(\omega - \omega_1)) + \delta(2\pi(\omega + \omega_1)) \right] \)

And \( y(t) = A \cos(\omega_1 t) \)
An Important Theorem In Spectral Analysis

• Sampling Theorem: A real, band limited, finite energy signal can be reconstructed from samples taken at the Nyquist rate.
  
  – Assume that the spectrum of a signal $s(t)$ is zero outside the interval $|f| < B$.
  – If samples $s_n$ are taken at the Nyquist rate of $f_s = 1/t_s = 2B$, then $s(t)$ can be reconstructed from its samples using the formula

$$s(t) = \sum_{n=-\infty}^{\infty} s_n \frac{\sin[2\pi B(t - nt_s)]}{2\pi B(t - nt_s)}$$

  – Sampling at less than the Nyquist rate can result in overlap of periodic extensions in frequency domain ("aliasing"), resulting in distortion of the reconstructed signal.
Effects Of Sampling

• When a real signal is sampled, copies of its spectrum appear at multiples of the sampling frequency => aliasing if signal is not band-limited. But any time-limited signal is *not* band-limited!

• Effects of aliasing can be abated by using time-windowing, filtering, and overlapping of data. This process somewhat increases the bandwidth and complexity of a signal processor, but greatly diminishes aliasing distortion.

• Digital signal processors usually work with a finite set of samples of a signal and another finite set of samples of the signal’s spectrum called the Discrete Fourier Transform (DFT).

• The Fast Fourier Transform (FFT) is a computationally efficient method of computing the DFT.
Effects Of Sampling (Illustration)

Band limited signal spectrum

Sampled signal spectrum
Sampling @ Nyquist rate

Sampled signal spectrum
Sampling < Nyquist rate

Overlap of spectral copies = aliasing

Sampled signal spectrum
Sampling < Nyquist rate

Overlap of spectral copies = aliasing

Sampled signal spectrum
Sampling > Nyquist rate

Overlap of spectral copies = aliasing

Sampled signal spectrum
Sampling < Nyquist rate

Overlap of spectral copies = aliasing

Sampled signal spectrum
Sampling > Nyquist rate

Overlap of spectral copies = aliasing

Sampled signal spectrum
Sampling < Nyquist rate

Overlap of spectral copies = aliasing

Sampled signal spectrum
Sampling > Nyquist rate

Overlap of spectral copies = aliasing
Random Signals: Correlation

- Let $x(t)$ and $y(t)$ be two random signals from processes that are stationary (statistics do not vary with time) and ergodic (statistics of the process can be estimated by taking time-averages of representative signals).

- The \textit{cross-correlation} of $x$ and $y$ is a measure of how “statistically alike” $x$ and $y$ are at particular spacings in time:

$$R_{xy}(\tau) = E[ x(t) y^*(t - \tau) ] \sim \frac{1}{T} \int_0^T x(t) y^*(t - \tau) dt$$

For stationary, ergodic signals correlated over a long time $T$

- The \textit{autocorrelation} of $x$, $R_x(\tau)$, is a measure of how statistically alike $x$ is to itself at different times.
Spectra of Random Signals

- The Fourier transforms of $R_x$ and $R_{xy}$ are the power spectral density, $S_x(\omega)$, of the process $x$, and the cross-power spectral density of the processes $x$ and $y$, $S_{xy}(\omega)$, respectively. They are measures of the frequency spectral content of the process $x$, and the shared spectral content of $x$ and $y$, respectively.
Spectra of Filtered Random Signals

- If a random process $x$ with power spectral density $S_x(\omega)$ is passed through a linear, time-invariant filter with transfer function $H(f)$, then the power spectral density of the output process $y$ is given by:

$$S_y(f) = |H(f)|^2 S_x(f)$$
Spatial Signal Processing (Beamforming)
Analogy Between Spatial Filtering (Beamforming) and Time-Frequency Processing

Goals of Spatial Filtering:

1. Increase SNR for plane wave signals in ambient ocean noise.
2. Resolve (distinguish between) plane wave signals arriving from different directions.
3. Measure the direction from which plane wave signals are arriving.

Goals of Time-Frequency Processing:

1. Increase SNR for narrowband signals in broadband noise.
2. Resolve narrowband signals at different frequencies.
3. Measure the frequency of narrowband signals.
Time-Frequency Filtering and Beamforming

- Broadband noise spectrum
- Sine wave at $f_1$
- Sine wave at $f_0$
- Narrowband filter at $f_0$
- Ambient noise angular density
- Plane wave at $\psi_1$
- Plane wave at $\psi_0$
- Narrow spatial filter at $\psi_0$

Frequency

Spatial angle $\psi$
SNR Calculation: Time-Frequency Filtering

Define
\[ \alpha \exp(i2\pi f_0 t) \equiv \text{Complex sinusoid at } f_0 \]
\[ \alpha^2 \delta(f - f_0) \equiv \text{Signal power spectral density (W/Hz)} \]
\[ |N(f)| \equiv \text{Noise power spectral density (W/Hz)} \]
\[ |H(f)|^2 \equiv \text{Filter power response} \]

Signal Power Is:
\[ P_s = \alpha^2 \int_{-\infty}^{\infty} \delta(f - f_0) |H(f)|^2 \, df = \alpha^2 |H(f_0)|^2 \text{ (watts)} \]

If we assume
\[ |H(f)|^2 = \begin{cases} 1, & -\frac{\beta}{2} \leq f - f_0 \leq \frac{\beta}{2} \\ 0, & \text{otherwise} \end{cases} \]
Idealized rectangular filter with bandwidth \( \beta \)
Then the noise power is:

$$P_N = \int_{-\infty}^{\infty} |N(f)| \cdot |H(f)|^2 df = N_0 \beta \text{ (watts)}$$

Where $N_0$ is the average noise level in band.

And SNR is:

$$SNR = \frac{P_s}{P_N} = \frac{\alpha^2}{N_0 \beta}$$

Note that SNR can be increased by narrowing the bandpass filter.
SNR Calculation: Spatial Filtering

Define

\[ \alpha \exp(i2\pi\Omega_0 x) \equiv \text{Acoustic plane wave arriving at angle } \Omega_0 \]
\[ \alpha^2 \delta(\Omega - \Omega_0) \equiv \text{Signal power angular density (W/steradian)} \]
\[ |N(\Omega)| \equiv \text{Noise power angular density (W/steradian)} \]
\[ |G(\Omega)|^2 \equiv \text{Spatial filter angular power response} \]

Signal Power Is:

\[ P_s = \alpha^2 \int_{4\pi} ^{\Omega_0} \delta(\Omega - \Omega_0)|G(\Omega)|^2 d\Omega = \alpha^2 |G(\Omega_0)|^2 \text{ (watts)} \]

If we assume

\[ |G(\Omega)|^2 = \begin{cases} 
  1, & -\frac{\beta}{2} \leq |\Omega - \Omega_0| \leq \frac{\beta}{2} \\
  0, & \text{otherwise}
\end{cases} \text{ Idealized “cookie cutter” beam pattern with width } \beta \]
Then the noise power is:

\[ P_N = \int_{4\pi} |N(\Omega)| \cdot |G(\Omega)|^2 \, d\Omega = \beta \cdot K_0 \]  (watts)

K₀ is the average noise intensity in beam

And SNR is:

\[ \text{SNR} = \frac{P_s}{P_N} = \frac{\alpha^2}{\beta \cdot K_0} \]

Again, SNR can be increased by narrowing “filter”
Array Gain and Directivity Calculations

Define

\[
\text{Array Gain} = \frac{SNR_{\text{Array}}}{SNR_{\text{OH}}}
\]

Assume

- plane wave signal
- arbitrary noise distribution

For the omnidirectional hydrophone,

\[
|G(\Omega)|^2 = 1 \quad \text{for all } \Omega
\]

Then

\[
SNR_{\text{OH}} = \frac{P_s}{P_N} = \frac{\alpha^2}{4\pi} \int_4^\pi \delta(\Omega - \Omega_0) |G_{\text{OH}}(\Omega)|^2 d\Omega
\]

\[
= \frac{\alpha^2}{4\pi} \int_4^\pi |N(\Omega)|^2 d\Omega
\]
Array Gain and Directivity Calculations (Cont’d)

For the array, assume it is steered in the direction of \( \Omega_0 \) and that \( |G^2(\Omega_0)|^2 = 1 \)

Then

\[
SNR_{array} = \frac{P_s}{P_N} = \frac{\alpha^2 \int \delta(\Omega - \Omega_0) |G_{array}(\Omega)|^2 d\Omega}{\frac{4\pi}{\int |N(\Omega)| \cdot |G_{array}(\Omega)|^2 d\Omega}} = \frac{\alpha^2}{\frac{4\pi}{\int |N(\Omega)| \cdot |G_{array}(\Omega)|^2 d\Omega}}
\]

