OPTIMAL DISTURBANCE ATTENUATION FOR DISCRETE-TIME SWITCHED AND MARKOVIAN JUMP LINEAR SYSTEMS

JI-WOONG LEE† AND GEIR E. DULLERUD‡

Abstract. An exact condition for uniform stabilization and disturbance attenuation for switched linear systems is given in the discrete-time domain via the union of an increasing family of linear matrix inequality conditions. Associated with each Markovian jump linear system is a switched linear system, so we obtain a necessary and sufficient condition for almost sure uniform stabilization and disturbance attenuation for Markovian jump linear systems as well. The results lead to semidefinite programming–based controller synthesis techniques, from which optimal finite-path-dependent linear dynamic output feedback controllers arise naturally. In particular, under the notion of path-by-path optimal disturbance attenuation, finite-path-dependent controllers can outperform the usual mode-dependent ones.

Key words. discrete linear inclusions, dynamic output feedback, \(H^\infty\) control, linear matrix inequalities, linear time-varying systems, semidefinite programming

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1. Introduction. The switched linear system is defined as a family of linear time-varying systems whose parameters vary within a single finite set, and it serves as an abstraction of hybrid systems, where continuous system dynamics and discrete events coexist and depend on each other in complex ways [31]. On the other hand, the Markovian jump linear system is a linear system whose parameters jump according to the state transitions of a finite-state Markov chain, and it typically arises in the context of networked control systems, where the feedback loop is subject to random delays [32]. The parameters of these systems are indexed, where the indices are called the modes of the systems. This paper focuses on the discrete-time domain and considers the problems of uniform disturbance attenuation for switched linear systems and almost sure uniform disturbance attenuation for Markovian jump linear systems, where the uniformity refers to the stability and \(\ell^2\)-gain of the sequences of modes, called switching sequences, that are admissible. We develop semidefinite programming formulations [41] for the solutions to these problems, and their generalizations, without any assumption on the parameters or admissible switching sequences.

The most fundamental problem of discrete-time switched linear systems is determining the asymptotic stability of the discrete linear inclusion (i.e., the family of linear time-varying systems over all infinite sequences of parameters in a given finite set) [12, 21]. This problem basically involves checking the stability of all possible (uncountably many) infinite matrix products due to the fact that the finiteness conjecture (i.e., the conjecture that it suffices to check the stability of periodic products) is generally false [8]. The problem is hence considered undecidable [7]; more precisely,
it is considered semidecidable (i.e., there exists an algorithm that is guaranteed to correctly determine, after a finite amount of computation, the stability of switched linear systems that are indeed stable) [6]. Recently, the authors gave a restatement of the result of [6] and introduced an increasing family of linear matrix inequality conditions whose union characterizes the uniform stability and stabilizability of discrete-time switched linear systems [30]. This result generalizes the multiple Lyapunov function approaches that have provided the most useful tools for stability analysis [9].

This paper complements the previous work on uniform stabilization described above and extends it to disturbance attenuation. The problem of disturbance attenuation reduces to that of $\mathcal{H}_\infty$ control of linear time-invariant systems [16, 26] if the given set of system parameters is a singleton. On the other hand, if the family of admissible switching sequences is a singleton, the problem reduces to that of $\ell^2$-induced norm minimization for linear time-varying systems [36, 4], where the set of possible parameter values is finite. Therefore, our work draws on the results in the $\mathcal{H}_\infty$ control and $\ell^2$-gain minimization literature. In particular, from the linear operator inequality approach for analyzing linear time-varying systems [17], we deduce that the Riccati difference inequality associated with any stable and contractive linear time-varying system admits a uniformly stabilizing solution that has a finite memory of past parameters. This finite-memory property leads to our analysis results. On the other hand, our controller synthesis results are based on a straightforward extension of the linear matrix inequality approach originally developed for linear time-invariant systems [33, 20].

We relax the standard restriction to mode-dependent controllers (i.e., controllers that perfectly observe the present mode of the system but do not recall past modes) and consider finite-path-dependent controllers (i.e., controllers that not only perfectly observe the present mode but also have a finite memory of past modes). This relaxation, along with the aforementioned finite-memory property, enables us to derive a complete characterization of the existence of finite-path-dependent linear dynamic output feedback controllers that stabilize the switched linear system and achieve a uniform disturbance attenuation level. This characterization is given in terms of the “union” of an increasing family of systems of linear matrix inequality conditions. Although our result inherits the semidecidable nature of the underlying problem, this limitation is not likely to pose difficulties in practice, as examples show that it usually suffices to check the feasibility of the first few systems of linear matrix inequalities from an increasing family. Moreover, the result is amenable to the standard linear matrix inequality–based controller synthesis technique, from which admissible finite-path-dependent controller syntheses arise naturally.

The notion of finite-path-dependent controllers not only serves as a relaxation to achieve an exact condition for disturbance attenuation but also is required for optimality under the notion of path-by-path disturbance attenuation. Roughly speaking, the path-by-path performance is defined as a finite family of disturbance attenuation levels over all admissible switching paths of a given length. This notion of performance is a natural extension of that of uniform disturbance attenuation and can be used to improve upon the performance of a controller synthesis that achieves a given uniform disturbance attenuation level. As long as the path-by-path optimality is concerned, examples show that finite-path-dependent controllers can outperform mode-dependent controllers.

The result described above carries over to the almost sure uniform disturbance attenuation for Markovian jump linear systems because the uniformity requirement for stability and performance enables one to consider each Markovian jump linear
system as a switched linear system, where the underlying Markov chain of the former defines the switching path constraint of the latter. An immediate consequence is that, as long as almost sure uniform disturbance attenuation is concerned, the performance of Markovian jump linear systems is intrinsically robust against variations that preserve the directed graph of the underlying Markov chain (i.e., against sparsity pattern-preserving variations from the given transition probability matrix and initial distribution). On the other hand, under the notion of path-by-path disturbance attenuation, one can reduce the conservatism associated with this robustness property while still maintaining almost sure uniform stability; two Markov chains with the same directed graph can result in widely different path-by-path performances, depending on the actual values of initial and transition probabilities.

A key contribution of this paper to the control of switched linear systems and Markovian jump linear systems is that exact "control-oriented" conditions for (almost sure) stabilization and disturbance attenuation are provided. These conditions are control oriented in the sense that they lead to semidefinite programming-based techniques, which render optimal controllers very efficiently. There has been little work done on the control of disturbance attenuation performances of switched linear systems, other than that some partial analysis results in the continuous-time domain exist [44, 23]. On the other hand, the usual approach in the literature to disturbance attenuation for Markovian jump linear systems has been based on the notion of stochastic stability in both continuous time [28, 34, 13] and discrete time [11, 19, 10, 37]. These results, however, though either partly satisfactory or exact, are not well-suited for efficient optimal controller synthesis. For instance, in the case of independent and identically distributed switching, an exact linear matrix inequality-based synthesis condition for discrete-time jump systems has been obtained in [38]; for general Markov switching, this condition is only sufficient. Moreover, even though stochastically stable Markovian jump linear systems are almost surely stable [29], the usual approach does not guarantee almost sure disturbance attenuation.

The paper is organized as follows. Section 2 analyzes the performance of general linear time-varying systems. Then, based on this analysis, main results on uniform disturbance attenuation for switched linear systems and Markovian jump linear systems are derived in sections 3 and 4, respectively. Section 5 introduces the notion of stochastic stability in both continuous time [28, 34, 13] and discrete time [11, 19, 10, 37]. These results, however, though either partly satisfactory or exact, are not well-suited for efficient optimal controller synthesis. For instance, in the case of independent and identically distributed switching, an exact linear matrix inequality-based synthesis condition for discrete-time jump systems has been obtained in [38]; for general Markov switching, this condition is only sufficient. Moreover, even though stochastically stable Markovian jump linear systems are almost surely stable [29], the usual approach does not guarantee almost sure disturbance attenuation.

Notation. If $X \in \mathbb{R}^{m \times n}$, the range (or image) of $X$ is denoted by $\text{Im} X$, the null space (or kernel) of $X$ by $\text{Ker} X$, and the rank of $X$ by $\text{rank} X$; denoted by $N(X)$ is any particular full-rank matrix such that $\text{Im} N(X) = \text{Ker} X$. The indicator matrix $1(X) = (\mu_{ij}) \in \{0, 1\}^{m \times n}$ of an $X = (x_{ij}) \in \mathbb{R}^{m \times n}$ is such that $\mu_{ij} = 1$ if $x_{ij} \neq 0$ and $\mu_{ij} = 0$ if $x_{ij} = 0$. The matrices whose entries are all 0 (resp., all 1) are denoted by $0$ (resp., $1$) whenever $m$ and $n$ are understood. If $X, Y \in \mathbb{R}^{n \times n}$ are symmetric and $X - Y$ is positive definite (resp., nonnegative definite), we write $X > Y$ (resp., $X \geq Y$). The identity matrix is denoted by $I$ with $n$ understood.

For $x \in \mathbb{R}^n$, denoted by $\|x\|$ is the Euclidean vector norm $\|x\| = \sqrt{x^T x}$ of $x$. If $X \in \mathbb{R}^{m \times n}$, the Euclidean vector norm induces the spectral norm $\|X\|$ of $X$ given by $\|X\| = \sup \{\sqrt{\lambda} : \lambda$ is an eigenvalue of $X^T X\}$. Given a symmetric positive definite matrix $X \in \mathbb{R}^{n \times n}$, $\|x\|_X$ is the Hilbert norm of $x \in \mathbb{R}^n$ defined by $\|x\|_X = \sqrt{x^T X x}$. If $x = (x(0), x(1), \ldots)$ is a sequence in $\mathbb{R}^n$, then we write $x \in l^2(\mathbb{R}^n)$ whenever the $\ell^2$ norm of $x$, defined by $\|x\| = \sqrt{\sum_{s=0}^{\infty} \|x(s)\|^2}$, is finite.
2. Analysis of linear time-varying systems. In this section, we analyze the disturbance attenuation performance of the discrete-time linear time-varying system and the asymptotic property of the associated Riccati difference equation. We choose a notation that is compatible with the latter sections of this paper. Let a subset \( \mathcal{G} \) of \( \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{l \times n} \times \mathbb{R}^{l \times m} \) and a sequence \( \theta = (\theta(0), \theta(1), \ldots) \) in \( \{0, 1, \ldots\} \) be as follows:

\[
\mathcal{G} = \{(A_0, B_0, C_0, D_0), (A_1, B_1, C_1, D_1), \ldots, \theta = (0, 1, \ldots)\}.
\]

With (2.1), we have \((2.1)\) determines the state sequence that the uniform contractiveness is equivalent to the condition that the \( \ell^2 \)-induced gain from \( w \) to \( z \) be less than one; that is, for some \( \gamma \in (0, 1) \), \( \|z\| \leq \gamma \|w\| \) whenever \( x(0) = 0 \) and \( w \in \ell^2(\mathbb{R}^m) \). The infimum of all \( \gamma > 0 \) that satisfy (2.5) is called the \( \ell^2 \)-induced norm of the system \((\mathcal{G}, \theta)\).

