Path-by-Path Optimal Control of Switched and Markovian Jump Linear Systems

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Joint work with G. E. Dullerud (UIUC) and P. P. Khargonekar (UF)
Flight Modes of Migrating Raptors

- **Cruise** (straight line flight):
  \[ x(t + 1) = f_1(x(t), w(t)) \]
- **Thermalling** (circling flight):
  \[ x(t + 1) = f_2(x(t), w(t)) \]

State-Space Representation

- **State Equation**: \[ x(t + 1) = f_{\theta(t)}(x(t), w(t)) \]
- **Switching Sequences**: \( \theta = (\theta(0), \theta(1), \ldots) \in \Theta \subset \{1, 2\}^{\infty} \)

- **Filtering Problem**: Track flying birds under arbitrary switching between flight modes.
Introduction
Motivating Example: Markovian Jump Systems

Macroeconomic Modes

- **Fast Growth:**
  \[ x(t + 1) = f_1(x(t), u(t), w(t)) \]
- **Slow Growth:**
  \[ x(t + 1) = f_2(x(t), u(t), w(t)) \]

State-Space Representation

- **State Equation:**
  \[ x(t + 1) = f_{\theta(t)}(x(t), u(t), w(t)) \]
- **Markov Switching:**
  \[ P(\theta(t + 1) = j | \theta(t) = i) = p_{ij} \]

- **Control Problem:** Use optimal monetary policy to control unemployment, inflation, etc.
Introduction
Control Theoretic Issues

Finite-Horizon Problems

■ Emphasis on short-term optimization
■ Not concerned with stability

Infinite-Horizon Problems

■ Emphasis on long-term planning and stability
■ Looks too much ahead. ⇒ conservative

Our Goals

■ Optimize finite-horizon performance.
■ Maintain stability.
### Summary of Contributions

#### Previous Work

**Receding-Horizon Control**

- Short-term optimization vs. long-term planning tradeoff
- Long control horizon required for stability
- Restricted to mode-dependent controllers

**Discrete-Time Switched/Markovian Jump Linear Systems**

- Convex programming–based exact analysis and synthesis
- Path-dependent controllers
- Restricted to worst-case/long-term average performance optimization
Summary of Contributions
This Work

Receding-Horizon-Type Control

- No restriction to control horizon for stability
- Path-dependent controllers
- Offline optimization

Discrete-Time Switched/Markovian Jump Linear Systems

- Convex programming–based exact analysis and synthesis
- Path-by-path Pareto-optimal performance for switched systems
- Short-term average performance optimization for Markovian jump systems
Definitions
State-Space Representations of Plant and Controller

Switched Linear Plant

\[
\begin{align*}
x(t + 1) &= A_{\theta(t)}x(t) + B_{1,\theta(t)}w(t) + B_{2,\theta(t)}u(t), \\
z(t) &= C_{1,\theta(t)}x(t) + D_{11,\theta(t)}w(t) + D_{12,\theta(t)}u(t), \\
y(t) &= C_{2,\theta(t)}x(t) + D_{21,\theta(t)}w(t)
\end{align*}
\]

Path-Dependent Linear Dynamic Output Feedback Controller

\[
\begin{align*}
x_K(t + 1) &= A_{K,(\theta(t-L),...,\theta(t))}x_K(t) + B_{K,(\theta(t-L),...,\theta(t))}y(t), \\
u(t) &= C_{K,(\theta(t-L),...,\theta(t))}x_K(t) + D_{K,(\theta(t-L),...,\theta(t))}y(t),
\end{align*}
\]

where \( L \in \{0, 1, \ldots \} \) is TBD.
**Definitions**

**Closed-Loop Systems**

**Closed-Loop State**

\[ \tilde{x}(t) = [x(t) \ x_K(t)]^T \]

**Closed-Loop Modes**

\[ \theta_L(t) = \begin{cases} (0, \ldots, 0, \theta(0), \ldots, \theta(t)) & \text{if } t < L; \\ \text{\(L-t\) times} \\ (\theta(t-L), \ldots, \theta(t)) & \text{if } t \geq L. \end{cases} \]