Putting these together yields

\[
AG = \frac{\int |N(\Omega)| d\Omega}{\frac{4\pi}{\int |N(\Omega)| \cdot |G_{array}(\Omega)|^2 d\Omega}}
\]
Array Gain and Directivity Calculations (Cont’d)

If the noise is isotropic (the same from every direction)

\[ N(\Omega) = K \]

Then the Array Gain (AG) becomes the Directivity Index (DI), a performance index for the array that is independent of the noise field.

\[
DI = \frac{4\pi}{\int_{4\pi} G_{array}(\Omega)^2 d\Omega}
\]

Array Gain and Directivity Index are usually expressed in decibels.
Line Hydrophone Spatial Response

A plane wave has wavelength

$$\lambda = \frac{c}{f},$$

where $f$ is the frequency

and $c$ is the speed of sound.
Line Hydrophone Spatial Response (Cont’d)

The received signal is

\[ s(t + \frac{x \sin \psi}{c}) \]

at the origin

\[ s(t) \]

at point \( x \)

Let the hydrophone’s response or sensitivity at the point \( x \) be \( g(x) \). Then, the total hydrophone response is

\[ s_{out}(t) = \int_{-L/2}^{L/2} g(x)s(t + \frac{x \sin \psi}{c})dx \]
Line Hydrophone Spatial Response (Cont’d)

Using properties of the Fourier Transform:

\[ S(f) = \int s(t) e^{-i2\pi ft} dt \]

And:

\[ \int_{-\infty}^{\infty} s(t + \frac{x \sin \psi}{c}) e^{-i2\pi ft} dt = \exp(\frac{i2\pi fx \sin \psi}{c}) S(f) \]

Or:

\[ s(t + \frac{x \sin \psi}{c}) = \int_{-\infty}^{\infty} \exp(\frac{i2\pi fx \sin \psi}{c} + i2\pi ft) S(f) df \]
Line Hydrophone Spatial Response (Cont’d)

Thus, the total hydrophone response can be written:

\[
s_{out}(t) = \int_{-L/2}^{L/2} g(x) \int_{-\infty}^{\infty} \exp\left(\frac{i 2 \pi f x \sin \psi}{c}\right) + i 2 \pi f t \right) S(f) df dx
\]

\[
= \int_{-\infty}^{\infty} S(f) \left[ \int_{-L/2}^{L/2} g(x) \exp\left(\frac{i 2 \pi f x \sin \psi}{c}\right) dx \right] e^{i 2 \pi f t} df
\]

\[
= \int_{-\infty}^{\infty} S(f) G\left(\frac{f \sin \psi}{c}\right) e^{i 2 \pi f t} df
\]

Where

\[
G\left(\frac{f \sin \psi}{c}\right) = \int_{-L/2}^{L/2} g(x) \exp\left[ i 2 \pi \left(\frac{f \sin \psi}{c}\right) x \right] dx
\]

We call \(g(x)\) the aperture function and \(G((f \sin \psi)/c)\) the pattern function. They are a Fourier Transform pair.
Response To Plane Wave

An Example:

Unit Amplitude Plane Wave from direction $\psi_0$:

$$s(t) = e^{i2\pi f_0 t} \quad S(f) = \delta(f - f_0)$$

The Line Hydrophone Response is:

$$s_{out}(t) = \int_{-\infty}^{\infty} \delta(f - f_0) G\left(\frac{f \sin \psi_0}{c}\right) e^{i2\pi f t} df$$

$$= G\left(\frac{f_0 \sin \psi_0}{c}\right) e^{j2\pi f_0 t}$$

Note that the output is the input signal modulated by the value of the pattern function at $\psi_0$
Response To Plane Wave (Cont’d)

The pattern function is the same as the angular power response defined earlier.

Sometimes we use electrical angle \( u \):

\[
    u = \frac{f \sin \psi}{c} = \frac{\sin \psi}{\lambda}
\]

instead of physical angle \( \psi \).
Uniform Aperture Function

Consider a uniform aperture function

\[ g(x) = \begin{cases} \frac{1}{L}, & -\frac{L}{2} \leq x \leq \frac{L}{2} \\ 0, & \text{otherwise} \end{cases} \]

The pattern function is:

\[ G(u) = \int_{-\infty}^{\infty} g(x) e^{j2\pi ux} \, dx = \frac{1}{L} \int_{-L/2}^{L/2} e^{j2\pi ux} \, dx = \frac{e^{j2\pi Lu/2} - e^{j2\pi Lu/2}}{j2\pi Lu} \]

\[ = \frac{e^{j2\pi Lu/2} - e^{j2\pi Lu/2}}{j2\pi Lu} \quad \Rightarrow \quad \frac{e^{j2\pi Lu} - 1}{j2\pi Lu} \bigg|_{-L/2}^{L/2} = \frac{\sin(\pi Lu)}{\pi Lu} \]

\[ \equiv \text{sinc}(Lu) \]
Rectangular Aperture Function and Pattern Function
Array Main Lobe Width (Beamwidth)

3 dB (Half-Power) Beamwidth

To find it, solve

\[ G \left( \frac{\psi_{3\,dB}}{2} \right)^2 = \frac{1}{2} \]

for \( \psi_{3\,dB} \)
Beamwidth Example: Uniform Weighting

\[
\sin \left( \frac{\pi Lu_{3dB}}{2} \right) = \frac{1}{\sqrt{2}}
\]

\[
\frac{\pi Lu_{3dB}}{2} = 1.39
\]

\[
\psi_{3dB} = \sin^{-1}(\lambda u_{3dB}) = \sin^{-1}\left( \frac{2 \cdot 1.39 \cdot \lambda}{\pi L} \right)
\]

\[
= \sin^{-1}\left( .885 \frac{\lambda}{L} \right) \approx .885 \frac{\lambda}{L} \quad \text{radians}
\]

\[
\approx 50 \frac{\lambda}{L} \quad \text{deg}
\]

For \( \lambda \ll L \). Note the effect of increasing \( L \).
Discrete Elements: Line Array

Aperture Function:

\[ g(x) = \sum_{n=1}^{N} a_n \delta(x - x_n) \]

For \( N = 2M + 1 \) (Odd), and uniform spacing \( d \),

\[ g(x) = \sum_{n=-M}^{M} a_n \delta(x - nd) \]

Pattern Function: (N odd, uniform spacing)

\[ G(u) = \int_{-\infty}^{\infty} g(x) e^{j2\pi ux} \, dx \]

\[ = \int_{-\infty}^{\infty} \sum_{n=-M}^{M} a_n \delta(x - nd) e^{j2\pi ux} \, dx \]

\[ = \sum_{n=-M}^{M} a_n e^{j2\pi n u} \]
Uniformly Weighted Discrete Line Array

Assume \( a_n = \frac{1}{N} \), uniform spacing, \( n \) odd.

Temporarily define:

\[ r \equiv e^{j2\pi du} \]

then

\[ G(r) = \frac{1}{N} \sum_{n=-M}^{M} r^n = a_n \frac{1}{N} \frac{r^{N/2} - r^{-N/2}}{r^{1/2} - r^{-1/2}} \]

or

\[ G(u) = \frac{1}{N} \frac{\sin(\pi uNd)}{\sin(\pi ud)} \]
Pattern Function For Uniform Discrete Line Array

\[
\text{Pattern Function: } \text{sinc}(Lu)\]

\[
\frac{1}{N} \left( \frac{\sin \pi NdL}{\sin \pi du} \right)
\]

\[
Nd = \frac{L}{N}
\]
Notes On Pattern Function For Discrete Line Array

• u Only has physical significance over the range \(-\frac{L}{\lambda}, \frac{L}{\lambda}\)

• The region \(-\frac{L}{\lambda} \leq u \leq \frac{L}{\lambda}\) may include more than one main lobe if \(d \geq \lambda\), which causes ambiguity (called grating lobes).