**Definition 2.2.** The system \((\mathcal{G}, \theta)\) is said to be uniformly (strictly) contractive if there exists a \( \gamma \in (0, 1) \) such that, whenever \( x(t_0) = 0 \),

\[
\sum_{s=t_0}^{t} \|z(s)\|^2 \leq \gamma^2 \sum_{s=t_0}^{t} \|w(s)\|^2
\]

for \( t \geq t_0 \geq 0 \) and for \( w \in \ell^2(\mathbb{R}^m) \).

**Remark 1.** It is clear that the uniform contractiveness is equivalent to the condition that the \( \ell^2 \)-induced gain from \( w \) to \( z \) be less than one; that is, for some \( \gamma \in (0, 1) \), \( \|z\| \leq \gamma \|w\| \) whenever \( x(0) = 0 \) and \( w \in \ell^2(\mathbb{R}^m) \). The infimum of all \( \gamma > 0 \) that satisfy (2.5) is called the \( \ell^2 \)-induced norm of the system \((\mathcal{G}, \theta)\).

**Lemma 2.3.** Let \( \mathcal{G} \) and \( \theta \) be as in (2.1); let \( \mathcal{G} \) be bounded. The following are equivalent:

(a) The system \((\mathcal{G}, \theta)\) is uniformly exponentially stable and uniformly strictly contractive.

(b) There exist \( \alpha_1, \beta_1 > 0 \), and \( X_t \in \mathbb{R}^{n \times n} \), \( t = 0, 1, \ldots \), such that, for all \( t \),

\[
\alpha_1 I \leq X_t \leq \beta_1 I;
\]

\[
\begin{bmatrix}
A_t & B_t \\
C_t & D_t
\end{bmatrix}^T
\begin{bmatrix}
X_{t+1} & 0 \\
0 & I
\end{bmatrix}
\begin{bmatrix}
A_t & B_t \\
C_t & D_t
\end{bmatrix}
- \begin{bmatrix}
X_t & 0 \\
0 & I
\end{bmatrix}
\leq -\alpha_1 I.
\]

\[
\begin{bmatrix}
A_t & B_t \\
C_t & D_t
\end{bmatrix}
\begin{bmatrix}
Y_t & 0 \\
0 & I
\end{bmatrix}
\begin{bmatrix}
A_t & B_t \\
C_t & D_t
\end{bmatrix}^T
- \begin{bmatrix}
Y_{t+1} & 0 \\
0 & I
\end{bmatrix}
\leq -\alpha_2 I.
\]

(c) There exist \( \alpha_2, \beta_2 > 0 \), and \( Y_t \in \mathbb{R}^{n \times n} \), \( t = 0, 1, \ldots \), such that, for all \( t \),

\[
\alpha_2 I \leq Y_t \leq \beta_2 I;
\]

\[
\begin{bmatrix}
A_t & B_t \\
C_t & D_t
\end{bmatrix}
\begin{bmatrix}
Y_t & 0 \\
0 & I
\end{bmatrix}
\begin{bmatrix}
A_t & B_t \\
C_t & D_t
\end{bmatrix}^T
- \begin{bmatrix}
Y_{t+1} & 0 \\
0 & I
\end{bmatrix}
\leq -\alpha_2 I.
Moreover, if either (b) or (c) holds, one may take

\[ X_t = Y_t^{-1} \]

for all \( t \).

**Proof.** The equivalence of (a) and (b) is shown in [17, Thm. 11] via an operator theoretic approach. A simple Schur complement argument, together with the matrix inversion formula, shows that (b) and (c) are equivalent via relation \((2.8)\). \(\Box\)

**Proposition 2.4.** Let \( \mathcal{A} \) be as in \((2.3)\) and bounded; let \( \theta \) be as in \((2.1)\). Then the system \((\mathcal{A}, \theta)\) is uniformly exponentially stable if and only if there exist \( \alpha_1, \beta_1 > 0 \) and \( X_t \in \mathbb{R}^{n \times n} \) (resp., \( \alpha_2, \beta_2 > 0 \) and \( Y_t \in \mathbb{R}^{n \times n} \)), \( t = 0, 1, \ldots \), such that

\[
\alpha_1 I \leq X_t \leq \beta_1 I; \quad A_t^T X_{t+1} A_t - X_t \leq -\alpha_1 I \quad (\text{resp., } \alpha_2 I \leq Y_t \leq \beta_2 I; \quad A_t Y_t A_t^T - Y_{t+1} \leq -\alpha_2 I)
\]

for all \( t \geq 0 \).

**Proof.** Set the matrices \( B_i, C_i, D_i, i \geq 0 \), to zero in Lemma 2.3. \(\Box\)

Lemma 2.3 is a time-varying version of the classical Kalman–Yacubovich–Popov (KYP) lemma (see, e.g., [35, 18]). Inequality \((2.6b)\) is called the (extended) KYP inequality, and \((2.7b)\) is its dual form. The solutions of these inequalities are obtained by solving the associated Riccati difference equations. Let \( \mathcal{S} \) be the set of all symmetric matrices in \( \mathbb{R}^{n \times n} \). For \( i = 0, 1, \ldots \), let \( \mathcal{X}_i \) be the set of symmetric matrices \( X \in \mathbb{R}^{n \times n} \) such that

\[ W_i(X) = I - D_i^T D_i - B_i^T XB_i \]

is invertible. Define \( \mathcal{S}_i : \mathcal{X}_i \to \mathcal{S}, i = 0, 1, \ldots \), by

\[ \mathcal{S}_i(X) = A_i^T X A_i + C_i^T C_i + (A_i^T X B_i + C_i^T D_i) W_i(X)^{-1} (B_i^T X A_i + D_i^T C_i) \]

for \( X \in \mathcal{X}_i \). Similarly, for \( i = 0, 1, \ldots \), let \( \mathcal{Y}_i \) be the set of symmetric matrices \( Y \in \mathbb{R}^{n \times n} \) such that

\[ V_i(Y) = I - D_i D_i^T - C_i Y C_i^T \]

is invertible, and define \( \mathcal{R}_i : \mathcal{Y}_i \to \mathcal{S}, i = 0, 1, \ldots \), by

\[ \mathcal{R}_i(Y) = A_i Y A_i^T + B_i B_i^T + (A_i Y C_i^T + B_i D_i^T) V_i(Y)^{-1} (C_i Y A_i^T + D_i B_i^T) \]

for \( Y \in \mathcal{Y}_i \).

**Lemma 2.5.** Let \( \mathcal{G} \) and \( \theta \) be as in \((2.1)\); let \( \mathcal{G} \) be bounded. The following are equivalent:

(a) The system \((\mathcal{G}, \theta)\) is uniformly exponentially stable and uniformly strictly contractive.

(b) There exist \( \varepsilon_1, \delta_1, \eta_1 > 0 \) such that for all \( T = 0, 1, \ldots \), and \( \varepsilon \in [0, \varepsilon_1] \), the equation

\[
X^{(\varepsilon; T)}_0 = \mathcal{S}_i(X^{(\varepsilon; T)}_{t+1}) + \varepsilon I,
\]

with the terminal condition \( X^{(\varepsilon; T)}_{T+1} = \varepsilon I \), satisfies for \( t = 0, \ldots, T \) that

\[
W_i(X^{(\varepsilon; T)}_{t+1}) \geq \eta_1 I; \quad \varepsilon I \leq X^{(\varepsilon; T)}_t \leq X^{(\varepsilon; T+1)}_t \leq \delta_1 I.
\]
with the initial condition $Y_{t_0}^{(e,t_0)} = eI$, satisfies for $t = t_0, t_0 + 1, \ldots$ that

\begin{equation}
    \mathcal{V}_i(Y_t^{e,t_0}) \geq \eta_2 I; \quad eI \leq Y_t^{e,t_0+1} \leq Y_{t+1}^{e,t_0} \leq \delta_2 I.
\end{equation}

Moreover, if (b) holds, then one may take $X_t = \lim_{t \to \infty} X_t^{(e,T)}$ in (2.6); if (c) holds, then one may take $Y_t = Y_t^{e,(0)}$ in (2.7).

Equation (2.9a) is the (generalized) Riccati difference equation associated with the KYP inequality, and (2.10a) is its dual form. The latter evolves forward in time and so provides an explicit recursive expression for computing the solution to the KYP inequalities (2.6)–(2.7). Similar, but less explicit, results from operator theoretic and so provides an explicit recursive expression for computing the solution to the KYP inequalities (2.6)–(2.7). Similar, but less explicit, results from operator theoretic points of view exist [22, 27]. The proof of Lemma 2.5 is based on the following standard lemma. For $Y \in \mathbb{Y}_i$, let

\begin{equation}
    \mathcal{A}_i(Y) = A_i + (A_i Y C_i^T + B_i D_i^T) \mathcal{V}_i(Y)^{-1} C_i.
\end{equation}

**Lemma 2.6.** Let $Y^{(1)}, Y^{(2)} \in \mathbb{Y}_i$; let $\Delta^{(12)} = Y^{(1)} - Y^{(2)}$. Then

\begin{equation}
    \mathcal{R}_i(Y^{(1)}) - \mathcal{R}_i(Y^{(2)}) = \mathcal{A}_i(Y^{(2)}) \Delta^{(12)} \mathcal{A}_i(Y^{(2)})^T + \mathcal{A}_i(Y^{(2)}) \mathcal{C}_i \mathcal{V}_i(Y^{(1)})^{-1} C_i \Delta^{(12)} \mathcal{A}_i(Y^{(2)})^T = \mathcal{A}_i(Y^{(1)}) \Delta^{(12)} \mathcal{A}_i(Y^{(2)})^T.
\end{equation}