**Closed-Loop System**

\[ \begin{align*} \tilde{x}(t+1) &= \tilde{A}_{\theta_L(t)} \tilde{x}(t) + \tilde{B}_{\theta_L(t)} w(t), \\
z(t) &= \tilde{C}_{\theta_L(t)} \tilde{x}(t) + \tilde{D}_{\theta_L(t)} w(t) \end{align*} \]
Definitions
Stability

### Uniform Exponential Stability of Switched Systems
Closed-loop system is uniformly exponentially stable if, whenever \( w = 0 \), there exist \( c > 0 \) and \( \lambda \in (0, 1) \) s.t.
\[
\|\tilde{x}(t)\| \leq c \lambda^{t-s} \|\tilde{x}(s)\|
\]
for \( t \geq s \), \( \tilde{x}(s) \in \mathbb{R}^{n+nK} \), \( \theta \in \Theta \).

### Almost Sure Uniform Exponential Stability of Markovian Jump Systems
Closed-loop system is a.s. uniformly exponentially stable if, whenever \( w = 0 \), there exist \( c > 0 \) and \( \lambda \in (0, 1) \) s.t., with probability one,
\[
\|\tilde{x}(t)\| \leq c \lambda^{t-s} \|\tilde{x}(s)\|
\]
for \( t \geq s \), \( \tilde{x}(s) \in \mathbb{R}^{n+nK} \).
Path-By-Path Performance of Switched Systems

Closed-loop system satisfies $T$-step path-by-path performance levels $\gamma(i_0,\ldots,i_T) > 0$ if, whenever $\tilde{x}(0) = 0$ and $w$ is white,

$$\frac{1}{T+1} \sum_{t=s}^{s+T} \mathbb{E} \|z(t)\|^2 < \gamma^2(\theta(s),\ldots,\theta(s+T)), \quad \forall s \geq 0, \forall \theta \in \Theta.$$

Finite-Step Average Performance of Markovian Jump Systems

Closed-loop system satisfies $T$-step average performance level $\gamma > 0$ if, whenever $\tilde{x}(0) = 0$ and $w$ is white,

$$\frac{1}{T+1} \sum_{t=s}^{s+T} \mathbb{E} \|z(t)\|^2 < \gamma^2, \quad \forall s \geq 0.$$
Uniform Exponential Stability of Switched Systems

Switched system \( x(t+1) = A_{\theta(t)}x(t), \theta \in \Theta \), is uniformly exponentially stable iff there exist \( M \geq 0 \) and \( Y_{(i_1,...,i_M)} > 0 \) s.t.

\[
A_{i_M}Y_{(i_0,...,i_{M-1})}A_{i_M}^T - Y_{(i_1,...,i_M)} < 0
\]

for all \((i_0, \ldots, i_M)\) “occurring” in \( \Theta \).

- **\( M = 0 \)** (Common Lyapunov function):
  \[
  A_{\theta(t)}YA_{\theta(t)}^T - Y < 0, \quad \forall t \geq 0
  \]

- **\( M = 1 \)** (Mode-dependent Lyapunov function):
  \[
  A_{\theta(t)}Y_{\theta(t-1)}A_{\theta(t)}^T - Y_{\theta(t)} < 0, \quad \forall t \geq 0
  \]

- **\( M = 2 \)** (Path-dependent Lyapunov function):
  \[
  A_{\theta(t)}Y_{(\theta(t-2),\theta(t-1))}A_{\theta(t)}^T - Y_{(\theta(t-1),\theta(t))} < 0, \quad \forall t \geq 0
  \]
Admissible Switching Sequences of Markovian Jump Systems

- \( P = (p_{ij}) \), transition probability matrix
- \( p = (p_i) \), initial probability distribution
- \( \Theta(P, p) = \{ \theta : p_{\theta(0)} > 0, \ p_{\theta(t)}\theta(t+1) > 0 \} \)

Almost Sure Uniform Exponential Stability of Markovian Jump Systems

Markovian jump system \( x(t+1) = A_{\theta(t)}x(t) \) is a.s. uniformly exponentially stable iff there exist \( M \geq 0 \) and \( Y(i_1, \ldots, i_M) > 0 \) s.t.