General trade-offs in array design:
1) Want \(L\) large so that beamwidth is small and resolution is good
2) Want \(d \leq \lambda\) to avoid grating lobes.
3) Since \(L=Nd\), or \(N=L/d\), increasing \(L\) and decreasing \(d\) both cause \(N\) to increase, which costs more money
Effects of Array Shading (Non-Uniform Aperture Function)

• Shading reduces sidelobe levels at the expense of widening the main lobe.
• For other aperture functions:

<table>
<thead>
<tr>
<th>Aperture</th>
<th>$\psi_{3dB}$, degrees</th>
<th>First sidelobe level, dB</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangular</td>
<td>$50\lambda/L$</td>
<td>-13.3</td>
</tr>
<tr>
<td>Circular</td>
<td>$58\lambda/L$</td>
<td>-17.5</td>
</tr>
<tr>
<td>Parabolic</td>
<td>$66\lambda/L$</td>
<td>-22.0</td>
</tr>
<tr>
<td>Triangular</td>
<td>$73\lambda/L$</td>
<td>-26.5</td>
</tr>
</tbody>
</table>
Beam Steering

Want to shift the peak of the pattern function from \( u = 0 \) to \( u = u_0 \). What is the aperture function needed to accomplish this?

\[
g(x) = \int G(u) e^{-j2\pi ux} \, du
\]

\[
g'(x) = \int G(u - u_0) e^{-j2\pi ux} \, du
\]

\[
= e^{j2\pi u_0 x} \int G(p) e^{-j2\pi px} \, dp
\]

\[
= e^{j2\pi u_0 x} g(x)
\]

Beam steering is accomplished by multiplying the non-steered aperture function by a unit amplitude complex exponential, which is just a delay whose value depends on \( x \).
Beam Steering For Discrete Arrays

The steered aperture function becomes

\[ g(x) = \sum_{n=1}^{N} g(nd) \delta(x - nd)e^{j2\pi ndu_0} \]

\[ = ...g(-d)e^{-j2\pi ndu_0} \delta(x - d) + g(0) \delta(x) + g(d)e^{j2\pi ndu_0} \delta(x + d) + ... \]

The steered physical angle is \( \psi_0 = \sin^{-1}(\lambda u_0) \).

The phase shift at element \( n \) is equivalent to a time shift:

\[ 2\pi ndu_0 = \frac{2\pi nd \sin \psi_0}{\lambda} = 2\pi f \frac{nd \sin \psi_0}{c} = 2\pi f \tau_n. \]

where \( \tau_n = \frac{nd \sin \psi_0}{c} \)

Is the time shift which must be applied to the \( nth \) element.
Beam steering produces a projected aperture. Since reducing the aperture increases the beam width, beam steering causes the width of the (steered) main lobe to increase. The lobe distorts (fattens) more on the side of the beam toward which the beam is being steered.
Beam Steering: An Example

For a uniformly weighted, evenly spaced \((d=\lambda/2)\), 8 element array, the pattern function is

\[
G(u - u_0) = \frac{1}{N} \frac{\sin[\pi Nd (u - u_0)]}{\sin[\pi d (u - u_0)]}
\]

To find the beamwidth, set

\[
|G(u^+ - u_0)|^2 = |G(u_0 - u^-)|^2 = \frac{1}{2}
\]

Which yields

\[
\pi d (u^+ - u_0) = \pi d (u_0 - u^-) = 0.175
\]

Using \(d = \lambda/2\) and \(u = \sin \psi / \lambda\) the condition is

\[
\sin \psi^+ - \sin \psi_0 = \sin \psi_0 - \sin \psi^- = 0.1114
\]
As an example, take $N=8$, $(d=\lambda/2)$.

**Case 1:**  \[ \sin \psi_0 = 0 \quad \sin \psi^+ = \sin \psi^- = 0.1114 \]

\[ \Rightarrow \psi^+ = 6.4^0 \quad \psi^- = 6.4^0 \quad \psi_{3dB} = 12.8^0 \]

Note:  \[ 50 \frac{\lambda}{L} = 50 \frac{\lambda}{nd} = \frac{50}{4} = 12.5^0 \]
Beam Steering: An Example (Cont’d)

Case 2: \( \psi_0 = 45^0 \quad \sin \psi_0 = 0.707 \)

\( \psi^+ = \sin^{-1} (0.1114 + 0.707) = 54.9^0 \)

\( \psi^- = \sin^{-1} (0.1114 - 0.707) = 36.6^0 \)

\( \psi_{3dB} = 18.3^0 \quad \text{(wider)} \)

But, beam is not symmetrical:

\( 54.9^0 - 45^0 = 9.9^0 \)

\( 45^0 - 36.6^0 = 8.4^0 \)

Notice that there is no grating lobe until \( \psi_0 = 90^0 \)

at which point there is a grating lobe at \( \psi = -90^0 \)

This is why \( d = \frac{\lambda}{2} \) is optimal
Beam Pattern Effects
Number of Elements (Array Size)
Beam Pattern
1 Element Line Array – $\lambda/2$ spacing

Array Pattern Function For 1 Element Array
Beam Pattern
2 Element Line Array – \( \lambda/2 \) spacing
Beam Pattern
3 Element Line Array – $\lambda/2$ spacing
Beam Pattern
4 Element Line Array – $\lambda/2$ spacing

Array Pattern Function For 4 Element Array
Beam Pattern
5 Element Line Array – $\lambda/2$ spacing

Array Pattern Function For 5 Element Array
Beam Pattern
6 Element Line Array – $\lambda/2$ spacing
Beam Pattern
7 Element Line Array – $\lambda/2$ spacing
Beam Pattern
8 Element Line Array – $\lambda/2$ spacing
Beam Pattern
9 Element Line Array – $\lambda/2$ spacing

Array Pattern Function For 9 Element Array
Beam Pattern
10 Element Line Array – $\lambda/2$ spacing
Beam Pattern Effects
Element Spacing (Sampling)
Beam Pattern
10 Element Line Array – \( \lambda/10 \) spacing

Array Pattern Function For 10 Element Array, \( d=0.1\lambda \).
Beam Pattern
10 Element Line Array – $\lambda/4$ spacing
Beam Pattern
10 Element Line Array – $\lambda/2$ spacing
Beam Pattern
10 Element Line Array – 3/4 \( \lambda \) spacing

Array Pattern Function For 10 Element Array, \( d=0.75\lambda \).
Beam Pattern
10 Element Line Array – $\lambda$ spacing
Beam Pattern
10 Element Line Array – 3/2 \( \lambda \) spacing
Beam Pattern Effects
Beam Pointing
Beam Pattern
10 Element Line Array – $\lambda/2$ spacing – 0 deg pointing
Beam Pattern
10 Element Line Array – $\lambda/2$ spacing – 5 deg pointing

Array Pattern Function For 10 Element Array, $\theta_0 = 5$ deg
Beam Pattern
10 Element Line Array – $\lambda/2$ spacing – 10 deg pointing

Array Pattern Function For 10 Element Array, $\theta_0 = 10$ deg
Beam Pattern
10 Element Line Array – $\lambda/2$ spacing – 20 deg pointing

Array Pattern Function For 10 Element Array, $\theta_0 = 20$ deg
Beam Pattern
10 Element Line Array – $\lambda/2$ spacing – 30 deg pointing

Array Pattern Function For 10 Element Array, $\theta_0 = 30$ deg
Beam Pattern
10 Element Line Array – $\lambda/2$ spacing – 45 deg pointing

Array Pattern Function For 10 Element Array, $\theta_0 = 45$ deg
Beam Pattern
10 Element Line Array – $\lambda/2$ spacing – 60 deg pointing

Array Pattern Function For 10 Element Array, $\theta_0 = 60$ deg
Beam Pattern
10 Element Line Array – $\lambda/2$ spacing – 90 deg pointing
Three-Dimensional Arrays

A three dimensional array has $M$ transducers placed at the vector locations $\vec{r}_n$. The beam pattern is given by:

$$b(f, \vec{v}) = \sum_{n=1}^{M} a_n e^{i k \vec{r}_n \cdot \vec{v}}$$

Where $\vec{v}$ is the unit direction vector in polar (azimuth $\phi$ – elevation $\theta$) coordinates,

$$\vec{r}_n \cdot \vec{v} = r_{nx} \cos \phi \cos \theta + r_{ny} \sin \phi \cos \theta + r_{nz} \sin \theta$$

$a_n$ is the element weight of the $n^{th}$ element, and $k = \frac{2 \pi f}{c}$ is the wavenumber.