**Proof.** It follows from [22, Lem. 11, p. 77] that, for $Y \in \mathbb{Y}_i$, we may write

\begin{equation}
    \mathcal{R}_i(Y) = \overline{A}_i Y \overline{A}_i^T + \overline{Q}_i + \overline{A}_i Y C_i^T \mathcal{V}_i(Y)^{-1} C_i Y \overline{A}_i^T
\end{equation}

with

\begin{align*}
    \overline{A}_i &= A_i + B_i D_i^T (I - D_i D_i^T)^{-1} C_i, \\
    \overline{Q}_i &= B_i B_i^T + B_i D_i^T (I - D_i D_i^T)^{-1} D_i B_i^T.
\end{align*}

Then [15, Lem. 3.1] leads to (2.12a). Since

\begin{equation}
    \mathcal{V}_i(Y^{(1)})^{-1} - \mathcal{V}_i(Y^{(2)})^{-1} - \mathcal{V}_i(Y^{(2)})^{-1} C_i \Delta^{(12)} C_i^T \mathcal{V}_i(Y^{(1)})^{-1} = 0,
\end{equation}

it is easily seen that

\begin{align*}
    \mathcal{A}_i(Y^{(2)}) &= \mathcal{A}_i(Y^{(1)})(I - \Delta^{(12)} C_i^T \mathcal{V}_i(Y^{(2)})^{-1} C_i), \\
    (I - \Delta^{(12)} C_i^T \mathcal{V}_i(Y^{(2)})^{-1} C_i)(I + \Delta^{(12)} C_i^T \mathcal{V}_i(Y^{(1)})^{-1} C_i) &= I.
\end{align*}

These equalities and (2.12a) yield (2.12b). □

Lemma 2.6 is useful in proving asymptotic properties of the Riccati difference equation. In particular, an immediate consequence of (2.12a) is that $\mathcal{R}_i(Y^{(1)}) \geq \mathcal{R}_i(Y^{(2)})$ whenever $Y^{(1)}, Y^{(2)} \in \mathbb{Y}_i$ and $Y^{(1)} \geq Y^{(2)}$. 
Proof of Lemma 2.5. To first show that (c) is equivalent to (a), suppose that (c) holds. Let $\varepsilon = \varepsilon_2$. Since $\mathcal{G}$ is bounded, (2.10) implies that
\[
A_t Y_t^{(\varepsilon,t_0)} A_t^T + B_t B_t^T - Y_{t+1}^{(\varepsilon,t_0)} + \alpha I \\
+ (A_t Y_t^{(\varepsilon,t_0)} C_t^T + B_t D_t^T) V_t (Y_t^{(\varepsilon,t_0)} - \alpha I)^{-1} (C_t Y_t^{(\varepsilon,t_0)} A_t^T + D_t B_t^T) \leq 0
\]
for some small $\alpha > 0$. Then the Schur complement formula yields
\[
\begin{bmatrix} A_t & B_t \\ C_t & D_t \end{bmatrix} \begin{bmatrix} Y_t^{(\varepsilon,t_0)} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_t & B_t \\ C_t & D_t \end{bmatrix}^T - \begin{bmatrix} Y_t^{(\varepsilon,t_0)} & 0 \\ 0 & I \end{bmatrix} \leq -\alpha I.
\]
Putting $\alpha_2 = \min\{\alpha, \varepsilon\}$, $\beta_2 = \varepsilon_2$, and $Y_t = Y_t^{(\varepsilon,0)}$, we obtain condition (c) of Lemma 2.3, which in turn leads to the uniform stability and contractiveness of $(\mathcal{G}, \theta)$.

Conversely, suppose that $(\mathcal{G}, \theta)$ is uniformly stable and contractive so that condition (c) of Lemma 2.3 holds. Taking the Schur complement of $C_t Y_t C_t^T + D_t D_t^T - I$ from (2.7b), we obtain
\[
V_t(Y_t) \geq \alpha_2 I; \quad R_t(Y_t) + \alpha_2 I \leq Y_{t+1}
\]
for all $t$. Choose an arbitrary $\varepsilon \in [0, \alpha_2]$. For all $t_0 \geq 0$ and all $t \geq t_0$, define $Y_{t_0}^{(\varepsilon,t_0)} = \varepsilon I$ and $Y_{t_0+1}^{(\varepsilon,t_0)} = R_t(Y_t^{(\varepsilon,t_0)}) + \varepsilon I$. Since $Y_t \geq \alpha_2 I$ for all $t$, we have $Y_{t_0} \geq Y_{t_0}^{(\varepsilon,t_0)}$; by induction, together with (2.12a), we obtain
\[
V_t(Y_t) \geq \alpha_2 I; \quad \varepsilon I \leq Y_t^{(\varepsilon,t_0)} \leq Y_t \leq \beta_2 I
\]
for $t \geq t_0 \geq 0$. Moreover, $Y_t^{(\varepsilon,t_0)} \geq \varepsilon I$ for all $t$, and so we have $Y_{t_0+1}^{(\varepsilon,t_0)} \leq Y_{t_0+1}^{(\varepsilon,t_0)}$ from (2.12a); by induction, we obtain that
\[
Y_{t+1}^{(\varepsilon,t_0)} \leq R_t(Y_t^{(\varepsilon,t_0)}) + \varepsilon I = Y_{t+1}^{(\varepsilon,t_0)}
\]
for $t \geq t_0 \geq 0$. Putting $\varepsilon_2 = \alpha_2$, $\delta_2 = \beta_2$, and $\eta_2 = \alpha_2$, we obtain condition (c). This concludes the proof of the equivalence of (a) and (c). The proof that (a) and (b) are equivalent is analogous, and so is omitted.

\[ \Box \]

\textbf{Theorem 2.7.} Let $\mathcal{G}$ and $\theta$ be as in (2.1); let $\mathcal{G}$ be bounded. Suppose that the system $(\mathcal{G}, \theta)$ is uniformly exponentially stable and uniformly strictly contractive so that condition (c) of Lemma 2.5 holds. For $\varepsilon \in (0, \varepsilon_2)$ and $t_0 \geq 0$, let $Y_{t_0}^{(\varepsilon,t_0)}$ and $A_i(\cdot)$, $i = t_0, t_0 + 1, \ldots$, be as in (2.10a) and (2.11), respectively, where $Y_{t_0}^{(\varepsilon,t_0)} = \varepsilon I$.
Then the following hold:

(a) For each $\varepsilon \in (0, \varepsilon_2)$ and $t_0 \geq 0$, define
\[
A^{(\varepsilon,t_0)} = \{A_i(Y_i^{(\varepsilon,t_0)}): i = t_0, t_0 + 1, \ldots\}, \quad \theta^{(t_0)} = (t_0, t_0 + 1, \ldots).
\]
Then each system $(A^{(\varepsilon,t_0)}, \theta^{(t_0)})$ is uniformly exponentially stable. Moreover, for each $\varepsilon \in (0, \varepsilon_2)$, there exist $c_\varepsilon \geq 1$ and $\lambda_\varepsilon \in (0, 1)$ such that
\[
\|A_{t-1}(Y_{t-1}^{(\varepsilon,t_0)}) \cdots A_{t_0}(Y_{t_0}^{(\varepsilon,t_0)})\| \leq c_\varepsilon \lambda_\varepsilon^{t-t_0}
\]
for $t > t_0 \geq 0$. 


(b) For each $\varepsilon \in (0, \varepsilon_2)$, there exist a nonnegative integer $M$ and $\alpha_2, \beta_2 > 0$ such that (2.7) is satisfied with

\begin{equation}
Y_t = \begin{cases} 
Y_t^{(\varepsilon,0)} & \text{for } t < M; \\
Y_t^{(\varepsilon,t-M)} & \text{for } t \geq M
\end{cases}
\end{equation}

for $t \geq 0$.

**Proof.** Fix $\varepsilon, \varepsilon' \in (0, \varepsilon_2)$ such that $\varepsilon < \varepsilon'$. For $t \geq t_0 \geq 0$, let $Y_t^{(t_0)} = Y_t^{(\varepsilon',t_0)} - Y_t^{(\varepsilon,t_0)}$. Then (2.12a) yields that

$Y_t^{(t_0)} \geq A_t(Y_t^{(\varepsilon,t_0)}) Y_t^{(t_0)} A_t(Y_t^{(\varepsilon,t_0)})^T + (\varepsilon' - \varepsilon)I$

for all $t \geq t_0$. Since $\varepsilon' - \varepsilon > 0$ and $(\varepsilon' - \varepsilon)I \leq Y_t^{(t_0)} \leq (\delta_2 - \varepsilon)I$ for $t \geq t_0 \geq 0$, we have that, with $X_t^{(t_0)} = (Y_t^{(t_0)})^{-1}$, there exist $\alpha_1, \beta_1 > 0$ such that

$\alpha_1 I \leq X_t^{(t_0)} \leq \beta_1 I; \quad A_t(Y_t^{(\varepsilon,t_0)})^T X_t^{(t_0)} A_t(Y_t^{(\varepsilon,t_0)}) - X_t^{(t_0)} \leq -\alpha_1 I$

for all $t$. The uniform stability of the systems $(A^{(\varepsilon,t_0)}, \theta^{(t_0)})$ follows from Proposition 2.4. Due to [30, Lem. 4], inequality (2.14) holds for $t > t_0 \geq 0$ with $c_\varepsilon = \sqrt{\beta_1 / \alpha_1}$ and $\lambda_\varepsilon = \sqrt{1 - \alpha_1 / \beta_1}$. Thus assertion (a) holds true.

