A Computational Stability Analysis of Discrete-Time Piecewise Linear Systems

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Stability analysis problem for nonlinear systems is done by finding a Lyapunov function.

- Only sufficient conditions.
- Finding a suitable Lyapunov function is difficult.

**Approximating nonlinear systems**

In each region the nonlinear system is linearized around a nominal point.

\[
x(t + 1) = \begin{cases} 
  A_1 x(t) + b_1 & \text{if } x(t) \in D_1; \\
  \vdots \\
  A_N x(t) + b_N & \text{if } x(t) \in D_N, 
\end{cases}
\]

where \( \bigcup_{i=1}^{N} D_i = \mathbb{R}^n \).
Introduction
Motivating Applications

- The plant with each controller is a linear subsystem.
- Stability of each subsystem is not enough for the stability of the closed loop system.
- The closed loop system should be stable for all switching structures.

Supervisory Control

\[
\dot{x}(t) = Ax + Bu,
\]

\[
u(t) = \begin{cases} 
K_1x(t) & \text{if } x(t) \in D_1; \\
\vdots & \\
K_Nx(t) & \text{if } x(t) \in D_N,
\end{cases}
\]
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Introduction

Problem Formulation

### Piecewise linear system

Given \( A_1, \ldots, A_N \in \mathbb{R}^{n \times n} \), define \( \mathcal{A} = \{ A_1, \ldots, A_N \} \). Also let \( \mathcal{D} = \{ D_1, \ldots, D_N \} \) be a partition of \( \mathbb{R}^n \) into a number of polyhedral cells; i.e. \( \bigcup_{i=1}^{N} D_i = \mathbb{R}^n \) and \( D_i \cap D_j = \emptyset \) whenever \( i \neq j \). Then the pair \( (\mathcal{A}, \mathcal{D}) \) defines the \textit{discrete-time piecewise linear system} represented by

\[
x(t + 1) = A_{\theta(t)} x(t)
\]

for \( x(0) \in \mathbb{R}^n \) and \( t = 0, 1, \ldots \), where \( \theta(t) = i \) whenever \( x(t) \in D_i \).

### Stability notion

Let \( C \subset \mathbb{R}^n \). \( (\mathcal{A}, \mathcal{D}) \) is said to be \textit{uniformly exponentially stable} on \( C \) if there exist \( c \geq 1 \) and \( \lambda \in (0, 1) \) such that

\[
\| x(t) \| \leq c \lambda^{t-t_0} \| x(t_0) \|
\]

for all \( x(t_0) \in C \) and for all \( t, t_0 \in \{0, 1, \ldots \} \) with \( t \geq t_0 \).
Introduction

Problem Formulation

- **Common Lyapunov function approach** [H. P. Horisberger and P. R. Belanger, 1976; B. R. Barmish, 1985; S. Boyd and Q. Yang, 1989; etc]:
  
  Used for quadratic stabilizability of uncertain systems $\Rightarrow$ Quadratic Stability.

- **Multiple Lyapunov function approach** [M. S. Branicky, 1994]:
  
  First used for stability analysis of switched and hybrid systems $\Rightarrow$ Piecewise quadratic Stability.

  - Piecewise quadratic (PWQ) Lyapunov functions to deal with performance analysis and optimal control problems [Johansson and Rantzer, 1998]
  
  - Lyapunov function with basis functions for discrete-time systems
    
    $V(x) = x^T P_i x$ where $P_i(x) = \sum_{j=0}^{N} P_{i(j)} \rho_{i(j)}(x)$ [Ferrari-Trecate et al., 2002]

    **Only sufficient conditions**
Stability analysis problem

For a given \((A, D)\) find the “biggest” subset \(C\) of the state space \(\mathbb{R}^n\) such that \((A, D)\) is uniformly exponentially stable on \(C\).

Remark: Stability under state-dependent switching: Common or Multiple Lyapunov functions \(\Rightarrow\) Only sufficient conditions for stability.

Separation approach

Two subproblems are solved separately.