Beamsteering to an arbitrary direction vector $\vec{v}_0$ is done as in the line array, by simply substituting $(\vec{v} - \vec{v}_0)$ for $\vec{v}$ in the beam pattern function.
Decision Theory and Receiver Design
Signal Detection and Performance Estimation

- **Problem**: How should received signals be processed in order to detect signals in noise? What kind of detection performance can be expected?
- **The approach to solution**:
  - Must be statistical, since noise is involved
  - Implement hypothesis testing
Hypothesis Testing

- Possible Hypotheses:
  - $H_0$: Only noise is present
  - $H_1$: Signal is present in addition to noise

- Steps in forming hypotheses:
  - Process array output to obtain a detection statistic $x$.
  - Calculate the a posteriori probabilities $P(H_0|x)$ and $P(H_1|x)$.
  - Pick the hypothesis whose probability is the highest: the maximum a posteriori, or MAP estimate.

$$
\begin{align*}
P(H_1|x) \geq 1, & \quad \text{Choose } H_1 \\
P(H_0|x) < 1, & \quad \text{Choose } H_0
\end{align*}
$$
Hypothesis Testing (Cont’d)

• Equivalently, we can use Bayes’ rule to write:

\[
P(H_1 \mid x)P(x) = P(x \mid H_1)P(H_1)
\]

\[
P(H_0 \mid x)P(x) = P(x \mid H_0)P(H_0)
\]

• \(P(H_1)\) and \(P(H_0)\) are called a priori probabilities

• Then the test can be written:

\[
\frac{P(H_1 \mid x)}{P(H_0 \mid x)} = \frac{P(x \mid H_1)P(H_1)}{P(x \mid H_0)P(H_0)} \begin{cases} \geq 1, & \text{Choose } H_1 \\ < 1, & \text{Choose } H_0 \end{cases}
\]
Hypothesis Testing (Cont’d)

• An equivalent test is

\[
\frac{P(x | H_1)}{P(x | H_0)} \begin{cases} 
\leq \frac{P(H_0)}{P(H_1)}, & \text{Choose } H_1 \\
> \frac{P(H_0)}{P(H_1)}, & \text{Choose } H_0 
\end{cases}
\]

• \( \frac{P(x | H_1)}{P(x | H_0)} \equiv \lambda(x) \) is called the likelihood ratio
Aside: Bayes’ Rule and Notation

- Probability density functions are often used to describe continuous random variables:
  \[ P(x_0) = \int_{-\infty}^{x_0} p(x)dx \]

- Bayes’ Rule as written for probabilities also holds for probability density functions (pdf).

- A compact notation is used in what follows:
  \[ p(x/H_1) \equiv p_1(x) \quad p(x/H_0) \equiv p_0(x) \]

- Likelihood ratio test written in terms of probability density functions

  \[ \frac{p_1(x)}{p_0(x)} \begin{cases} \leq \frac{P(H_0)}{P(H_1)}, & \text{Choose } H_1 \\ > \frac{P(H_0)}{P(H_1)}, & \text{Choose } H_0 \end{cases} \]
A First Example: Constant Signal

- The possible inputs are:

  \[ H_0 : \quad x(t) = n(t) \quad \text{(Noise only)} \]
  \[ H_1 : \quad x(t) = \mu + n(t) \quad \text{(Signal plus noise)} \]

If \( n(t) \) is Gaussian distributed and \( \mu \neq 0 \), then

- \( p(x \mid H_1) = p_1(x) \) and \( p(x \mid H_0) = p_0(x) \) are as shown below:

\[
p_0(x) = (2\pi \sigma^2)^{-1/2} \exp\left(-\frac{x^2}{2\sigma^2}\right)
\]
\[
p_1(x) = (2\pi \sigma^2)^{-1/2} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right)
\]
• At time t, we receive a signal $x(t)$. Knowing $p_0(x)$ and $p_1(x)$, we can calculate the likelihood ratio

$$\lambda(x) = \frac{p_1(x)}{p_0(x)}$$

and compare it to a threshold

$$\lambda_0 = \frac{P(H_0)}{P(H_1)}$$

and decide accordingly:

$$\lambda(x) \begin{cases} \geq \lambda_0, & \text{Choose } H_1 \\ < \lambda_0, & \text{Choose } H_0 \end{cases}$$

• Note that $\gamma$ is the value of $x$ at which $\lambda(x) = \lambda_0$ in the figure.
Errors and Correct Decisions

- **The possible errors are:**
  - **False Alarm:** We choose $H_1$ when $H_0$ is the right answer.
  - **False Dismissal:** We choose $H_0$ when $H_1$ is the right answer.

- **The possible correct decisions are:**
  - **Detection:** We choose $H_1$ when it is the right answer.
  - **Correct Dismissal:** We choose $H_0$ when it is the right answer.
Probabilities of Errors and Correct Decisions

- \( P_{FA} = \int_{\gamma}^{\infty} p_0(x) \, dx \)
- \( P_{FD} = \int_{-\infty}^{\gamma} p_1(x) \, dx \)
- \( P_D = \int_{\gamma}^{\infty} p_1(x) \, dx \)
- \( P_{CD} = \int_{-\infty}^{\gamma} p_0(x) \, dx \)

Note: \( P_{CD} + P_{FA} = 1 = P_{FD} + P_D \) because \( \int_{-\infty}^{\infty} p(x) \, dx = 1 \)
Neyman-Pearson Criterion

- Usually we don’t know $P(H_1)$ and $P(H_0)$ and thus cannot calculate $\lambda_0$ from their ratio.
- Instead, we can specify a desired $P_{FA}$, or false alarm rate, and use it to obtain $\gamma$.

$$
P_{FA} = \int_{\gamma}^{\infty} p_0(x)dx = \text{specified false alarm probability}
$$

- Then we can calculate

$$
\lambda_0 = \frac{p_1(\gamma)}{p_0(\gamma)}
$$

or just compare $x$ to $\gamma$ directly.
For each sample $x_i = x(t_i)$, the probabilities are:

$$p_0(x_i) = (2\pi \sigma^2)^{-1/2} \exp\left(-\frac{x_i^2}{2\sigma^2}\right), \quad p_1(x_i) = (2\pi \sigma^2)^{-1/2} \exp\left(-\frac{(x_i - \mu)^2}{2\sigma^2}\right)$$
If we have a set of $M$ multiple, independent samples,

$$\bar{x} = x_1, x_2, \ldots$$

then their joint probability density functions under $H_1$ and $H_0$ are

$$p_0(\bar{x}) = (2\pi \sigma^2)^{-M/2} \prod_{i=1}^{M} \exp\left(-\frac{x_i^2}{2\sigma^2}\right)$$

$$= (2\pi \sigma^2)^{-M/2} \exp\left(-\sum_{i=1}^{M} \frac{x_i^2}{2\sigma^2}\right)$$

and

$$p_1(\bar{x}) = (2\pi \sigma^2)^{-M/2} \exp\left(-\sum_{i=1}^{M} \frac{(x_i - \mu)^2}{2\sigma^2}\right)$$
Same Example: Multiple Samples (Cont’d)

• The likelihood ratio becomes:

\[
\lambda(\bar{x}) = \frac{p_1(\bar{x})}{p_0(\bar{x})} = \exp\left( - \sum_{i=1}^{M} \frac{(x_i - \mu)^2 - x_i^2}{2\sigma^2} \right) = \exp\left( \frac{\mu M}{\sigma^2} y - \frac{\mu^2 M}{2\sigma^2} \right)
\]

where \( y = \frac{1}{M} \sum_{i=1}^{M} x_i \) is the mean value of the samples.

• Note that each \( x_i \) is Gaussian with mean 0 under \( H_0 \) or \( \mu \) under \( H_1 \). Also, each \( x_i \) has variance \( \sigma^2 \) under both \( H_0 \) and \( H_1 \).

• Then \( y \) is also Gaussian, with the same mean, but with variance \( \frac{\sigma^2}{M} \).
Same Example: Multiple Samples (Cont’d)

• $y$ is a detection statistic (i.e. it is a sufficient statistic)

• Using the Neyman-Pearson criterion, the probability of a false alarm
  $$P_{FA} = \int_{\gamma}^{\infty} p_0(y) dy$$

  can be used to obtain a threshold $\gamma$ for $y$.

• Note that using $y = \frac{1}{M} \sum_{i=1}^{M} x_i$ satisfies our intuition that the
  receiver should counter the effects of noise by averaging the samples.
Second Example: Arbitrary But Known Signal

- Possible receiver inputs are:

\[ H_0: x(t) = n(t) \quad \text{(Noise only)} \]

\[ H_1: x(t) = s(t) + n(t) \quad \text{(Signal plus noise)} \]

- If the signal is present, we know its shape exactly.

- Assume we have \( M \) samples \( s_i = s(t_i) \) in the interval \((0, T)\). The probabilities are:

Under \( H_0 \):

\[ p_0(\bar{x}) = (2\pi \sigma^2)^{-M/2} \exp \left( -\sum_{i=1}^{M} \frac{x_i^2}{2\sigma^2} \right) \]

Under \( H_1 \):

\[ p_1(\bar{x}) = (2\pi \sigma^2)^{-M/2} \exp \left( -\sum_{i=1}^{M} \frac{(x_i - s_i)^2}{2\sigma^2} \right) \]
• The likelihood ratio is:

\[ \lambda(\bar{x}) = \frac{p_1(\bar{x})}{p_0(\bar{x})} = \exp\left( -\sum_{i=1}^{M} \frac{(x_i - s_i)^2 - x_i^2}{2\sigma^2} \right) = \exp\left( \frac{1}{\sigma^2} \sum_{i=1}^{M} x_is_i - \frac{1}{2\sigma^2} \sum_{i=1}^{M} s_i^2 \right) \]

• The second term can be calculated before receiving the samples.