To prove (b), pick a nonnegative integer $M$ such that $c_\varepsilon^2 \lambda_\varepsilon^{2M} < \varepsilon / (\delta_2 - \varepsilon)$. Then, using (2.12b), we deduce that there exists an $\varepsilon'' \in (0, \varepsilon)$ such that

\begin{align*}
R_t(Y_t^{(\varepsilon,t-M)}) - R_t(Y_t^{(\varepsilon,t-M+1)}) &= A_t(Y_t^{(\varepsilon,t-M)}) \cdots A_t(Y_t^{(\varepsilon,M)}) (R_{t-M+1}(\varepsilon I) - \varepsilon I) \\
&\quad \times A_t(Y_t^{(\varepsilon,t-M+1)})^T \cdots A_t(Y_t^{(\varepsilon,t-M+1)})^T \\
&\leq (\delta_2 - \varepsilon)c_\varepsilon^2 \lambda_\varepsilon^{2M} I \\
&\leq \varepsilon'' I
\end{align*}

for $t \geq M$; or equivalently,

$R_t(Y_t^{(\varepsilon,t-M)}) - Y_{t+1}^{(\varepsilon,t-M+1)} \leq -(\varepsilon - \varepsilon'') I$

for $t \geq M$. On the other hand,

$R_t(Y_t^{(\varepsilon,0)}) - Y_{t+1}^{(\varepsilon,0)} = -\varepsilon I \leq -(\varepsilon - \varepsilon'') I.$

Therefore, $Y_t$ defined by (2.15) satisfies

$R_t(Y_t) - Y_{t+1} \leq -(\varepsilon - \varepsilon'') I$

for all $t \geq 0$. Using the Schur complement formula, we have that there exists an $\alpha > 0$ such that

$[A_t \ B_t] [Y_t \ 0] [A_t \ B_t]^T - \begin{bmatrix} Y_{t+1} & 0 \\ 0 & I \end{bmatrix} \leq -\alpha I$

holds for $t \geq 0$. Putting $\alpha_2 = \min\{\alpha, \varepsilon\}$ and $\beta_2 = \delta_2$, we see that $Y_t$ satisfies (2.7) for all $t \geq 0$, and hence assertion (b) holds. \qed
Remark 2. Using (2.13), we may write
\[ Y_{t+1}^{(\varepsilon,t_0)} = \overline{A}_tY_t^{(\varepsilon,t_0)}\overline{X}_t^T + (\overline{Q}_t + \varepsilon I) + \overline{A}_tY_t^{(\varepsilon,t_0)}C_t^TY_t^{(\varepsilon,t_0)}Y_t^{(\varepsilon,t_0)} \] for \( \varepsilon \in (0,\varepsilon_2) \) and \( t \geq t_0 \geq 0 \). Let
\[ \mathcal{G}' = \{(\overline{A}_0,\overline{Q}_0 + \varepsilon I, C_0, 0), (\overline{A}_1, \overline{Q}_1 + \varepsilon I, C_1, 0), \ldots\}. \]
If \((\mathcal{G}, \Theta)\) is uniformly stable, it is easy to verify that \((\mathcal{G}', \Theta)\) is uniformly detectable and uniformly stabilizable. Then, by [1, 14], the system \((\mathcal{A}^{(\varepsilon,t_0)}, \Theta^{(t_0)})\) is uniformly stable for each \( t_0 \geq 0 \); it follows from (2.12b) that, for each \( t_0 \geq 0 \), \( Y_t^{(\varepsilon,t_0)} \) converges to the unique “moving equilibrium” (i.e., the maximal solution) of the Riccati difference equation (2.10a) as \( t - t_0 \to \infty \) [14]. However, part (a) of Theorem 2.7 says that this convergence is uniform in \((t,t_0)\), as the uniform stability of \((\mathcal{A}^{(\varepsilon,t_0)}, \Theta^{(t_0)})\) is again uniform in \( t_0 \).

It is known that, if a given linear time-varying system is uniformly stable, the corresponding Lyapunov inequality admits a solution that has a finite memory of past parameters (see, e.g., [30]). Part (b) of Theorem 2.7 is an extension of this and says that the KYP inequality (or equivalently, the Riccati inequality) associated with a uniformly stable and contractive linear time-varying system has a solution that has a finite memory of past parameters.

3. Control of switched linear systems. The switched linear system is a family of linear time-varying systems whose parameters vary within a single finite set. Fix a positive integer \( N \) and define
\[
\mathcal{G} = \{(A_1, B_1, C_1, D_1), \ldots, (A_N, B_N, C_N, D_N)\},
\]
where \( A_i \in \mathbb{R}^{n \times n}, B_i \in \mathbb{R}^{n \times m}, C_i \in \mathbb{R}^{l \times n}, D_i \in \mathbb{R}^{l \times m} \) for \( i = 1, \ldots, N \). Let \( \Omega \) be the set of all infinite sequences in \( \{1, \ldots, N\} \); each member of \( \Omega \) is called a switching sequence. If \( \Theta \) is a nonempty subset of \( \Omega \), then the pair \((\mathcal{G}, \Theta)\) defines the switched linear system, where \( \Theta \) is the set of admissible switching sequences: for initial states \( x(0) \), disturbance sequences \( w \), switching sequences \( \Theta \in \Theta \), and \( t \geq 0 \), the system \((\mathcal{G}, \Theta)\) has the state-space representation
\[
x(t+1) = A_{\Theta(t)}x(t) + B_{\Theta(t)}w(t),
\]
\[
z(t) = C_{\Theta(t)}x(t) + D_{\Theta(t)}w(t).
\]
If \( \Theta(t) = i \), then the system is said to be in mode \( i \) at time \( t \), and its parameters at time \( t \) are given by the quadruple \((A_i, B_i, C_i, D_i)\). When the set \( \Theta \) is equal to the entire set \( \Omega \), the pair \((\mathcal{G}, \Omega)\) defines the discrete linear inclusion, which is the switched linear system without a switching path constraint; on the other hand, if \( \Theta = \{\theta\} \) is a singleton, then the pair \((\mathcal{G}, \Theta)\) is nothing but the linear time-varying system \((\mathcal{G}, \theta)\). We require that the stability and contractiveness of the system \((\mathcal{G}, \Theta)\) be uniform over all switching sequences in \( \Theta \).

Definition 3.1. The system \((\mathcal{G}, \Theta)\) is said to be uniformly (exponentially) stable if there exist \( \varepsilon \geq 1 \) and \( \lambda \in (0, 1) \) such that, whenever \( w = 0 \), inequality (2.4) holds for \( t \geq t_0 \geq 0 \), for \( x(t_0) \in \mathbb{R}^n \), and for \( \Theta \in \Theta \).

Definition 3.2. The system \((\mathcal{G}, \Theta)\) is said to be uniformly (strictly) contractive if there exist \( \gamma \in (0, 1) \) such that, whenever \( x(t_0) = 0 \), inequality (2.5) holds for \( t \geq t_0 \geq 0 \), for \( w \in \ell^2(\mathbb{R}^m) \), and for \( \Theta \in \Theta \).
For the sake of convenience, we introduce a dummy mode 0 and think of each \( \theta \in \Theta \) as a two-sided sequence \( (\ldots, \theta(-1), \theta(0), \theta(1), \ldots) \) by putting \( \theta(t) = 0 \) for \( t < 0 \). Fix a nonnegative integer \( L \). Any finite sequence in \( \{0, \ldots, N\} \) will be called a \((finite)\ switching path\); in particular, elements of the set \( \{0, \ldots, N\}^{L+1} \) are switching paths of length \( L \) and are called \( L\)-paths. For \( \theta \in \Theta \) and \( t \geq 0 \), let

\[
\theta_L(t) = (\theta(t-L), \ldots, \theta(t)).
\]

An \( L\)-path \((i_0, \ldots, i_L)\) is said to occur in \( \Theta \) if \( \theta_L(t) = (i_0, \ldots, i_L) \) for some \( \theta \in \Theta \) and some \( t \geq 0 \). Each \( \theta \in \Theta \) generates an \( L\)-path switching sequence \( \theta_L \) defined by

\[
\theta_L = (\theta_L(0), \theta_L(1), \ldots).
\]

Denote the set of \( L\)-paths occurring in \( \Theta \) by \( L(\Theta) \) so that

\[
L(\Theta) = \{ \theta_L(t) : \theta \in \Theta, t \geq 0 \}.
\]

If \((i_0, \ldots, i_L) \in L(\Theta)\), then write

\[
(i_0, \ldots, i_L)_- = (i_0, \ldots, i_{L-1}), \quad (i_0, \ldots, i_L)_+ = (i_1, \ldots, i_L)
\]

for \( L > 0 \), and write \((i_0, \ldots, i_L)_- = (i_0, \ldots, i_L)_+ = 0 \) for \( L = 0 \). If \( L > 0 \), define \( M_L(\Theta) \) to be the smallest subset of \( L(\Theta) \) such that the following hold: \( \theta_L(t) \in M_L(\Theta) \) for all \( t \geq L \) and for all \( \theta \in \Theta \) and, for each \( j \in L_0(\Theta) \), there exists a switching path \((i_{0j}, \ldots, i_{L-1}) \in \{0, \ldots, N\}^L \) such that

\[
(i_{0j}, \ldots, i_{L-1}, \theta(0), \ldots, \theta(1)) = (\theta(0), \ldots, \theta(L-1)) \in M_L(\Theta)
\]

for all \( \theta \in \Theta \). If \( L = 0 \), then let \( M_0(\Theta) = L_0(\Theta) \). The sets \( M_L(\Theta) \), \( L = 0, 1, \ldots \), are unique and so are well defined. Let

\[
M_L(\Theta) = \{ i_- : i \in M_L(\Theta) \}, \quad L_L(\Theta) = \{ i_- : i \in L(\Theta) \}.
\]

In general, we have

\[
L(\Theta) \cap \{1, \ldots, N\}^{L+1} \subseteq M_L(\Theta) \subseteq L(\Theta) \subseteq \{0, \ldots, N\}^{L+1},
\]

\[
M_L(\Theta) \subseteq M_{L+1}(\Theta) \subseteq L_{L+1}(\Theta).
\]

**Example 1.** Let \( \Theta \) consist of three periodic sequences \((1,1,\ldots), (1,2,1,2,\ldots), \) and \((2,2,\ldots)\). To simplify notation, write \( i_0 \cdots i_L \) for \((i_0, \ldots, i_L)\). Then we have

\[
L_0(\Theta) = M_0(\Theta) = \{1, 2\}, \quad L_1(\Theta) = \{01, 02, 11, 12, 21, 22\},
\]

\[
L_2(\Theta) = \{001, 002, 011, 012, 022, 111, 121, 212, 222\},
\]

and so on. For \( L = 1 \), we can replace the one-path \((0,1)\) in \( L_1(\Theta) \) with \((1,1)\) or \((2,1)\); similarly, the one-path \((0,2)\) can be replaced with either \((1,2)\) or \((2,2)\). Hence, we have

\[
M_1(\Theta) = \{11, 12, 21, 22\}.
\]

However, for \( L = 2 \), if we choose any \((i_0, i_1) \neq (0,0)\), then we have that at least one of the two-paths \((i_0, i_1, 1)\), \((i_1, 1, 1)\), \((1, i_1, 2)\) does not belong to \( L_2(\Theta) \) and so \((0,0,1)\)
cannot be replaced with any of \((0, 1, 1), (1, 1, 1), (1, 2, 1) \in \mathcal{L}_2(\Theta)\); on the other hand, both the two-paths \((0, 0, 2)\) and \((0, 2, 2)\) can be replaced with \((2, 2, 2) \in \mathcal{L}_2(\Theta)\). Hence,

\[
\mathcal{M}_2(\Theta) = \{001, 011, 012, 111, 121, 212, 222\}.
\]

This \(\Theta\) will be used later in Examples 4 and 7.