- Characterization of stabilizing sets of switching sequences for \(A\) (without concerning switching structure).
- Generating switching structure–preserving state-space partitions for \((A, D)\) (without concerning stability).
Subproblem I

Stability of a Switching Sequences

A set $\Theta \in \{1, \ldots, N\}^\infty$ of switching sequences is said to be \textit{uniformly exponentially stabilizing} for $A$ if there exist $c \geq 1$ and $\lambda \in (0, 1)$ such that $x(t + 1) = A_{\theta(t)} x(t)$ satisfies
\[
\|x(t)\| \leq c \lambda^{t-t_0} \|x(t_0)\|
\]
over all $t_0 \geq 0$, $t \geq t_0$, $x(t_0) \in \mathbb{R}^n$, and all switching sequences $\theta \in \Theta$.

- For integers $L \geq 0$, $(i_0, \ldots, i_L)$ is called an $L$-path if $i_0, \ldots, i_L \in \{1, \ldots, N\}$.

Admissible Sets of L-Paths

A subset $\mathcal{N}$ of $\{1, \ldots, N\}^{L+1}$ is said to be admissible if each switching path in $\mathcal{N}$ leads to itself via switching paths in $\mathcal{N}$; e.g., $\mathcal{N} = \{(1, 1), (1, 2), (2, 1)\}$.
Subproblem I Cont’d

### \( \mathcal{A} \)-Admissible Sets

Let \( \mathcal{N} \) be an admissible set of \( L \)-paths. It is said to be an \( \mathcal{A} \)-admissible set of \( L \)-paths if there exist matrices \( X_j > 0 \) such that

\[
A_{i_L}^T X_{i_1 \ldots i_L} A_{i_L} - X_{i_0 \ldots i_{L-1}} < 0
\]

for all \((i_0, \ldots, i_L) \in \mathcal{N}\).

### Minimal and Maximal Sets

- Let \( \mathcal{N} \) be an admissible set. If the only admissible set \( \hat{\mathcal{N}} \) satisfying \( \hat{\mathcal{N}} \subset \mathcal{N} \) is \( \mathcal{N} \) itself, then \( \mathcal{N} \) is said to be a minimal set of \( L \)-paths.
- If \( \mathcal{N} \) is minimal and \( \mathcal{A} \)-admissible then it is said to be \( \mathcal{A} \)-minimal.
- Let \( \mathcal{N} \) be an \( \mathcal{A} \)-admissible set. If the only \( \mathcal{A} \)-admissible set \( \hat{\mathcal{N}} \) satisfying \( \mathcal{N} \subset \hat{\mathcal{N}} \) is \( \mathcal{N} \) itself, then \( \mathcal{N} \) is said to be \( \mathcal{A} \)-maximal.
Subproblem I Cont’d

Remark

- Finding an $\mathcal{A}$-minimal set of $L$-paths for some $L \Rightarrow$ Identifying a stabilizing periodic switching sequence.
- If $\mathcal{N}$ is $\mathcal{A}$-admissible set of $L$-paths then $\mathcal{N}$ yields a set $\Theta$ of switching sequences

$$\Theta = \{ (\theta(0), \theta(1), \ldots) : (\theta(t), \ldots, \theta(t+L)) \in \mathcal{N}, t = 0, 1, \ldots \}$$

which is uniformly exponentially stable.
- Finding $\mathcal{A}$-maximal sets of $L$-paths over all $L \Rightarrow$ Identifying all uniformly stabilizing sets of switching sequences.
**Graph Representation**

Let $G = (V, E)$ be a complete directed graph with $V = \{1, \ldots, N\}$ and $E = \{1, \ldots, N\}^2$. Then $G_L = (V_L, E_L)$ for $L = 0, 1, \ldots$ is constructed recursively as following:

- Put $V_0 = V$, $E_0 = E$ and $G_0 = G$.
- The vertices of $G_L$ are the edges of $G_{L-1}$ and the edges of $G_L$ are the possible paths of length $L + 1$. 

![Graph Representation Diagram]

$G_0$ and $G_1$ illustrate the graph representations for $L = 0$ and $L = 1$, respectively.
Elementary cycles and minimal sets

For integers $L > 0$ there is one to one correspondence between the family of minimal sets of $L$-paths and the family of elementary cycles in $G_{L-1}$.