• As we sample more finely in the interval \((0, T)\), the summation becomes the integral:

\[ \sum_{i=1}^{M} s_i^2 \rightarrow \int_{0}^{T} s^2(t)dt \equiv E_s \]

where \(E_s\) is the energy in the signal \(s\).
• The test statistic in this case is:

\[ y(\bar{x}) = \sum_{i=1}^{M} x_i s_i \rightarrow \int_{0}^{T} x(t)s(t)dt \]

• Note that the received signal \( x(t) \) is being **correlated** with the signal we are trying to detect \( s(t) \).

• Equivalently, we can filter \( x(t) \) using a filter with impulse response function

\[ h(t) = s(T - \tau) \]

as can be seen from this equation:

\[ \int_{0}^{T} h(\tau) x(T - \tau) d\tau = \int_{0}^{T} s(T - \tau) x(T - \tau) d\tau = \int_{0}^{T} s(t)x(t)dt = y \]

• A filter whose impulse response function is matched to the signal in this way is called a **matched filter**.
Test Statistic SNR

- We can define the SNR of $y$ to be:

$$\text{SNR}_y = \frac{[E(y_1) - E(y_0)]^2}{\text{var}(y_0)}$$

- The expected values of the test statistic $y$ under $H_0$ and $H_1$ are

$$E(y \mid H_0) = \bar{y}_0 = E\left[\int_0^T x(t)s(t)dt\right] = 0$$

$$E(y \mid H_1) = \bar{y}_1 = E\left[\int_0^T (x(t) + n(t))s(t)dt\right] = E_s$$

- The variance of $y$ under $H_0$ is (using the shorthand $y_0 = y \mid H_0$):

$$\text{var}(y_0) = E[(y_0 - \bar{y}_0)^2] = E\left[\int_0^T \int_0^T s(t)s(\tau)n(t)n(\tau)dtd\tau\right]$$
• Let \( n(t) \) be Gaussian white noise with spectral level \( \frac{N_0}{2} \), i.e.:

\[
R_{nn}(\tau) = \frac{N_0}{2} \delta(\tau)
\]

• Then

\[
\text{var}(y_0) = \frac{N_0}{2} \int \int s(t)s(\tau)\delta(t - \tau) dt d\tau = \frac{N_0 E_s}{2}
\]

• And so:

\[
\text{SNR}_y = \frac{2E_s}{N_0}
\]

• As long as \( n(t) \) is white Gaussian noise (WGN), there is no other receiver, i.e. no other test statistic \( y \), which has a higher SNR. For many other types of noise, the matched filter is optimal or near optimal as well. This is why the matched filter is used.
Third Example: Signal Known Except Amplitude and Start Time

• This is the most common case, in which we are
  – Looking for a target echo
  – Listening for a radiated signal

• Exact arrival time and signal amplitude are unknown. The hypotheses are:

\[
H_0 : \ x(t) = n(t) \quad \text{(Noise only)}
\]

\[
H_1 : \ x(t) = a \cdot s(t - t_0) + n(t) \quad (a, t_0 \ \text{unknown})
\]

• As before, \( T \) is the duration of \( s(t) \)
• We apply the signal to a matched filter. Under $H_1$, the output is

$$y(t) = \int_0^T h(\tau) x(t - \tau) d\tau$$

$$= \int_0^T s(T - \tau)[a \cdot s(t - \tau - t_0) + n(t - \tau)] \cdot d\tau$$

$$= a \cdot R_s(t - T - t_0) + \int_0^T s(T - \tau) n(t - \tau) d\tau$$

• The first term is the autocorrelation function as $s$ at a lag of $t - T - t_0$. It is maximum when $t = T + t_0$, the time corresponding to the end of the pulse arrival.

• The second term is random due to the noise.
• Assume $s(t)$ is a tone burst:

$$s(t) = \begin{cases} 
A\cos(2\pi f_0 t), & 0 \leq t \leq T \\
0, & \text{otherwise}
\end{cases}$$

• The autocorrelation function is:
• Autocorrelation function is written:

\[ R_s(p) = \begin{cases} 
\frac{A \cdot a}{2} (T - |p|) \cos(2\pi f_0 t), & -T \leq p \leq T \\
0, & \text{otherwise}
\end{cases} \]

• Can get the envelope of Rs(p) by squaring and low-pass filtering

\[ [R_s^2(p)]_{lpf} = \frac{A^2 a^2}{8} (T - \|p\|^2) \]

• This is maximum when \( p = (t-T-t_0) = 0 \) or \( t=T+t_0 \).
• Thus the peak in \([y^2(t)]_{lpf}\) occurs at \( t= T+t_0 \), and since we know \( T \), can get \( t_0 \)
• Therefore, we define a new test statistic $Z(t)$:

$$Z(t) \equiv [y^2(t)]_{lpf}$$

• The probability density functions of $Z(t)$ under $H_0$ and $H_1$ are shown by Burdic to be:

$$\rho_0(z) = \frac{1}{\sigma_y^2} \exp\left(-\frac{z}{\sigma_y^2}\right), \quad \sigma_y^2 = \frac{E_s N_0}{2}$$

$$\rho_1(z) = \frac{1}{\sigma_y^2} \exp\left(-S - \frac{z}{\sigma_y^2}\right) I_0\left(z\left(\frac{zS}{\sigma_y^2}\right)^{1/2}\right)$$

• Where $S \equiv \text{SNR}_y$ and $I_0(\cdot)$ is the zero-order modified Bessel function.
The probability density functions are plotted below

Can use the Neyman-Pearson criterion to get $\gamma$, then calculate $P_D$

$$P_{FA} = \int_{\gamma}^{\infty} p_0(z) \, dz$$
Fourth Example: Possible Doppler Shift

- Non-zero radial motion between a transmitter (or reflector) and receiver causes the frequency of the received signal to be shifted relative to the transmitted signal. This is called Doppler Shift.
- This complication is usually met by implementing a parallel bank of filters (or FFT), each matched to a different frequency.
Passive Broadband Detection

Want to detect targets with broadband signatures:

Assume we know the ambient noise power spectrum
Passive Broadband Detection (Cont’d)

- Use the receiver shown below, where $h_1(t)$ and $h_2(t)$ are filters whose impulse functions need to be determined.

![Diagram](image-url)
Passive Broadband Detection (cont.)

- It has been shown that the Eckart Filter is optimal for $h_1(t)$:

$$|H_1(f)|^2 = \frac{\psi_s(f)}{\psi_n(f)^2}$$

Eckart Filter

- Note: when the noise is white, $H_1(f)$ looks like $\psi_s(f)$. Otherwise, $H_1(f)$ is minimized when $\psi_n(f)$ is large

- The power spectrum of $y$ under $H_0$ and $H_1$ is then:

$$\Psi_{y_0}(f) = \Psi_n(f)|H_1(f)|^2 = \frac{\psi_s(f)}{\psi_n(f)}$$

$$\Psi_{y_1}(f) = (\psi_s(f) + \psi_n(f))|H_1(f)|^2 = \frac{\psi_s^2(f)}{\psi_n^2(f)} + \frac{\psi_s(f)}{\psi_n(f)}$$

and

$$\text{SNR}_y = \frac{\int_0^\infty [\Psi_{y_1}(f) - \Psi_{y_0}(f)]df}{\int_0^\infty \Psi_{y_0}(f)df} = \frac{\int_{-\infty}^\infty \Psi_s^2(f)}{\int_{-\infty}^\infty \Psi_n^2(f)} \frac{df}{\int_{-\infty}^\infty \Psi_s(f)df}$$
Passive Broadband Detection (cont.)

- Burdic shows that the SNR of the output of the envelope detector is
  \[ \text{SNR}_{y}^2 = \text{SNR}_{y}^2 \]

- The commonly-used post detection filter is an averager whose duration is as long as possible,
  \[ h_2(t) = \begin{cases} 
  \frac{1}{T}, & \quad -\frac{T}{2} \leq \tau \leq \frac{T}{2} \\
  0, & \quad \text{otherwise} 
\end{cases} \]

- The product of \( T\beta_\varepsilon \) is typically large, where \( \beta_\varepsilon \) is the effective noise bandwidth at the output of the pre-detection filter \( h_1(\tau) \), i.e. \( \beta_\varepsilon \) is the width of a rectangular filter which admits the same noise power. The frequency domain expression for \( \beta_\varepsilon \) is derived by Burdic in section 8-4 to be
  \[
  \beta_\varepsilon = \left( \frac{\int \Psi_n(f) |H_1(f)|^2 df}{\int \Psi_n^2(f) |H_1(f)|^4 df} \right)^2
  \]
Passive Broadband Detection (cont.)