**Example 2.** For each positive integer \(k\), let \(\Theta^{(k)}\) consist of a single sequence

\[
\theta^{(k)} = \left\{1, \ldots, 1, 2, \ldots, 2, 1, \ldots, 2, \ldots, 2, \ldots\right\},
\]

where \(k\) 1’s and \(k\) 2’s alternate in \(\theta^{(k)}\). Then

\[
\mathcal{L}_0(\Theta^{(k)}) = \mathcal{M}_0(\Theta^{(k)}) = \{1, 2\} \quad \text{for all } k;
\]

\[
\mathcal{L}_1(\Theta^{(k)}) = \begin{cases}
\{01, 12, 21\}, & k = 1; \\
\{01, 11, 12, 21, 22\}, & k \geq 2;
\end{cases} \quad \mathcal{M}_1(\Theta^{(k)}) = \begin{cases}
\{12, 21\}, & k = 1; \\
\{11, 12, 21, 22\}, & k \geq 2;
\end{cases}
\]

\[
\mathcal{L}_2(\Theta^{(k)}) = \begin{cases}
\{001, 012, 121\}, & k = 1; \\
\{001, 011, 112, 122, 211, 221\}, & k = 2; \\
\{001, 011, 111, 112, 122, 211, 212, 222\}, & k \geq 3;
\end{cases} \quad \mathcal{M}_2(\Theta^{(k)}) = \begin{cases}
\{121, 212\}, & k = 1; \\
\{112, 122, 211, 221\}, & k = 2; \\
\{111, 112, 122, 211, 221, 222\} & k \geq 3,
\end{cases}
\]

and so on. These sets will be used in Example 3. The two-path switching sequence generated by \(\theta^{(2)}\), for instance, is

\[
\theta^{(2)} = (001, 011, 112, 122, 211, 112, 122, 221, 221, \ldots).
\]

**Theorem 3.3.** Let \(G\) be as in (3.1); let \(\Theta \subset \Omega\) be nonempty. The system \((G, \Theta)\) is uniformly exponentially stable and uniformly strictly contractive if and only if there exist a nonnegative integer \(M\) and an indexed family \(\{X_j : j \in \mathcal{M}_M(\Theta)\}\) of symmetric positive definite matrices \(X_j \in \mathbb{R}^{n \times n}\) such that

\[
(3.4) \quad \begin{bmatrix} A_{iM} & B_{iM} \\ C_{iM} & D_{iM} \end{bmatrix}^T X_i + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} A_{iM} & B_{iM} \\ C_{iM} & D_{iM} \end{bmatrix} - \begin{bmatrix} X_i & 0 \\ 0 & I \end{bmatrix} < 0
\]

for all \(M\)-paths \(i = (i_0, \ldots, i_M) \in \mathcal{M}_M(\Theta)\).

**Proof.** To show sufficiency, fix an \(M\) and suppose (3.4) holds for \(i = (i_0, \ldots, i_M) \in \mathcal{M}_M(\Theta)\). Assume \(M > 0\) without loss of generality. By the definition of \(\mathcal{M}_M(\Theta)\), one can choose \((i_0, \ldots, i_{M-1}) \in \{0, \ldots, N\}^M; j \in \{\theta(0): \theta \in \Theta\}\), such that (3.3), with \(L\) replaced by \(M\), holds for all \(\theta \in \Theta\). Put

\[
X_t = \begin{cases} X_{(\theta(0), \ldots, \theta(0))} & \text{for } t = 0; \\
X_{(\theta(0), \ldots, \theta(0), \theta(0), \ldots, \theta(t-1))} & \text{for } 0 < t < M; \\
X_{(\theta(t-M), \ldots, \theta(t-1))} & \text{for } t \geq M.
\end{cases}
\]

Then, since \(\mathcal{M}_M(\Theta)\) is finite, one can choose \(\alpha, \beta > 0\) independently of \(\theta \in \Theta\) such that, in particular,

\[
\alpha I \leq X_t \leq \beta I; \quad A_{\theta(t)}^T X_{t+1} A_{\theta(t)} - X_t < -\alpha I
\]
for $t \geq 0$; hence it follows from Proposition 2.4 and the first half of the proof of Theorem 2.7 that the system $(G, \Theta)$ is uniformly stable. On the other hand, the finiteness of $M\theta$ implies that there exists an $\eta \in (0, 1)$, which is independent of $\theta$, such that

$$\sum_{i=0}^{\infty} \| z(t) \|^2 + \| x(t+1) \|^2 - \| x(t) \|^2 \leq (1-\eta) \| w(t) \|^2$$

for all $x(t) \in \mathbb{R}^n$ and $w(t) \in \mathbb{R}^m$ so that

$$\| z(t) \|^2 + \| x(t+1) \|^2 - \| x(t) \|^2 \leq (1-\eta) \| w(t) \|^2$$

for $t \geq 0$. If $t \geq t_0 \geq 0$ and $x(t_0) = 0$, then this inequality, as well as the fact that $X_i > 0$ for all $i \in \mathcal{M}_\theta$, leads to

$$\sum_{s=t_0}^{t} \| z(s) \|^2 \leq (1-\eta) \sum_{s=t_0}^{t} \| w(s) \|^2.$$

Putting $\gamma = \sqrt{1-\eta}$ yields (2.5) for $t \geq t_0 \geq 0$. Since $\gamma$ is independent of $\theta$, this inequality holds for all $\theta \in \Theta$, too. Hence the system $(G, \Theta)$ is uniformly contractive as well as uniformly stable. This proves sufficiency.

To show necessity, suppose that $(G, \Theta)$ is uniformly stable and contractive. Consider the augmented disturbance signal $\tilde{w}(t) = [w(t)^T v(t)^T]^T$ with $v(t) \in \mathbb{R}^n$, $t \geq 0$, and the perturbed system $(G^{(e)}, \Theta)$, where

$$G^{(e)} = \{(A_1, B_1^{(e)}, C_1, D_1^{(e)}), \ldots, (A_N, B_N^{(e)}, C_N, D_N^{(e)})\},$$

$$B_1^{(e)} = [B_1, \sqrt{e}] \in \mathbb{R}^{n \times (m+n)}, \quad D_1^{(e)} = [D_1, 0] \in \mathbb{R}^{l \times (m+n)}.$$

Then there exists a sufficiently small $\varepsilon > 0$, dependent on $\gamma$ in (2.5) but independent of $\Theta$, such that $(G^{(e)}, \Theta)$ is uniformly stable and contractive for all $e \in (0, \varepsilon)$. If we fix an $e \in (0, \varepsilon)$, then by Lemma 2.5, there exist $\delta_2, \eta_2 > 0$ such that the (dual) Riccati equation $Y_{t+1}(e,t_0) = R_{\theta(t)}(Y_t(e,t_0)) + eI$, with the initial condition $Y_t(e,t_0) = \varepsilon I$, satisfies

$$\forall_{t \geq t_0} \geq \eta_2 I; \quad \varepsilon I \leq Y_{t+1}(e,t_0) \leq \delta_2 I$$

for $t \geq t_0 \geq 0$ and for $\theta \in \Theta$. Part (b) of Theorem 2.7 and its proof then imply that there exists a nonnegative integer $M$, which depends only on $e, \delta_2, \eta_2$, and $G$, such that for some $a_2, \beta_2 > 0$, $Y_t$ given by (2.15), with $A_t = A_{\theta(t)}, B_t = B_{\theta(t)}, C_t = C_{\theta(t)},$ and $D_t = D_{\theta(t)}$, satisfies (2.7) for $\theta \in \Theta$ and $t \geq 0$. Then Lemmas 2.3 and 2.5 imply that the symmetric positive definite matrices $X_t$, $t \geq 0$, satisfying (2.6) can be taken to be of the form

$$X_t = f(\theta(t-M), \ldots, \theta(t-1))$$

for some function $f: \{0, \ldots, N\}^M \rightarrow \mathbb{S}$, where $\theta(s) = 0$ for $s < 0$. Putting

$$X_{\theta(t-M), \ldots, \theta(t-1)} = f(\theta(t-M), \ldots, \theta(t-1))$$
leads to (3.4) for \( i = (i_0, \ldots, i_M) \in \mathcal{L}_M(\Theta) \). Since \( \mathcal{M}_M(\Theta) \subset \mathcal{L}_M(\Theta) \), and since the path length \( M \) is independent of \( \Theta \), we obtain the desired result. \( \square \)

**Remark 3.** In Theorem 3.3 the number of linear matrix inequalities (3.4) to solve simultaneously, given \( M \), is equal to the cardinality of \( \mathcal{M}_M(\Theta) \) and bounded above by \( \sum_{k=1}^{M+1} N_k \); the number of matrix variables to solve for is equal to the cardinality of \( \mathcal{M}_M(\Theta) \). In particular, if either \( \Theta = \Omega \) or \( \Theta \in \Theta \) for some \( \Theta \) which is recurrent with respect to \( \{1, \ldots, N\} \), so that every finite switching path in \( \{1, \ldots, N\} \) occurs infinitely many times in \( \Theta \), then the cardinality of \( \mathcal{M}_M(\Theta) \) is precisely \( N^{M+1} \).

**Remark 4.** If \( N = 1 \), then we have \( \Theta = \{(1, 1, \ldots)\} \), and the set \( \mathcal{M}_M(\Theta) \) is a singleton for each \( M \), and so Theorem 3.3 reduces to the classical KYP lemma.

If inequalities (3.4), with \( X_j > 0 \) for \( j \in \mathcal{M}_M(\Theta) \), are feasible for some \( M \), then it is also feasible when \( M \) is replaced with any integer greater than \( M \). Hence Theorem 3.3 characterizes the performance of switched linear systems via the countably infinite union of an increasing family of systems of linear matrix inequalities. For uniform stability and contractiveness, not only is each member of this family sufficient, but also the union of the family is necessary.