**Remark 1:** If for a $K > 0$ the sequence $(i_0, \ldots, i_{L-1}), (i_1, \ldots, i_L), \ldots, (i_K, \ldots, i_{K+L-1})$ forms an elementary cycle in $G_{L-1}$ then the set

$$ \{(i_0, \ldots, i_L), \ldots, (i_{K-1}, \ldots, i_{K+L-1})\} $$

is a minimal set of $L$-paths.

**Remark 2:** Johnson’s algorithm is used to produce the elementary cycles.
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Subproblem II

**L-path partition of state space**

- \( \mathcal{D}_0 = \{D_1, \ldots, D_N\} \)
- Define \( D_{(i_0, \ldots, i_L)} \subset \mathbb{R}^n \) recursively by
  \[
  D_{(i_0, \ldots, i_L)} = \{ x \in D_{(i_0, \ldots, i_{L-1})} : A_{i_0} x \in D_{(i_1, \ldots, i_L)} \}
  \]
  for all \( L \) and \( (i_0, \ldots, i_L) \in \{1, \ldots, N\}^{L+1} \).
- The switching structure preserving *L-path partition* of \( \mathbb{R}^n \):
  \[
  \mathcal{D}_L = \left\{ D_{(i_0, \ldots, i_L)} : (i_0, \ldots, i_L) \in \{1, \ldots, N\}^{L+1} \right\}.
  \]
The interior of the cell $D_{(i_0, \ldots, i_L)}$ is defined to be the set of all $x \in \mathbb{R}^n$

$$E(i_0, \ldots, i_L)x + e(i_0, \ldots, i_L) < 0.$$  

For constructing $D_{(i_0, \ldots, i_L, i_{L+1})}$, the following set of inequalities should be solved:

$$E(i_0, \ldots, i_L)x + e(i_0, \ldots, i_L) < 0,$$

$$E(i_1, \ldots, i_{L+1})A_{i_0}x + e(i_1, \ldots, i_{L+1}) < 0.$$  

Some of the inequalities are redundant which can be removed by linear programming.

### Removing Redundant Inequalities

Check whether $s^T x \leq t$ is redundant in the system of inequalities $Ex \leq e$. For $f(x) = s^T x$

$$\max f(x)$$

subject to $Ex \leq e$, $s^T x \leq t + 1$,

If $x^*$ is a feasible solution and $f(x^*) \leq t$ then $s^T x \leq t$ is redundant.
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switching sequences generated by cells

- Choose $D_{11} \in \mathcal{D}_1$ with nonempty interior $\Rightarrow (1, 1)$
- Choose $D_{112} \in \mathcal{D}_2$ with nonempty interior $\Rightarrow (1, 2)$
- Choose $D_{121} \in \mathcal{D}_2$ with nonempty interior $\Rightarrow (2, 1)$...

- $\theta = (1, 1, 2, 1, \ldots)$ is a switching sequence generated by $D_{11}$ and $\mathcal{D}_2$.
- $\theta_L = ((1, 1), (1, 2), (2, 1), \ldots)$ is a chain of $L$-paths generated by $D_{11}$ and $\mathcal{D}_2$ for $L = 1$.

Theorem

Let $D_{(i_0, \ldots, i_L)} \in \mathcal{D}_L$ be nonempty. If each chain of $L$-paths generated by $D_{(i_0, \ldots, i_L)}$ and $\mathcal{D}_{L+1}$ has a limit set that is contained in $\mathcal{A}$-maximal set of $L$-paths then $(\mathcal{A}, \mathcal{D})$ is uniformly exponentially stable on $D_{(i_0, \ldots, i_L)}$. 

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Proposed Algorithm

Generate $C_0 \subset C_1 \subset \ldots$ s.t. for each $L$, $(\mathcal{A}, \mathcal{D})$ is $C_L$-uniformly exponentially stable.