\[
A_{i_M} Y(i_0, \ldots, i_{M-1}) A_{i_M}^T - Y(i_1, \ldots, i_M) < 0
\]

for all \((i_0, \ldots, i_M)\) “occurring” in \( \Theta(P, p) \).
Path-by-Path Performance of Switched Linear Systems

The switched system
\[
\begin{align*}
    x(t+1) &= A_{\theta(t)}x(t) + B_{\theta(t)}w(t), \\
    z(t) &= C_{\theta(t)}x(t) + D_{\theta(t)}w(t)
\end{align*}
\]

is uniformly exponentially stable and satisfies \( T \)-step path-by-path performance levels \( \gamma(i_0,\ldots,i_T) > 0 \) iff there exist \( M \geq 0 \) and \( Y(j_1,\ldots,i_M) > 0 \) s.t.

\[
A_{i_M}Y(i_0,\ldots,i_{M-1})A_{i_M}^T - Y(i_1,\ldots,i_M) < -B_{i_M}B_{i_M}^T,
\]

\[
\frac{1}{T+1} \sum_{t=M}^{M+T} \text{tr} \left( C_{it}Y(i_{t-M},\ldots,i_{t-1})C_{it} + D_{it}D_{it}^T \right) < \gamma^2(i_M,\ldots,i_{M+T})
\]

for all \( (i_0,\ldots,i_{M+T}) \) “occurring” in \( \Theta \).
Results
Performance Analysis

Finite-Step Average Performance of Markovian Jump Linear Systems

Let $\mathbf{p} = \mathbf{pP}$. The Markovian jump system is a.s. uniformly exponentially stable and satisfies $T$-step average performance level $\gamma > 0$ iff there are $\gamma(i_0, \ldots, i_T) > 0$ s.t.

$$\mathbf{A}_{i_M} \mathbf{Y}(i_0, \ldots, i_{M-1}) \mathbf{A}_{i_M}^T - \mathbf{Y}(i_1, \ldots, i_M) < -\mathbf{B}_{i_M} \mathbf{B}_{i_M}^T,$$

$$\frac{1}{T+1} \sum_{t=M}^{M+T} \text{tr} \left( \mathbf{C}_{i_t} \mathbf{Y}(i_{t-M}, \ldots, i_{t-1}) \mathbf{C}_{i_t} + \mathbf{D}_{i_t} \mathbf{D}_{i_t}^T \right) < \gamma^2(i_{M}, \ldots, i_{M+T})$$

for all $(i_0, \ldots, i_{M+T})$ “occurring” in $\Theta(\mathbf{P}, \mathbf{p})$, and s.t.

$$\sum_{(i_0, \ldots, i_T) \text{ “occurring” in } \Theta(\mathbf{P}, \mathbf{p})} \pi(i_0, \ldots, i_T) \gamma^2(i_0, \ldots, i_T) \leq \gamma^2$$

with $T$-step probabilities $\pi(i_0, \ldots, i_T) = p_{i_0} p_{i_0 i_1} \cdots p_{i_{T-1} i_T}$. 
Synthesis Condition for Switched Linear Systems

There exists a controller s.t. the closed-loop system is uniformly exponentially stable and satisfies $T$-step path-by-path performance levels $\gamma(i_0,\ldots,i_T) > 0$ iff there exist $M \geq 0$, $R_{(j_1,\ldots,j_M)} > 0$, $S_{(j_1,\ldots,j_M)} > 0$, $Z_{(j_0,\ldots,j_M)} > 0$, and $W_{(j_0,\ldots,j_M)}$ s.t.

$$H(i_0,\ldots,i_M) + F_{i_M}^T W(i_0,\ldots,i_M) G_{i_M} + G_{i_M}^T W_{(i_0,\ldots,i_M)} F_{i_M} < 0,$$

$$\hat{H}(i_0,\ldots,i_M) + \hat{F}_{i_M}^T W(i_0,\ldots,i_M) \hat{G}_{i_M} + \hat{G}_{i_M}^T W_{(i_0,\ldots,i_M)} \hat{F}_{i_M} < 0,$$

$$\frac{1}{T+1} \sum_{t=M}^{M+T} \text{tr} Z_{(i_t-M,\ldots,i_t)} < \gamma^2_{(i_M,\ldots,i_{M+T})}$$

for all $(i_0,\ldots,i_{M+T})$ “occurring” in $\Theta$, where $F_i$, $G_i$, $\hat{F}_i$, $\hat{G}_i$ are const. and $H_{\ldots}$, $\hat{H}_{\ldots}$ are linear in $R_{\ldots}$, $S_{\ldots}$, $Z_{\ldots}$, and $W_{\ldots}$. 
Results
Controller Synthesis