- Using the Eckert Filter

\[
\beta_\varepsilon = \frac{\int \frac{\Psi_s(f)}{\Psi_n(f)} df}{\int \frac{\Psi_s^2(f)}{\Psi_n^2(f)} df}
\]

- Given large \( T\beta_\varepsilon \), Burdic shows that the SNR at the averager output is

\[
\text{SNR}_z = T\beta_\varepsilon \text{SNR}_y^2 = T\beta_\varepsilon \text{SNR}_y^2
\]

- Using the expressions for \( \text{SNR}_y \) and \( \beta_\varepsilon \)

\[
\text{SNR}_z = T \left[ \int \frac{\Psi_s(f)}{\Psi_n(f)} df \right]^2 \left[ \int \frac{\Psi_s^2(f)}{\Psi_n^2(f)} df \right]^2 = T \int \frac{\Psi_s^2(f)}{\Psi_n^2(f)} df
\]

- Note the effect on \( \text{SNR}_z \) of increasing \( T \).
Passive Narrowband Detection

- Want to detect targets that emit pure tone signatures:

- Receiver is shown below (essentially a spectrum analyzer)

![Diagram of Passive Narrowband Receiver Mechanization](image)
Passive Narrowband Detection (Cont’d)

• Typically implemented by Fournier transforming the input signal. Second filter is an integrator ("averager"). Long averages are usually employed, so that $T\beta >> 1$.

If:

$\Phi_s(f) = a^2 \delta(f - f_0)$

Filter:

$|H_1(f)|^2 = \begin{cases} 
1, & -\frac{\beta}{2} \leq f - f_0 \leq \frac{\beta}{2} \\
0, & \text{otherwise}
\end{cases}$

Noise Spectrum:

$\Phi_n(f) \approx \text{Constant around } f_0$
Passive Narrowband Detection (Cont’d)

- Then
  \[ \text{SNR}_y = \frac{a^2}{\psi_n(f_0) \cdot \beta} \]
- As before, the SNR of the test statistic \( Z \) is
  \[ \text{SNR}_z = T\beta \text{ SNR}_y^2 \]
- Putting these together
  \[ \text{SNR}_z = \frac{T}{\beta} \left( \frac{a^2}{\psi_n(f_0)} \right)^2 \]
Receiver Operating Characteristic Curves

- ROC curves graphically display the relationship between test statistic SNR, $P_{FA}$, and $P_D$. Each receiver has its own ROC curve.

The one below is for large time-bandwidth products.

**Figure 13-10** Receiver operating characteristic curves for $p_0(z)$ and $p_r(z)$ both Gaussian. SNR ($z$) = $d$ for indicated $P_D$ and $P_R$. 
Signal Parameter Estimation

- Now consider the case where we want to estimate
  - arrival time $\tau$
  - signal frequency, $f$
  - signal arrival angle, $\psi$
  of a signal $s$ with energy $E_s$

- We want unbiased estimators. For example, if $\tau_0$ is the true arrival time, and $\hat{\tau}$ is the estimate, then $\hat{\tau}$ is unbiased if
  \[ E(\hat{\tau}) = \tau_0 \]

- The other desirable feature of an estimator is that it have variance that decreases asymptotically with increasing SNR. Low variance means small error.
Minimum Bounds on Variances

• It can be shown that the minimum possible (Cramer-Rao) variance of the time delay estimate is

\[
\text{var}(\hat{\tau}) = \frac{1}{2E_s \frac{\beta_s^2}{N_0}}
\]

• Where \( \beta_s^2 \) is the mean-square bandwidth of the signal

• Note that the variance of the time delay estimate is reduced (which is good) by increasing signal bandwidth and input SNR
Minimum Bounds on Variances (Cont’d)

• Similarly, the Cramer-Rao lower bound for frequency estimates is

\[ \text{var}(\hat{f}) = \frac{1}{2E_s T_0^2} \]

\[ = \frac{1}{2E_s N_0} T_0^2 \]

Where \( T_0^2 \) is the mean-square signal duration.

• The variance of the frequency estimate is reduced by increasing signal duration and input SNR.
Minimum Bounds on Variances (Cont’d)

• Finally, the Cramer-Rao lower bound for arrival angle estimates for a line array of length $L$ is

$$
\text{var}(\hat{\psi}) = \frac{1}{2E_s \cdot \frac{\pi^2 L^3}{N_0 \cdot 2\lambda^3}}
$$

where $\lambda$ is the wavelength of the signal: $\lambda = c/f$, where $c$ is the speed of sound and $f$ is the frequency.

• The variance of the arrival angle estimates is reduced by increasing the aperture size and by higher input SNR.
Additional Material
Spectra Of Real Signals

• In general, the Fourier transform of any signal is a complex function of a real variable, the frequency.
• The Fourier transform of a real signal is *conjugate symmetric*: \( S^*(f) = S(-f) \)
• Because of this, spectra of real signals have negative frequency components that are equal in magnitude to the positive frequency components, but opposite in phase.
• Thus, all spectral information about a real signal is contained in the positive frequencies. Negative frequency components are redundant!
• To fully take advantage of this in signal processing, need the concept of analytic signals.
Analytic Signals

- Real signals are often represented, for analysis, as the real part of a complex signal:
  \[ x(t) = A\cos(\omega_0 t + \phi) = \Re\{ A\exp[i(\omega_0 t + \phi)] \} \]

- Often, the variation of the signal near its center frequency is of interest. Can rewrite the complex signal as:
  \[ f(t) = A\exp[i(\omega_0 t + \phi)] = Ae^{i\phi} - e^{i\omega_0 t} \]

  This representation conveniently encapsulates information about the center frequency, amplitude and phase of a signal. This representation is a special case of an analytic signal.

- If the spectrum of the complex envelope is confined to a narrow band of frequencies, then the spectrum of \( f(t) \) is confined to positive frequencies.
Properties of Analytic Signals

- Whereas real signals have both positive and negative frequency components, the analytic signal representation only has the positive set.
- Analytic signals are convenient for use in modern complex array processors and are well suited to Fourier analysis.
- Drawback: real and imaginary, or “in-phase” and “quadrature” samples must be taken.
- This is more than made up for in simplicity of processing and bandwidth savings in other parts of processing stream.
Processing of Analytic Signals

• To create an analytic signal representation $s_a(t)$ of a signal $s(t)$, it is necessary to create a fictitious complex signal.

• The real part of the analytic signal is the real signal $s(t)$. The imaginary part of the analytic signal is called the *Hilbert transform* of $s(t)$.

• For narrowband signals, samples of the Hilbert transform of $s(t)$ can be obtained by sampling the signal again 90° out-of-phase of the “real” samples.

• The analytic signal can be conveniently digitally shifted in frequency and subsampled to further decrease bandwidth requirements.

• Only the information-containing part of the signal need be processed, not the negative frequencies or the empty space down to DC.
Sampling with Analytic Signal Output

Analytic sampling can also be accomplished with phase shifters
Array Gain: Discrete Line Array

Noise Model:

\[ |N(\phi)| = \begin{cases} 
K \sin \phi, & 0^\circ \leq \phi \leq 90^\circ \\
KL_B \sin(-\phi), & -90^\circ \leq \phi \leq 0^\circ 
\end{cases} \quad (L_B \approx 0.1) \]
Array Gain: Discrete Line Array (Cont’d)

Nine-element vertical line array gain versus elevation steering angle in a surface-generated ambient noise field
Array Performance Analysis

- Want to write $\text{SNR}_{\text{array}}$ in the form of the sonar equation and include signal and filter bandwidth effects.
- Signal power spectrum/ angular density at the source: $\Psi_s(f, \Omega)$
- Source level at frequency $f$, direction $\Omega$:

$$SL(f, \Omega) = 10 \log \left[ \frac{\Psi_s(f, \Omega)}{\Psi_{\text{ref}}} \right]$$

- Transmission loss at range $r$: $T(r)$
- Signal level received by the array:

$$S_{\text{array}}(f) = 10 \log \left[ \frac{\int \Psi_s(f, \Omega) |G_{\text{array}}(f, \Omega)|^2 d\Omega}{\Psi_{\text{ref}} \cdot T(r)} \right]$$
Array Performance Analysis (Cont’d)