The condition in Theorem 3.3 simplifies if we focus on the uniform stability only. For nonnegative integers \( L \) and \( \Theta \subset \Omega \), let \( \mathcal{N}_L(\Theta) \) be the largest subset of \( \mathcal{M}_L(\Theta) \) satisfying the following: For each \( (i_0, \ldots, i_L) \in \mathcal{N}_L(\Theta) \), there exist an integer \( M > L \) and a switching path \( (i_0, \ldots, i_M) \) such that \( (i_M-L, \ldots, i_M) = (i_0, \ldots, i_L) \) and \( (i_0, \ldots, i_L) \in \mathcal{N}_L(\Theta) \) for \( 0 \leq t \leq M - L \). Then we have

\[
\mathcal{N}_L(\Theta) \subset \mathcal{M}_L(\Theta) \cap \{1, \ldots, N\}^{M+1}.
\]

For example, if

\[
\Theta = \{(1, 2, 2, 2, \ldots), (1, 1, 2, 2, \ldots), (1, 1, 1, 2, 2, \ldots), \ldots\},
\]

then \( \mathcal{N}_0(\Theta) = \{1, 2\} \), \( \mathcal{N}_1(\Theta) = \{11, 22\} \), \( \mathcal{N}_2(\Theta) = \{111, 222\} \), and so on.

**Corollary 3.4.** Let \( \mathcal{G} \) be as in (3.1); let \( \Theta \subset \Omega \) be nonempty. The system (\( \mathcal{G}, \Theta \)) is uniformly exponentially stable if and only if there exist a nonnegative integer \( M \) and matrices \( X_j > 0 \) such that

\[
A^T_{i_M} X_{i+} A_{i_M} - X_{i-} < 0
\]

for all \( i = (i_0, \ldots, i_M) \in \mathcal{N}_M(\Theta) \).

**Proof.** Set the matrices \( B_i, C_i, D_i; i = 1, \ldots, N \), to zero in (3.4) to obtain (3.5). Since \( \mathcal{N}_M(\Theta) \subset \mathcal{M}_M(\Theta) \), it suffices to show sufficiency. If (3.5) holds for all \( i = (i_0, \ldots, i_M) \in \mathcal{N}_M(\Theta) \), perform the following algorithm:

0. Set \( \mathcal{M}_M(\Theta) = \mathcal{N}_M(\Theta) \).

1. If \( \mathcal{M}_M(\Theta) = \mathcal{M}_M(\Theta) \), then stop; otherwise, choose an \( i = (i_0, \ldots, i_M) \in \mathcal{M}_M(\Theta) \) such that \( (i_0, \ldots, i_M, i_{M+1}) \in \mathcal{M}_M(\Theta) \) for some \( i_{M+1} \in \{1, \ldots, N\} \). (By definition of \( \mathcal{M}_M(\Theta) \), such an \( M \)-path \( i \) exists.)

2. If \( i \notin \mathcal{M}_M(\Theta) \), then choose an \( \alpha > 0 \) such that \( A^T_{i_M} X_{i+} A_{i_M} - \alpha I < 0 \), put \( X_{i-} = \alpha I \), and go to step 4.

3. Choose an \( \alpha > 0 \) such that \( A^T_{i_M} X_{i+} A_{i_M} - \alpha X_{i-} < 0 \), and substitute \( X_{i-} \) with \( \alpha X_{i-} \). (By definition of \( \mathcal{N}_M(\Theta) \), \( i_- \) is not equal to \( i_+ \).) Whenever there exists an integer \( L > 0 \) and a switching path \( (i_L, \ldots, i_{M+1}) \) such that \( (i_{L-1}, \ldots, i_{M+L-1}) \in \mathcal{M}_M(\Theta) \) for all \( 0 \leq t \leq L \), then substitute \( X_{i_{L-1}, \ldots, i_{M+L-1}} \) with \( \alpha X_{i_{L-1}, \ldots, i_{M+L-1}} \), too. (Again, by definition of \( \mathcal{N}_M(\Theta) \), such a path \( (i_L, \ldots, i_{M+L-1}) \) cannot be equal to \( i_+ \).)
4. Substitute \( \tilde{M}_M(\Theta) \) with \( \tilde{M}_M(\Theta) \cup \{i\} \) and go to step 1.

Since the cardinality of \( M_M(\Theta) \) is finite, we will have reconstructed an entire family \( \{X_j: j \in M_M(\Theta)\} \) of matrices \( X_j > 0 \) at the termination of this algorithm so that (3.5) holds for all \( i = (i_0, \ldots, i_M) \in M_M(\Theta) \).

Remark 5. If \( \Theta \) is defined via a strongly connected directed graph, then Corollary 3.4 reverts to the stability analysis result in [30].

Now, consider the set

\[
\mathcal{T} = \{(A_i, B_{1,i}, B_{2,i}, C_{1,i}, C_{2,i}, D_{11,i}, D_{12,i}, D_{21,i}): i = 1, \ldots, N\}
\]

with \( A_i \in \mathbb{R}^{n \times n}, B_{1,i} \in \mathbb{R}^{n \times m_1}, B_{2,i} \in \mathbb{R}^{n \times m_2}, C_{1,i} \in \mathbb{R}^{l_1 \times n}, C_{2,i} \in \mathbb{R}^{l_2 \times n}, D_{11,i} \in \mathbb{R}^{l_1 \times m_1}, D_{12,i} \in \mathbb{R}^{l_1 \times m_2}, D_{21,i} \in \mathbb{R}^{l_2 \times m_2} \) for \( i = 1, \ldots, N \). If \( \Theta \subseteq \Omega \) and is nonempty, then the pair \((\mathcal{T}, \Theta)\) defines the controlled switched linear system represented by

\[
\begin{align*}
    x(t+1) &= A_{\theta(t)}x(t) + B_{1,\theta(t)}w(t) + B_{2,\theta(t)}u(t), \\
    z(t) &= C_{1,\theta(t)}x(t) + D_{11,\theta(t)}w(t) + D_{12,\theta(t)}u(t), \\
    y(t) &= C_{2,\theta(t)}x(t) + D_{21,\theta(t)}w(t).
\end{align*}
\]

Given the initial state \( x(0) \), disturbance sequence \( w = (w(t)) \), control sequence \( u = (u(t)) \), and switching sequence \( \Theta \subseteq \Theta \), this system of equations defines the evolution of the state \( x(t) \), controlled output \( z(t) \), and measured output \( y(t) \) for \( t \geq 0 \). Based on the analysis result given by Theorem 3.3, we will be deriving a necessary and sufficient condition for controller synthesis.

We make the standard assumption that the mode \( \theta(t) \) is perfectly observed at each time instant \( t \); however, relaxing the standard restriction to mode-dependent controllers (i.e., controllers that do not recall past modes), we consider all controllers that have a finite memory of past modes as well as a perfect observation of the current mode. Fix a nonnegative integer \( L \). Let

\[
\Theta_L = \{\theta_L : \theta \in \Theta\}
\]

be the set of \( L \)-path switching sequences generated by \( \Theta \); let

\[
\mathcal{K} = \{(A_{K,i}, B_{K,i}, C_{K,i}, D_{K,i}): i \in \mathcal{L}_L(\Theta)\}
\]

with \( A_{K,i} \in \mathbb{R}^{n_K \times n_K}, B_{K,i} \in \mathbb{R}^{n_K \times l_2}, C_{K,i} \in \mathbb{R}^{m_2 \times n_K}, D_{K,i} \in \mathbb{R}^{m_2 \times l_2} \) for \( i \in \mathcal{L}_L(\Theta) \). Then the pair \((\mathcal{K}, \Theta_L)\) defines the \( L \)-path-dependent (linear output feedback) controller (of order \( n_K \)), which determines the control sequence \( u \) according to

\[
\begin{align*}
x_K(t+1) &= A_{K,\theta_L(t)}x_K(t) + B_{K,\theta_L(t)}y(t), \\
u(t) &= C_{K,\theta_L(t)}x_K(t) + D_{K,\theta_L(t)}y(t)
\end{align*}
\]

given the initial controller state \( x_K(0) \) and \( L \)-path switching sequence \( \theta_L \in \Theta_L \). Controllers that are \( L \)-path-dependent for some nonnegative integer \( L \) shall be said to be \( \text{finite-path-dependent} \); zero-path-dependent controllers are called \( \text{mode-dependent} \). The dependence of these controllers on the past measurements \( y(0), \ldots, y(t) \) at each time instant \( t \) is encoded in the partition

\[
K_i = \begin{bmatrix} A_{K,i} & B_{K,i} \\ C_{K,i} & D_{K,i} \end{bmatrix} \in \mathbb{R}^{(n_K+m_2) \times (n_K+l_2)}, \quad i \in \mathcal{L}_L(\Theta).
\]
Given a finite-path-dependent controller \((\mathcal{K}, \Theta_L)\), where \(L\) is the path length, let

\[
\tilde{A}_i = \hat{A}_{iL} + \hat{B}_{2,iL} \hat{K}_i \hat{C}_{2,iL}, \quad \tilde{B}_i = \hat{B}_{1,iL} + \hat{B}_{2,iL} \hat{K}_i \hat{D}_{21,iL},
\]

(3.10)

\[
\tilde{C}_i = \hat{C}_{1,iL} + \hat{D}_{12,iL} \hat{K}_i \hat{D}_{21,iL} \hat{D}_{iL} = \hat{D}_{11,iL} + \hat{D}_{12,iL} \hat{K}_i \hat{D}_{21,iL}
\]

for \(i = (i_0, \ldots, i_L) \in \mathcal{L}_L(\Theta)\), with

\[
\hat{A}_i = \begin{bmatrix} A_i & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{(n+n_K) \times (n+n_K)},
\]

\[
\hat{B}_{1,i} = \begin{bmatrix} B_{1,i} \\ 0 \end{bmatrix} \in \mathbb{R}^{(n+n_K) \times m_1}, \quad \hat{B}_{2,i} = \begin{bmatrix} 0 & B_{2,i} \\ I & 0 \end{bmatrix} \in \mathbb{R}^{(n+n_K) \times (n_K+m_2)},
\]

\[
\hat{C}_{1,i} = \begin{bmatrix} C_{1,i} & 0 \end{bmatrix} \in \mathbb{R}^{l_1 \times (n+n_K)}, \quad \hat{C}_{2,i} = \begin{bmatrix} 0 & I \\ C_{2,i} & 0 \end{bmatrix} \in \mathbb{R}^{(n_K+l_2) \times (n+n_K)},
\]

\[
\hat{D}_{12,i} = \begin{bmatrix} 0 & D_{12,i} \end{bmatrix} \in \mathbb{R}^{l_1 \times (n_K+m_2)}, \quad \hat{D}_{21,i} = \begin{bmatrix} 0 & D_{21,i} \end{bmatrix} \in \mathbb{R}^{(n_K+l_2) \times m_1}
\]

for \(i \in \{1, \ldots, N\}\). Let

\[
\mathcal{T}_\mathcal{K} = \{ (\hat{A}_i, \hat{B}_i, \hat{C}_i, \hat{D}_i) : i \in \mathcal{L}_L(\Theta) \}.
\]

If we define the closed-loop state by

\[
\dot{x}(t) = [x(t)^T \ x_K(t)^T]^T \in \mathbb{R}^{n+n_K},
\]

then the closed-loop system \((\mathcal{T}_\mathcal{K}, \Theta_L)\) has the representation

(3.11)

\[
\dot{x}(t+1) = \hat{A}_{\theta_L(t)} \dot{x}(t) + \hat{B}_{\theta_L(t)} w(t),
\]

\[
z(t) = \hat{C}_{\theta_L(t)} \dot{x}(t) + \hat{D}_{\theta_L(t)} w(t)
\]

for each \(L\)-path switching sequence \(\theta_L \in \Theta_L\).