0. Set $C_{-1} = \emptyset$ and $L = 0$.
1. Obtain $\mathcal{D}_{L+1}$.
2. Generate all $\mathcal{A}$-maximal sets of $L$-paths.
   Computational cost of this step is high $\Rightarrow$ Can be avoided for special cases which will be introduced at the end of the talk.
3. Let $C_L$ be the union of $C_{L-1}$ and all $D_{(i_0, \ldots, i_L)}$ such that each chain of $L$-paths generated by $D_{(i_0, \ldots, i_L)}$ and $\mathcal{D}_L$ has a limit set contained in an $\mathcal{A}$-maximal set of $L$-paths.
4. If $\mathcal{D}_{L+1} = \mathcal{D}_L$, then stop. Otherwise increment $L$ to $L + 1$ and go to Step 1.
Example I

\[ A_1 = \begin{bmatrix} -0.5 & 0 \\ -1.5 & -0.5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.5 & -1 \\ 0 & 1.5 \end{bmatrix}, \]

\[ D_1 = \{ [x_1 \ x_2]^T \in \mathbb{R}^2 : x_1 \leq 1 \}, \]

\[ D_2 = \{ [x_1 \ x_2]^T \in \mathbb{R}^2 : x_1 > 1 \}, \]

- **L = 0**: \( C_0 = \emptyset \).
- **L = 1**: \( D_{11} = D_{111} \Rightarrow \) The generated chain \( (11, 11, \ldots) \). \( \{11\} \) is the \( A \)-maximal set \( \Rightarrow C_1 = D_{11} \).
• $L = 2$: $D_{211} = D_{2111} \Rightarrow x(1) \in D_{111}$ for $x(0) \in D_{211}$. 
  \{111, 112, 122, 221, 211, 212\} is the $A$-maximal set. 
  $\Rightarrow C_2 = D_{11} \cup D_{211}$.

• $L = 3$: $D_{1211} = D_{12111}$ and $D_{2111} = D_{21111} \Rightarrow x(2) \in D_{1111}$ for 
  $x(0) \in D_{1211}$. $A$-maximal sets are: 
  \{1111, 1112, 1121, 1122, 1212, 1221, 2111, 2112, 2122, 2211, 2212\}, 
  \{1111, 1112, 1121, 1211, 1212, 1221, 2111, 2112, 2121, 2122, 2211, 2212\}. 
  $\Rightarrow C_3 = C_4 = \cdots = D_{11} \cup D_{211} \cup D_{1211}$.
Example II

\[ A_1 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \ A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \]

\[ D_1 = \{ [x_1 \ x_2]^T \in \mathbb{R}^2 : x_1 \leq 1 \}, \]

\[ D_2 = \{ [x_1 \ x_2]^T \in \mathbb{R}^2 : x_1 > 1 \}, \]

- \( L = 0 \): \( C_0 = \emptyset \).
- \( L = 1 \): \( D_{12} = D_{121}, \ D_{21} = D_{211} \). \( \mathcal{A} \)-maximal set is \( \{12, 21\} \)
  \( D_{11} = D_{111} \cup D_{112} \)

The limit sets are not in \( \mathcal{A} \)-maximal set \( \Rightarrow \) No conclusion.
Example II Cont’d

• $L = 2$: $D_{112} = D_{1121}, D_{121} = D_{1211}$ and $D_{211} = D_{2111}$.  
  $\mathcal{A}$-maximal set is $\{112, 121, 122, 211, 212, 221\}$  
  $D_{111} = D_{111} \cup D_{112}$  
  The limit sets are not in $\mathcal{A}$-maximal set $\Rightarrow$ No conclusion.

• $D_{11...1}$ is divided to $D_{11...11}$ and $D_{11...12}$ indefinitely.

• $\{11...1\}$ is not in $\mathcal{A}$-maximal sets for any path-length $L$.

• $C_L = \emptyset$ for all $L \geq 0$
Summary

- A computational tool to analyze the stability of piecewise linear system without incurring conservatism.
- Solving two separate subproblems and combining them for stability analysis.

Questions

- Is the “biggest” $C$ obtained when $D_L = D_{L+1}$?
- Is it possible to obtain the “biggest” $C$ even if $D_L \neq D_{L+1}$ for any $L$?
- Under what condition, $C = \lim_{L \to \infty} C_L$ is the “biggest”?