- Assume a plane wave signal, with the array steered in the direction of the source. Then

\[
S_{\text{array}}(f) = 10 \log \left[ \frac{\Psi_s(f) G_{\text{array}}(f)^2}{\Psi_\text{ref} \cdot T(r)} \right]
\]

- The total received signal power is:

\[
S_{\text{array}}(f) = 10 \log \left[ \int_{-\infty}^{\infty} \frac{\Psi_s(f) G_{\text{array}}(f)^2 \, df}{\Psi_\text{ref} \cdot T(r)} \right]
\]

- Assume \( |G_{\text{array}}(f)|^2 \) is rectangular and centered at \( f_c \), with width \( \beta \). Then:

\[
S_{\text{array}} = 10 \log \left[ \frac{\Psi_s(f_c)}{\Psi_\text{ref} \cdot T(r)} \right]
\]

\[
= \text{SL}(f_c) - \text{TL}(r) + 10 \log \beta
\]

where

\[
\text{TL}(r) = 10 \log \left[ \frac{T(r)}{r_{\text{ref}}} \right], \quad r_{\text{ref}} = 1.
\]
Array Performance Analysis (Cont’d)

- Ambient noise power spectrum/angular density at frequency $f$, direction $\Omega$:
  \[ |N(\Omega, f)| \]

- Ambient noise spectral level at frequency $f$ at the input of the array:
  \[
  NL_i(f) = 10 \log \left[ \frac{\int 4\pi |N(\Omega, f)| d\Omega}{\Psi_{\text{ref}}} \right]
  \]

- Ambient noise spectral level received by array at frequency $f$:
  \[
  NL_{\text{array}}(f) = 10 \log \left[ \frac{\int 4\pi |N(\Omega, f)| \cdot |G_{\text{array}}(\Omega, f)|^2 d\Omega}{\Psi_{\text{ref}}} \right]
  \]

- Total ambient noise power received by the array:
  \[
  NL_{\text{array}} = 10 \log \left[ \frac{\int_{-\infty}^{\infty} \int 4\pi |N(\Omega, f)| \cdot |G_{\text{array}}(\Omega, f)|^2 d\Omega df}{\Psi_{\text{ref}}} \right]
  \]
Array Performance Analysis (Cont’d)

- Now assume that $|G_{array}(\Omega,f)|^2$ is a rectangular (bandpass) filter, centered at $f_c$, with width $\beta$. Then:

$$NL_{array} = 10 \log \left[ \frac{\beta \int |N(\Omega,f_c)| \cdot |G(\Omega)|^2 \, d\Omega}{4\pi \Psi_{ref}} \right]$$

- Recall that the array gain is

$$AG(f_c) = 10 \log \left[ \frac{\int |N(\Omega,f_c)| \, d\Omega}{4\pi \int |N(\Omega,f_c)| \cdot |G_{array}(\Omega,f_c)|^2 \, d\Omega} \right] = NL_i(f_c) + 10 \log \beta - NL_{array}(f_c)$$

- Using the $AG$, we can write

$$NL_{array} = NL_i(f_c) - AG(f_c) + 10 \log \beta$$

- Then

$$SNR = S_{array} - NL_{array} = SL(f_c) - TL(r) - NL_i(f_c) + AG(f_c)$$

Note that higher array gain increases SNR.
Directivity Index For A Discrete Line Array

- Uniform weighting
- \( f_0 \) is the frequency at which \( d = \lambda / 2 \)
- When \( f = f_0 \), \( DI = 10 \log N \)
- When \( f > f_0 \), \( DI \) oscillates about \( 10 \log N \). However, grating lobes cause ambiguity.
- When \( f < f_0 \), \( DI \) drops at about 3dB per octave. End fire beam has higher DI, but broader beamwidth, hence poorer directional resolution.

DI for a discrete line array, broadside and endfire with \( N=9 \).
Target Angle Estimation Using Split Apertures

The aperture functions are:

\[
g(x) = \begin{cases} 
\frac{2}{L}, & -\frac{L}{2} \leq x \leq \frac{L}{2} \\
0, & \text{otherwise} 
\end{cases}
\]

\[
g_R(x) = g(x - \frac{L}{4}), \quad 0 \leq x \leq \frac{L}{2}
\]

\[
g_L(x) = g(x + \frac{L}{4}), \quad -\frac{L}{2} \leq x \leq 0
\]
Target Angle Estimation Using Split Apertures (Cont’d)

Pattern functions are:

\[ G(u) = \int_{-L/4}^{L/4} g(x) e^{j2\pi ux} \, dx \]

\[ = \frac{\sin(\pi Lu/2)}{\pi Lu/2} \equiv \sin c(Lu/2) \]

And:

\[ G_R(u) = \int_{0}^{L/2} g(x - L/4) e^{j2\pi ux} \, dx \]

\[ = \exp(j\pi Lu/2) \int_{-L/4}^{L/4} g(y) e^{j2\pi uy} \, dy \]

\[ = \exp(j\pi Lu/2) G(u) \]

And similarly:

\[ G_L(u) = \exp(-j\pi Lu/2) G(u) \]
Target Angle Estimation Using Split Apertures (Cont’d)

The signals out of the arrays were previously shown to be:

\[ s_R(t) = \int_{-\infty}^{\infty} g_R(x) s(t + \frac{x \sin \psi}{c}) dx \]

\[ = \int_{-\infty}^{\infty} S(f) G_R(\frac{f \sin \psi}{c}) e^{j2\pi ft} df \]

For a plane wave at angle \( \psi_0 \) and frequency \( f_0 \):

\[ S(f) = \delta(f - f_0) \]

\[ s_R(t) = G_R(\frac{f_0 \sin \psi_0}{c}) e^{j2\pi f_0 t} \]

\[ = G(\frac{f_0 \sin \psi_0}{c}) \exp( j2\pi f_0 t + j\pi Lu / 2 ) \]

And similarly:

\[ s_L(u) = G(\frac{f_0 \sin \psi_0}{c}) \exp( j2\pi f_0 t - j\pi Lu / 2 ) \]

\( j\pi Lu \) is the phase difference between the two signals.
Target Angle Estimation Using Split Apertures (Cont’d)

We use the phase difference between the signals coming out of the left and right arrays to estimate the arrival angle $\psi_0$. To get the phase difference, multiply the two signals together:

$$s_L(t)s_R^*(t) = \left| G\left(\frac{f_0\sin\psi_0}{c}\right) \right|^2 \exp\left(\frac{j\pi L \sin\psi_0}{\lambda}\right)$$

$$= \left| G\left(\frac{f_0\sin\psi_0}{c}\right) \right|^2 \left[ \cos\left(\frac{\pi L \sin\psi_0}{\lambda}\right) + j\sin\left(\frac{\pi L \sin\psi_0}{\lambda}\right) \right]$$

To find the angle $\psi_0$, compute

$$\frac{\text{Im}\{s_L(t)s_R^*(t)\}}{\text{Re}\{s_L(t)s_R^*(t)\}} = \tan\left(\frac{\pi L \sin\psi_0}{\lambda}\right)$$

And solve for $\psi_0$:

$$\psi_0 = \sin^{-1}\left[ \frac{\lambda}{\pi L} \tan^{-1}\left( \frac{\text{Im}\{s_L(t)s_R^*(t)\}}{\text{Re}\{s_L(t)s_R^*(t)\}} \right) \right]$$
Split Aperture Mechanization

Below is the mechanization for processing split-aperture array outputs to estimate an arrival angle \( \psi_0 = \sin^{-1}\left(\frac{\lambda \phi}{\pi L}\right) \) where \( \phi \) is the phase difference between the two signals.