- **Synthesis Condition for Markovian Jump Linear Systems:**
  Similar

- **Recovery of Controller Coefficients:** Straightforward
  [Scherer et al. (1997); Masubuchi et al. (1998)]

- **Path-Dependent Controller:**

  \[
  x_K(t + 1) = A_K,\theta(t-L),...,\theta(t))x_K(t) + B_K,\theta(t-L),...,\theta(t))y(t),
  \]

  \[
  u(t) = C_K,\theta(t-L),...,\theta(t))x_K(t) + D_K,\theta(t-L),...,\theta(t))y(t),
  \]

  where one can take \( L = M \) and \( x_K(t) \in \mathbb{R}^n \).
Numerical Illustration
A Markovian Jump Linear Plant

- **Mode 1:**
  \[ x(t + 1) = 0.5x(t) + [1 \ 0]w(t), \quad y(t) = x(t) + [1 \ 0]w(t) \]

- **Mode 2:**
  \[ x(t+1) = x(t) + [1 \ 0]w(t) + u(t), \quad y(t) = x(t) + [0 \ 1]w(t) \]

- **Mode 3:**
  \[ x(t + 1) = 0.5x(t) + [1 \ 0]w(t), \quad y(t) = x(t) + [0 \ 1]w(t) \]

- **Common Output to Regulate:**
  \[ z(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t) \]

- **Transition Probability Matrix:**
  \[ P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

- **Initial Probability Distribution:**
  \[ p = \begin{bmatrix} 1/3 \\ 1/6 \\ 1/2 \end{bmatrix} \Rightarrow p = pP \]
Numerical Illustration
Numerical Solution

- **Infimum** $T$-step average performance level $\gamma$ for path length $M$:

<table>
<thead>
<tr>
<th>$M$</th>
<th>$T = 0$</th>
<th>$T = 2$</th>
<th>$T = 4$</th>
<th>$T = 6$</th>
<th>$T = 8$</th>
<th>$T = 10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.3280</td>
<td>1.3280</td>
<td>1.3280</td>
<td>1.3280</td>
<td>1.3280</td>
<td>1.3280</td>
</tr>
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<td>1.1837</td>
<td>1.1837</td>
<td>1.1837</td>
<td>1.1837</td>
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</tr>
<tr>
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<td>1.1824</td>
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</tr>
<tr>
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<td>1.1829</td>
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</tr>
<tr>
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<td>1.1828</td>
<td>1.1822</td>
<td>1.1819</td>
<td>1.1817</td>
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</tr>
<tr>
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<td>1.1837</td>
<td>1.1828</td>
<td>1.1822</td>
<td>1.1819</td>
<td>1.1817</td>
<td>1.1817</td>
</tr>
</tbody>
</table>

- **Consistency**: Infimum $\gamma$ converges to the optimal infinite-horizon performance level 1.1808 as $T$ increases with $M = 4$.

- **Controller**: First-order controller whose coefficients at time $t$ depend on $(\theta(t - 4), \ldots, \theta(t))$. 
Conclusions
Optimal Control of Switched and Markovian Jump Linear Systems

Path-By-Path Optimal Control

- $T$-step path-by-path performance for switched systems
- $T$-step average performance for Markovian jump systems
- Guaranteed stability for all forward lengths $T$ and for sufficiently large, often small, backward lengths $M$

Special Cases

- If $N > 1$, as $T \to \infty$, $T$-step avg. performance $\to$ infinite-horizon LQG performance
- If $N = 1$ (i.e., LTI case), then $T$-step path-by-path performance $\to$ infinite-horizon LQG performance for all $T$

**Question**: What are good applications?