Finite State Simulations and Bisimulations for Discrete-time Piecewise Affine System

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Introduction

Motivating Example: DC-DC Power Converter
Stability Analysis

- Construction of Lyapunov functions ⇒ common or multiple Lyapunov function approaches.

- Inherently conservative approaches:
  - If they fail ⇒ No alternative solution approach.

Reachability and Verification

- Abstraction of the infinite state hybrid system ⇒ Finite State Symbolic Models (Bisimulation).

- If a bisimulation is not obtained ⇒ No partial conclusion.
  - Some classes of hybrid systems admit bisimulation.
1. Construction of a nested sequence of symbolic models.
2. Analysis of stability for each symbolic model.

Our Approach

- Exploit the link between Lyapunov analysis and symbolic models.
- Using simulations for Nonconservative stability analysis.
- Existence of bisimulations.
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Introduction
Piecewise Affine (PWA) Systems

Discrete-time PWA System

\[ S = \{(A_1, b_1), \ldots, (A_N, b_N)\}. \quad D = \{D_1, \ldots, D_N\} \text{ a partition of } \mathbb{R}^n. \]

\[ x(t + 1) = A_{\theta(t)}x(t) + b_{\theta(t)} \quad \text{if } x(t) \in D_i. \]

State in terms of Initial state

Given a switching sequence \( \theta = (\theta(0), \theta(1), \ldots) \)

\[ x(t) = \Phi_\theta(t, t_0)x(t_0) + f_\theta(t, t_0) \]

where

\[ \Phi_\theta(t, t_0) = \begin{cases} I_n & \text{if } t = t_0 \\ A_{\theta(t-1)} \cdots A_{\theta(t_0)} & \text{if } t > t_0, \end{cases} \]

\[ f_\theta(t, t_0) = \sum_{s=t_0}^{t-1} \Phi_\theta(t, s + 1)b_{\theta(t)} \]
Introduction

Counterexample [Stanford & Urbano, 1994]

$x(t + 1) = \begin{cases} A_1x(t) & \text{if } x(t) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \text{ with } |x_1| > 3|x_2| \\ A_2x(t) & \text{otherwise,} \end{cases}$

$A_1 = \begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}.$

- With $c = (0.8)^{-1}$ and $\lambda = (0.8)^{1/6} \Rightarrow \|x(t)\| \leq c\lambda^t\|x(0)\|$.
- Small perturbation of state leads to divergence.
- There is no stabilizing switching sequence for this system [Stanford & Urbano, 1994].
- This examples motives us to use a stronger stability notion.
Introduction

Stability Definition

Stability of a Piecewise Affine system

Let $P \subseteq \mathbb{R}^n$. The PWA system is *uniformly exponentially stable on $P$* if there exist $c \geq 1$ and $\lambda \in (0, 1)$ such that, for all $\theta$ generated by initial states $x(0) \in P$

$$\|\Phi_\theta(t, t_0)\| \leq c\lambda^{t-t_0},$$

$$\|f_\theta(t, t_0)\| \leq c,$$

$$\|f_\theta(t, t_0)\| \rightarrow 0 \text{ as } t - t_0 \rightarrow \infty$$

Stability of a Switching Sequence

A set $\Theta$ is *uniformly exponentially stable* if there exist $c \geq 1$ and $\lambda \in (0, 1)$ such that for all $\theta \in \Theta$.

$$\|\Phi_\theta(t, t_0)\| \leq c\lambda^{t-t_0}$$
Introduction
Problem Statement: Our Approach

Stability Analysis Problem

For a given PWA system find the “biggest” subset $C$ of the state space $\mathbb{R}^n$ such that the PWA system is uniformly exponentially stable on $C$.