The input signals are \( s_R(t) = \cos(2\pi ft + \phi/2) \) and \( s_L(t) = \cos(2\pi ft - \phi/2) \).
Examples
Example: Passive Broadband

Given $T\beta = 1000$

Find: The required $\text{SNR}_y$ such that $P_D = 0.9$ and $P_{FA} = 10^{-4}$
Example: Passive Broadband (Cont’d)

- **Solution:** from the ROC curve $P_D$ and $P_{FA}$ cross at $\text{SNR}_z = 25$, using the relationship between $\text{SNR}_y$ and $\text{SNR}_z$ derived earlier for passive broadband detection.

\[
\text{SNR}_z = T \beta \text{SNR}_y^2 = 1000 \cdot \text{SNR}_y^2
\]

Or

\[
\text{SNR}_y = \left[ \frac{25}{1000} \right]^{\frac{1}{2}} = 0.158
\]

Or

\[
10 \log(\text{SNR}_y) = -8.0 \text{ dB}
\]
System Performance Analysis
Narrowband Passive Detection

- A narrowband passive receiver looks for target radiated tones. It is essentially a spectrum analyzer.
• At the output of the array, we saw that the noise and signal levels were

\[
NL_{array}(f) = NL_s(f) - AG(f) + 10\log{\beta}
\]

\[
SL_{array}(f_0) = SL(f_0) - TL(r, f_0) \quad \text{(for a pure tone at } f_0 \text{)}
\]

• Therefore, at the output of the filter centered at \( f_0 \), the SNR in dB is

\[
10\log(\text{SNR}_y) = SL(f_0) - TL(r, f_0) - NL_s(f_0) + AG(f_0) - 10\log{\beta}
\]

• For the narrowband passive receiver, we saw that

\[
\text{SNR}_y = \left[ \frac{\text{SNR}_z}{T\beta} \right]^{\frac{1}{2}}
\]

Or in dB

\[
10\log(\text{SNR}_y) = 5\log(\text{SNR}_z) - 5\log{T} - 5\log{\beta}
\]
Putting these together yields

$$5\log(SNR_Z) - 5\log T + 5\log \beta = SL(f_0) - TL(r, f_0) - NL_s(f_0) + AG(f_0)$$

This equation relates the test statistic $z$ SNR, the averaging time $T$, the filter bandwidth $\beta$, the source level $SL$, the transmission loss $TL$, the ambient noise level $NL_s$, and the array gain $AG$.

We are usually given an allowable false alarm time $\tau_{FA}$. Since there are $N$ channels, and 1 output each $T$ seconds, there are $N/T$ false alarms opportunities per second. The number of false alarms per second is then

$$P_{FA} \cdot \frac{N}{T} = \frac{1}{\tau_{FA}}$$

This equation can be solved for $P_{FA}$. Then, using $P_D$, we can get required $SNR_Z$. 

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Example: Passive Narrowband

• Given:
  Target Signal: Pure sinusoid at $f_1 = 500$ Hz with source level $SL = 125$ dB/$\mu$Pa@1m.
  Ambient noise level: $NL_s = +65$ dB/$\mu$ PA in 1 Hz band at 500 Hz. (Corresponds approximately to Sea State 3.)
  Spatial Filter: Horizontal line array with $L = 100$ ft.
  Predetection processor: Bank of contiguous narrowband filters with $\beta = 0.5$ Hz covering a 500 Hz band centered at 500 Hz.
  Postdetection processor: Linear integrators with $T = 100$ sec.
  Desired system $\bar{\tau}_{FA} : 1000$ sec.

• Find: The detection range for $P_D = 50\%$
Example: Passive Narrowband (Cont’d)

• Solution:
  From the previous slide,
  \[ P_{FA} = \frac{T}{N \cdot \tau_{FA}} = \frac{100}{1000 \cdot 1000} = 10^{-4} \]

• Since \( T\beta = 100 \cdot 0.5 = 50 \), use the ROC curve for large \( T\beta \) and \( P_D = 50\% \) to get \( d = 14 = \text{SNR}_Z \)

• The noise is isotropic, so \( AG = DI \) and \( DI \) is found by Burdic to be
  \[ DI \approx 10 \log \left( \frac{2L}{\lambda} \right) = 10 \log \left( \frac{2 \cdot 100}{10} \right) = 13dB \]
  \[ \lambda = \frac{c}{f} \approx \frac{5000}{500} \approx 10 \]
Example: Passive Narrowband (Cont’d)

- The use the equation derived earlier, solving for $TL$

\[
TL(r, f_0) = SL(f_0) - NL_s(f_0) + DI - 5 \cdot SNR_z + 5 \cdot \log\left(\frac{T}{\beta}\right)
\]

\[
= 125 - 65 + 13 - 10 \log(14) + 5 \log\left(\frac{100}{.5}\right)
\]

\[
= 78.8 \text{ dB}
\]

- Assuming spherical spreading with no absorption, the detection range is 8,700 meters.
Aside: Processing Gain Against Reverberation

• The spectrum of reverberation is very similar to that of the transmitted signal spectrum.
• For windowed tones, the spectrum is heavily concentrated around the transmit frequency (zero doppler shift). This is called the “reverb ridge”.
• So is the spectrum of the echo from a non-moving target. And target echo strength falls off faster than reverb strength.

=> Echoes of windowed tones from low-doppler targets are easily drowned out by reverberation.
• When the target is moving, the spectrum is shifted away from the concentrated reverb spectrum at zero doppler.
• It is easier to detect high doppler targets using tone pulses.
• Assume that sound backscattered by the ocean surface (called \textit{reverberation}) causes the noise or interference, against which we are trying to detect a target echo.
• The receiver for this example is simply one channel of the system shown in the figure on Slide 114.
Active Sonar: Echo Level

- The target echo level (in dB) is
  \[
  EL = SL - 2 \cdot TL(r) + TS
  \]

- Where \( TS \) is the target strength:
  \[
  TS \equiv \text{Target Strength} = \frac{\text{Scattered Power}}{\text{Incident Power}}
  \]
Active Sonar: Reverberation Level

- The reverberation level is

\[ RL(r) = SL - 2 \cdot TL(r) + S_s(\theta_g(r)) + 10\log\left(\frac{c\tau\gamma_B}{2}\right) \]

- Here \( \gamma_B \) is the effective, two-way **azimuthal beamwidth**, and

\[ \frac{c\tau}{2 \cdot r \cdot \gamma_B} \]

is the area of the surface which is insonified, called the **scattering patch**.
Active Sonar: Surface Scattering Strength

- The ocean surface scatters sound back toward the receiver. The amount scattered depends upon the grazing angle $\theta_g$, the surface condition, and the size of the insonified area. We have models for the scattered energy per unit surface area:

$$S_s = \text{surface scattering strength} = \frac{\text{scattered power}}{\text{unit surface area}}$$
Active Sonar: Signal to Interference Ratio

- Using the equations developed for $EL$ and $RL$, the target echo (desired signal) to reverberation (unwanted interference) level is called the signal to interference (SIR) level:

$$SIR = EL - RL = TS - S_s(\theta_g(r)) - 10\log\left(\frac{cR\gamma_B}{2}\right)$$

- Note that source level and transmission loss do not matter
Active Sonar: Example

• **Given:**

<table>
<thead>
<tr>
<th>System</th>
<th>10-ft-diameter baffled circular aperture</th>
</tr>
</thead>
<tbody>
<tr>
<td>Array</td>
<td>5 kHz</td>
</tr>
<tr>
<td>Transmit frequency</td>
<td>0.1 sec rectangular pulse</td>
</tr>
<tr>
<td>Pulse shape</td>
<td>+220 dB/ Pa at 1 m</td>
</tr>
<tr>
<td>Source level</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Target</th>
</tr>
</thead>
<tbody>
<tr>
<td>Target strength</td>
</tr>
<tr>
<td>Relative velocity</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Environment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Surface reverberation</td>
</tr>
</tbody>
</table>

• **Find:**
The detection range for $P_D = 0.5$ and $P_{FA} = 10^{-4}$
From Urick, Table 8.1 we get

\[ 10 \log(\gamma_B) = 10 \log \left( \frac{\lambda}{\pi D} \right) + 6.9, \quad \text{or} \quad \gamma_B = \frac{\lambda}{\pi D} \cdot 10^{-0.69} \]

From which \( \gamma_B = 0.156 \) radians.

The target echo and reverberation both fill the 0 Doppler frequency bin. The time-bandwidth product is 1. We get no advantage from a post-detection filter. Therefore,

\[ \text{SNR}_z = \text{SNR}_y^2 \]

where \( y \) is the filter output and \( z \) is the integrator output.
Active Sonar: Example (Cont’d)

- For $P_D = 0.5$ and $P_{FA} = 10^{-4}$, the graph below gives the required $\text{SNR}_z$ versus $T\beta$.

- Given $T\beta = 1$, we get that $\text{SNR}_z$ needs to be about 75.
Active Sonar: Example (Cont’d)

• Detection is achieved (with $P_D = 50\%$ and $P_{FA} = 10^{-4}$) when

$$\text{SIR} = EL - RL = 10 \log (\text{SNR}_y)$$

• Using the equation derived for SIR and solving for $r$ yields

$$10 \log (r) = TS - S_s - 10 \log \left( \frac{c \tau \gamma_B}{2} \right) - 10 \log (\text{SNR}_y)$$

$$= 10 - (-40) - 10 \log \left( \frac{1500 \cdot (0.1) \cdot (0.156)}{2} \right) - 9.4 = 29.9 \text{ dB}$$

or

$$r \approx 1000 \text{ m}.$$  

• Performance will be improved by decreasing $\tau$ or $\gamma_B$, but they must be large enough for the target to be insonified