Let \(N_L\) be the cardinality of \(\mathcal{L}_L(\Theta)\). If we label the elements of \(\mathcal{L}_L(\Theta)\) from 1 to \(N_L\), then each \(L\)-path switching sequence \(\theta_L = (\theta_L(0), \theta_L(1), \ldots) \in \Theta_L\) can be considered a closed-loop switching sequence in \(\{1, \ldots, N_L\}\); letting \(\theta_L(t) = 0\) for \(t < 0\), the closed-loop switching path \((\theta_L(t-L-M), \ldots, \theta_L(t))\) can be identified with the switching path \((\theta(t-L-M), \ldots, \theta(t))\) for each triple \((t, L, M)\) of nonnegative integers. This leads to the following identities for all integers \(L > 0\) and \(M \geq 0\):

(3.12)

\[
\mathcal{M}_M(\Theta_L) = \mathcal{L}_M(\Theta_L) = \mathcal{L}_{M+L}(\Theta).
\]

Hence, even if \(L > 0\), the closed-loop system \((\mathcal{T}_\mathcal{K}, \Theta_L)\) is a switched linear system, where the closed-loop modes are the \(L\)-paths in \(\mathcal{L}_L(\Theta)\), and the closed-loop \(M\)-paths are the \((M+L)\)-paths in \(\mathcal{L}_{M+L}(\Theta)\) for each nonnegative integer \(M\).

Lemma 3.5. Let \(\mathcal{T}\) be as in (3.6); let \(\Theta \subset \Omega\) be nonempty. Suppose that \(\mathcal{K}\) is finite-path-dependent as in (3.8) with some nonnegative integer \(L\). Then the closed-loop system \((\mathcal{T}_\mathcal{K}, \Theta_L)\) is uniformly exponentially stable and uniformly strictly contractive if and only if there exist an integer \(M \geq L\) and an indexed family \(\{X_j : j \in \mathcal{L}_M(\Theta)\}\) of symmetric positive definite matrices \(X_j \in \mathbb{R}^{(n+n_K) \times (n+n_K)}\) such that

(3.13)

\[
H_{(i_0, \ldots, i_M)} + G_{i_M}^T K_{(i_{M-L-\ldots, i_M})} F_{i_M} + F_{i_M}^T K_{(i_{M-L-\ldots, i_M})} G_{i_M} < 0
\]
for all \( M \)-paths \((i_0, \ldots, i_M) \in \mathcal{L}_M(\Theta)\), where

\[
H_i = \begin{bmatrix}
-X^{-1}_{i_M} & \hat{A}_{i_M} & \hat{B}_{1,i_M} & 0 \\
\hat{A}^T_{i_M} & -X_{i_M} & 0 & \hat{C}^T_{1,i_M} \\
\hat{B}^T_{i_M} & 0 & -I & D_{T1,i_M} \\
0 & \hat{C}_{1,i_M} & D_{11,i_M} & -I
\end{bmatrix} \in \mathbb{R}^{(2n+2n_K+m_1+l_1) \times (2n+2n_K+m_1+l_1)}
\]

for \( i = (i_0, \ldots, i_M) \in \mathcal{L}_M(\Theta) \) and where

\[
\begin{bmatrix}
F_i \\
G_i
\end{bmatrix} = \begin{bmatrix}
\hat{B}^T_{2,i} & 0 & 0 & \hat{D}^T_{12,i} \\
0 & \hat{C}_{2,i} & \hat{D}_{21,i} & 0
\end{bmatrix} \in \mathbb{R}^{(2n_K+m_2+l_2) \times (2n+2n_K+m_1+l_1)}
\]

for \( i \in \{1, \ldots, N\} \).

Proof. Let us first assume that \( L > 0 \). It follows from Theorem 3.3 and (3.12) that \((T_K, \Theta_L)\) is uniformly stable and contractive if and only if there exists a nonnegative integer \( M \) satisfying the following: either \( M = 0 \) and there is a single \( X_0 > 0 \) such that

\[
(3.14a)\quad \begin{bmatrix}
\bar{A}_i & \bar{B}_i \\
\bar{C}_i & \bar{D}_i
\end{bmatrix}^T \begin{bmatrix}
X_0 & 0 \\
0 & I
\end{bmatrix} \begin{bmatrix}
\bar{A}_i & \bar{B}_i \\
\bar{C}_i & \bar{D}_i
\end{bmatrix} - \begin{bmatrix}
X_0 & 0 \\
0 & I
\end{bmatrix} < 0
\]

for \( i = (i_0, \ldots, i_L) \in \mathcal{L}_L(\Theta) \), or \( M > 0 \) and there are \( X_{(j_1, \ldots, i_M+L)} > 0 \) such that

\[
(3.14b)\quad \begin{bmatrix}
\bar{A}_{(i_M, \ldots, i_M+L)} & \bar{B}_{(i_M, \ldots, i_M+L)} \\
\bar{C}_{(i_M, \ldots, i_M+L)} & \bar{D}_{(i_M, \ldots, i_M+L)}
\end{bmatrix}^T \begin{bmatrix}
X_{i_0} & 0 \\
0 & I
\end{bmatrix} \begin{bmatrix}
\bar{A}_{(i_M, \ldots, i_M+L)} & \bar{B}_{(i_M, \ldots, i_M+L)} \\
\bar{C}_{(i_M, \ldots, i_M+L)} & \bar{D}_{(i_M, \ldots, i_M+L)}
\end{bmatrix} - \begin{bmatrix}
X_{i_0} & 0 \\
0 & I
\end{bmatrix} < 0
\]

for \( i = (i_0, \ldots, i_M+L) \in \mathcal{L}_M+L(\Theta) \). If \( M = 0 \), then we may put \( X_{(j_1, \ldots, j_L)} = X_0 \) for all \( L \)-paths \((j_1, \ldots, j_L)\) and write (3.14a) as

\[
(3.14c)\quad \begin{bmatrix}
\bar{A}_i & \bar{B}_i \\
\bar{C}_i & \bar{D}_i
\end{bmatrix}^T \begin{bmatrix}
X_{i_0} & 0 \\
0 & I
\end{bmatrix} \begin{bmatrix}
\bar{A}_i & \bar{B}_i \\
\bar{C}_i & \bar{D}_i
\end{bmatrix} - \begin{bmatrix}
X_{i_0} & 0 \\
0 & I
\end{bmatrix} < 0;
\]

if (3.14c) holds for \( i \in \mathcal{L}_L(\Theta) \), with some \( X_{(j_1, \ldots, j_L)} > 0 \), then Lemma 2.3 and the finiteness of \( \mathcal{L}_L(\Theta) \) imply that \((T_K, \Theta_L)\) is uniformly stable and contractive. Now, as in [20, p. 431], rewrite (3.14b) and (3.14c) (the former for \( M > 0 \) and the latter for \( M = 0 \)) as inequalities of the form (3.13) using the decompositions (3.10) along with the Schur complement formula and an appropriate congruence transformation. Replacing \( M \) with \( M - L \) then yields the desired result.

If \( L = 0 \), then \( \mathcal{M}_M(\Theta_L) = \mathcal{M}_M(\Theta) \), which is not equal to \( \mathcal{L}_{M+L}(\Theta) = \mathcal{L}_M(\Theta) \) in general. However, the proof of Theorem 3.3 shows that the existence of \( X_j > 0 \), \( j \in \mathcal{M}_M(\Theta) \), such that (3.4) holds for all \( i = (i_0, \ldots, i_M) \in \mathcal{M}_M(\Theta) \), suffices for the existence of \( X_j > 0 \), \( j \in \mathcal{L}_M(\Theta) \), such that (3.4) holds for all \( i = (i_0, \ldots, i_M) \in \mathcal{L}_M(\Theta) \). Therefore, the proof of the result for \( L = 0 \) is identical to the case of \( L > 0 \). □

Note that Lemma 3.5 is stated in terms of the closed-loop \((M - L)\)-paths in \( \mathcal{L}_M(\Theta) \), \( M \geq L \), rather than the closed-loop \( M \)-paths in \( \mathcal{M}_M(\Theta_L) \), because the former are easier to deal with. Inequality (3.13) is amenable to the standard linear matrix
inequality embedding technique, originally developed for linear time-invariant systems [33, 20]. Finite-path-dependent controllers arise naturally from this technique.

Definition 3.6. The controller \((\mathbf{K}, \Theta_L)\) is said to be an admissible \((L\text{-path-dependent})\) synthesis \((\text{of order } n_K)\) for the system \((\mathbf{T}, \Theta)\) if the closed-loop system \((\mathbf{T}_{\mathbf{K}}, \Theta_L)\) is uniformly exponentially stable and uniformly strictly contractive.