- Construction of a nested sequence of symbolic models:
  - Ignore stability analysis.
  - Obtain switching structure-preserving state-space partitions.
  - Construct a sequence of symbolic models for PWA system s.t.
    \[ G_0 \succeq G_1 \succeq G_2 \succeq \ldots \succeq \text{PWA} \]

- Construction of Lyapunov functions:
  - Choose a symbolic model.
  - Obtain all generated switching sequences.
  - Solve Lyapunov inequalities over all these switching sequences.
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A state-space partition and the associated directed graph ⇒ Simulation.
Symbolic models
Switching Sequences

Switching Sequences Generated by Graphs

- Choose node (11) which represents $D_{11} \in D_1 \Rightarrow (1, 1)$
- There is an edge from (11) to (12) $\Rightarrow (1, 2)$
- There is an edge from (12) to (21) $\Rightarrow (2, 1)$

Then $\theta = (1, 1, 2, 1, \ldots)$ is a *switching sequence generated by* (11) or $D_{11}$. 

![Diagram of switching sequences generated by graphs](image)
Stabilizing Switching Sequences

Let $\Theta(G_L)$ be the set of all switching sequences generated by $G_L$:

$$\Theta(G_0) \supseteq \Theta(G_1) \supseteq \ldots \supseteq \Theta(\text{PWA})$$

**Theorem [Lee and Dullerud, 2006]**

A set $\Theta$ of switching sequences is uniformly exponentially stable if and only if there exist a $M \geq 0$ and matrices $X_{i_1 \ldots i_M} \succ 0$ s.t.

$$A_{i_M}^T X_{i_1 \ldots i_M} A_{i_M} - X_{i_0 \ldots i_{M-1}} \prec 0 \quad (1)$$

for all $(i_0, \ldots, i_M) \in \{1, \ldots, N\}^{M+1}$ “occurring” in $\Theta$.

For some $L$, (1) is satisfied for $\Theta(G_L) \implies$ (1) is satisfied for $\Theta(\text{PWA})$. 
Stability Analysis

The PWA system is stable on $D_{(i_0,\ldots,i_L)}$ if for all $(j_0,\ldots,j_L)$ that are the nodes of irreducible components of $G_L$ reachable from the node $(i_0,\ldots,i_L)$

$$A_{j_0}^T X_{(j_1,\ldots,j_L)} A_{j_0} - X_{(j_0,\ldots,j_{L-1})} \prec 0, \quad X_{(j_0,\ldots,j_L)} \succ 0$$

and $b_{j_0} = \ldots = b_{j_L} = 0$.

- If $G_L$ is bisimulation $\Rightarrow$ iff.

- Two irreducible components: (11) and (21, 12). Only (21, 12) is reachable form (22).

- The PWA system is stable on $D_{22}$ if $A_2 A_1$ is stable and $b_1 = b_2 = 0$. 
Stability Analysis: Nonconservatism

Let $P_L$ denote the union of all stable cells in $\mathcal{D}_L$. Suppose for some $i$, $D_i$ is bounded, contains the origin in the interior, $A_i$ is stable and $b_i = 0$. Then the PWA system is not uniformly exponentially stable on any subset of $\mathbb{R}^n \setminus (\bigcup_{L=0}^{\infty} P_L)$.

$$\Theta(G_0) \supseteq \Theta(G_1) \supseteq \ldots \rightarrow \Theta(PWA)$$
Stability Analysis
Illustrative Example: Stability Analysis via Simulations

\[ A_1 = \begin{bmatrix} 0.5 & -1 \\ -1 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.25 & 0 \\ 1 & 0.5 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0.5 & 0 \\ 1 & 1.5 \end{bmatrix}, \quad b_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad b_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \]

- \( P_1 = D_2 \).
- No conclusion for \( D_1 \) and \( D_3 \).
Stability Analysis
Illustrative Example Cont’d: Stability Analysis via Simulations

$\mathcal{D}_1$

$\mathcal{D}_2$

- $P_1 = P_0 \cup D_{12} \cup D_{32}$.
- PWA is not stable on $D_{11}$.
- No conclusion for $D_{13}$ and $D_{33}$. 
Stability Analysis
Illustrative Example Cont’d: Stability Analysis via Simulations

\[ D_2 \]

\[ D_3 \]

\[ P_\infty \]
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Existence of Bisimulations
Discrete Transitions [Vladimerou et al., 2008]

**Discrete Transitions**

Let $\theta = (\theta(0), \theta(1), \ldots)$. If $\theta(t) \neq \theta(t + 1) \Rightarrow$ a discrete transition.

**Finite Number of Discrete Transitions**

Let $Q \subset \mathbb{R}^n$ be a bounded invariant polyhedron. Suppose:

(a) $\exists \epsilon > 0$ and $\phi \in \mathbb{R}^n$ s.t.

$$\phi^T (A_i x + b_i - x) \geq \epsilon \|A_i x + b_i - x\|$$

whenever $x \in D_i \cap Q$.

(b) $\exists a_- \text{ and } a_+ \in \mathbb{R}$ s.t.

$$a_- \leq \phi^T x \leq a_+$$

for all $x \in Q$. 

\[ \begin{align*}
D_1 &\quad x_1 \\
D_2 &\quad x_2 \\
D_3 &\quad x_3 \\
D_4 &\quad x_4 \\
\end{align*} \]
Existence of Bisimulations
Discrete Transitions [Vladimerou et al., 2008] Cont’d

Finite Number of Discrete Transitions
Cont’d

(c) \( \exists \gamma > 0 \) s.t.

\[
\| A_i x + b_i - x \| \geq \gamma
\]

whenever \( x \in D_i \cap Q \) and \( A_i x + b_i \in D_j \cap Q \) with \( i \neq j \).

Then

number of discrete transitions \( \leq \frac{a_+ - a_-}{\epsilon \gamma} \).
Existence of Bisimulations

Finite State Bisimulation

Existence of a Bisimulation

Suppose for a piecewise affine system

- number of discrete transitions ≤ \( K \).
- \( \exists \tilde{L} \) s.t. \( D(i,\ldots,i) \in D_-^{\tilde{L}} = \emptyset \) or \( D(i,\ldots,i) \in D_-^{\tilde{L}} = D(i,\ldots,i) \in D_-^{\tilde{L}+1} \).

Then \( \Theta(G_L) = \Theta(\text{PWA}) \) for some \( L \leq \tilde{L}K \).

- All switching sequences are eventually constant.
Existence of Bisimulations
Illustrative Example: Existence of Bisimulations

Let

\[ A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1/4 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1/2 & 0 \\ 1/4 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1/4 & 0 \\ 1 & 1/2 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \]

\[ b_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 3/2 \\ 1/4 \end{bmatrix}, \quad b_3 = \begin{bmatrix} 1/4 \end{bmatrix}, \quad b_4 = \begin{bmatrix} 0 \end{bmatrix}, \]

\[ D_1 = \left\{ [x_1 \ x_2]^T \in \mathbb{R}^2 : x_1 < -1 \right\}, \]

\[ D_2 = \left\{ [x_1 \ x_2]^T \in \mathbb{R}^2 : -1 \leq x_1 < 0 \right\}, \]

\[ D_3 = \left\{ [x_1 \ x_2]^T \in \mathbb{R}^2 : 0 \leq x_1 < 1 \right\}, \]

\[ D_4 = \left\{ [x_1 \ x_2]^T \in \mathbb{R}^2 : 1 \leq x_1 \right\}. \]

With \( \gamma = 1, \ \varepsilon = 0.25, \ \phi = [1 \ 1/2]^T, \ a_- = -5/2 \) and \( a_+ = 5/2 \)

number of discrete transitions \( \leq 20 \)
Existence of Bisimulations
Illustrative Example Cont’d: Existence of Bisimulations

- For some $L \leq 20$, $\Theta(G_L) = \Theta(\text{PWA})$.
- $\Theta(G_2) = \Theta(\text{PWA})$.

\[ D_0 \]
\[ D_2 \]
\[ D_3 \]

\[ G_0 \]
\[ G_2 \]
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Summary

- Link between Lyapunov analysis and symbolic models.
- Simulations for Nonconservative stability analysis.
- Sufficient conditions for existence of bisimulations.

Future Work

- Application to real-world examples.
- Controller synthesis.