Theorem 3.7. Let \(\mathbf{T}\) be as in (3.6); let \(\Theta \subset \Omega\) be nonempty. Suppose that \(n_K \geq n\). There exists an admissible finite-path-dependent synthesis \((\text{of order } n_K)\) for the system \((\mathbf{T}, \Theta)\) if and only if there exist a nonnegative integer \(M > j\), and matrices \(\Theta_j \in \mathbb{R}^{n \times n}\) such that

\[
\begin{align}
(3.15a) & \quad N_{F,i,M}^T \begin{bmatrix} A_{iM} & B_{1,iM} \\ C_{1,iM} & D_{11,iM} \end{bmatrix} \begin{bmatrix} \mathbf{R}_i & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{iM} & B_{1,iM} \\ C_{1,iM} & D_{11,iM} \end{bmatrix}^T - \begin{bmatrix} \mathbf{R}_i & 0 \\ 0 & I \end{bmatrix} N_{F,i,M} < 0, \\
(3.15b) & \quad N_{G,i,M}^T \begin{bmatrix} A_{iM} & B_{1,iM} \\ C_{1,iM} & D_{11,iM} \end{bmatrix}^T \begin{bmatrix} \mathbf{R}_i & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{iM} & B_{1,iM} \\ C_{1,iM} & D_{11,iM} \end{bmatrix} - \begin{bmatrix} \mathbf{R}_i & 0 \\ 0 & I \end{bmatrix} N_{G,i,M} < 0, \\
(3.15c) & \quad \begin{bmatrix} \mathbf{R}_i & \mathbf{I} \\ \mathbf{I} & \mathbf{S}_i \end{bmatrix} \geq 0
\end{align}
\]

for all \(M\)-paths \(i = (i_0, \ldots, i_M) \in \mathcal{M}_M(\Theta)\), where

\[
N_{F,i} = N\left(\begin{bmatrix} B_{2,i} & D_{12,i} \end{bmatrix}\right), \quad N_{G,i} = N\left(\begin{bmatrix} C_{2,i} & D_{21,i} \end{bmatrix}\right)
\]

for \(i \in \{1, \ldots, N\}\). Moreover, if (3.15) holds for all \(i = (i_0, \ldots, i_M) \in \mathcal{M}_M(\Theta)\), then there exist a nonnegative integer \(L \leq M\) and matrices \(\mathbf{K}_{i_0, \ldots, i_L} \in \mathbb{R}^{(n_K + m_2) \times (n_K + m_2)}\) such that (3.13) holds for all \((i_0, \ldots, i_M) \in \mathcal{M}_M(\Theta)\) with

\[
X_j = \begin{bmatrix} \mathbf{S}_j & \mathbf{U}_j \mathbf{V}_j^\top \\ \mathbf{V}_j^\top \mathbf{U}_j & \mathbf{V}_j \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{R}_j & -\mathbf{R}_j \mathbf{U}_j \mathbf{V}_j^{-\frac{1}{2}} \\ -\mathbf{V}_j^{-\frac{1}{2}} \mathbf{U}_j^\top \mathbf{R}_j & \mathbf{V}_j^{-\frac{1}{2}} (1 + \mathbf{U}_j^\top \mathbf{R}_j \mathbf{U}_j) \mathbf{V}_j^{-\frac{1}{2}} \end{bmatrix} > 0
\]

for \(j \in \mathcal{M}_M(\Theta)\), where \(\mathbf{U}_j \in \mathbb{R}^{n \times n}\) and \(\mathbf{V}_j \in \mathbb{R}^{n \times n}\) are any matrices such that \(\mathbf{U}_j \mathbf{U}_j^\top = \mathbf{S}_j = \mathbf{R}_j^{-1}\) and \(\mathbf{V}_j > 0\); in particular, one may take \(L = M\).

Proof. It follows from [20, Lem. 3.1] that, with \(L = M\), inequality (3.13) in \(\mathbf{K}_{(i_M, \ldots, i_M)} = \mathbf{K}_{(i_0, \ldots, i_M)}\) is feasible if and only if

\[
N\left(\mathbf{F}_{i_M}\right)^\top \mathbf{H}_{(i_0, \ldots, i_M)} N\left(\mathbf{F}_{i_M}\right) < 0; \quad N\left(\mathbf{G}_{i_M}\right)^\top \mathbf{H}_{(i_0, \ldots, i_M)} N\left(\mathbf{G}_{i_M}\right) < 0.
\]

These inequalities are equivalent to (3.15) due to the Schur complement arguments in [20, sect. 5] together with the matrix inverse completion result in [33, Lem. 6.2]. The proof of the latter shows that \(X_j, j \in \mathcal{M}_M(\Theta)\), can be reconstructed from \(X_j\) and \(S_j\) through (3.16). Now, since \(\mathcal{M}_M(\Theta) \subset \mathcal{L}_M(\Theta)\), the existence of matrices \(\mathbf{K}_{(i_0, \ldots, i_M)}\) such that (3.13) holds for \((i_0, \ldots, i_M) \in \mathcal{L}_M(\Theta)\), implies the feasibility of (3.15) for \((i_0, \ldots, i_M) \in \mathcal{M}_M(\Theta)\).

To show the converse, suppose that there are a nonnegative integer \(M\) and matrices \(\mathbf{R}_j, S_j > 0, j \in \mathcal{M}_M(\Theta)\), such that (3.15) holds for \(i = (i_0, \ldots, i_M) \in \mathcal{M}_M(\Theta)\). Assume \(M > 0\) without loss of generality. Reconstruct \(X_j > 0, j \in \mathcal{M}_M(\Theta)\), through
Then there exist matrices $K(i_0,\ldots,i_M)$, $(i_0,\ldots,i_M) \in M_M(\Theta)$, such that (3.13), with $L = M$, holds for $(i_0,\ldots,i_M) \in M_M(\Theta)$. By the definition of $M_M(\Theta)$, for each $j \in L_0(\Theta)$, one can choose a switching path $(i_0^j,\ldots,i_{M-1}^j) \in \{0,\ldots,N\}^M$ such that (3.3), with $L = M$, holds for all $\theta \in \Theta$. If we put

$$X_{\theta_M(t)} = \begin{cases} X_{\{i_0^0,\ldots,i_{M-1}^0\}}, & t = 0; \\ X_{\{i_0^t,\ldots,i_{M-1}^t,\theta(0),\ldots,\theta(t)\}}, & 0 < t < M; \end{cases}$$

$$K_{\theta_M(t)} = \begin{cases} K_{\{i_0^0,\ldots,i_{M-1}^0\}}, & t = 0; \\ K_{\{i_0^t,\ldots,i_{M-1}^t,\theta(0),\ldots,\theta(t)\}}, & 0 < t < M, \end{cases}$$

for all $\theta_M \in \Theta_M$ and $t \geq 0$ such that $\theta_M(t) \notin M_M(\Theta)$, then we recover (3.13) for all $(i_0,\ldots,i_M) \in L_M(\Theta)$. The result then follows from Lemma 3.5.

**Remark 6.** Given a nonnegative integer $M$, the number of systems of linear matrix inequalities (3.15) to solve simultaneously is equal to the cardinality of $M_M(\Theta)$, and the feasibility of inequalities (3.15) is sufficient for the existence of an admissible $L$-path-dependent synthesis for some $L \leq M$. However, given a nonnegative integer $L$, the feasibility of (3.15) for some $M \leq L$ is sufficient but not necessary for the existence of an admissible $L$-path-dependent synthesis. In fact, even if the existence of an admissible $L$-path-dependent controller guarantees that inequalities (3.15) are feasible for some finite $M$, there is no upper bound on such an $M$. See Example 3.

**Remark 7.** In the case of reduced order controllers with $n_K < n$, the matrices $R_j, S_j, j \in M_M(\Theta)$, must satisfy

$$\text{rank} \begin{bmatrix} R_{i_0} & I \\ I & S_{i_N} \end{bmatrix} \leq n + n_K,$$

in addition to (3.15), for all $(i_0,\ldots,i_M) \in M_M(\Theta)$.

**Corollary 3.8.** Let $T$ be as in (3.6); let $\Theta \subset \Omega$ be nonempty. Suppose that $n_K \geq n$. There exists a finite-path-dependent linear output feedback controller of order $n_K$ that uniformly stabilizes the system $(T, \Theta)$ if and only if there exist a nonnegative integer $M$ and matrices $R_j, S_j > 0$ such that

\begin{align}
(3.17a) & \quad N(B^T_{i_M} (A_{i_M} R_{i_M} A^T_{i_M} - R_{i_M}) N(B^T_{i_M}) < 0, \\
(3.17b) & \quad N(C_{i_M})^T (A^T_{i_M} S_{i_M} A_{i_M} - S_{i_M}) N(C_{i_M}) < 0, \\
(3.17c) & \quad \begin{bmatrix} R_{i_M} & I \\ I & S_{i_M} \end{bmatrix} \geq 0
\end{align}

for all $(i_0,\ldots,i_M) \in N_M(\Theta)$.

**Proof.** In (3.15), set the matrices $B_{i,i}, C_{i,i}, D_{11,i}, D_{12,i},$ and $D_{21,i}$ to zero for all $i = 1,\ldots, N$, and obtain (3.17). The rest of the proof proceeds similarly to that of Corollary 3.4.

**Remark 8.** Corollary 3.8 reverts to the stabilization result in [30] if a strongly connected directed graph defines $\Theta$.

Suppose that a set of matrices $K_i, i \in M_L(\Theta)$, is obtained by solving (3.13) via Theorem 3.7 for some nonnegative integer $L \leq M$. If $L = 0$, then it follows from $M_0(\Theta) = L_0(\Theta)$ that we have all the matrices $K_i, i \in L_0(\Theta)$, that define an admissible zero-path-dependent controller synthesis. If $L > 0$, on the other hand,
then choose switching paths \( \{i_0^t, \ldots, i_{L-1}^t\} \in \{0, \ldots, N\}^L, \ j \in \mathcal{L}_0(\Theta) \), such that (3.3) holds for all \( \theta \in \Theta \), and put
\[
K_{\theta L}(t) = \begin{bmatrix} \theta(0) \\ \vdots \\ \theta(L-1) \end{bmatrix}
\]
whenever \( \theta_L \in \Theta_L, \ t < L \), and \( \theta_L(t) \notin \mathcal{M}_L(\Theta) \); then we recover all matrices \( K_i, \ i \in \mathcal{L}_L(\Theta) \), that define an admissible \( L \)-path-dependent controller synthesis.

**Example 3.** Let \( \theta^{(k)}(t) \) be as in Example 2 for positive integers \( k \). Let
\[
\mathcal{T} = \{(1, -1, 1, -1, 1, -1, -1), (1, 0, 0, 0, 0, 0, 0)\}
\]
and \( \Theta^{(k)} = \{\theta^{(k)}\} \) so that the controlled system \( (\mathcal{T}, \Theta^{(k)}) \) has the representation
\[
\begin{align*}
x(t+1) & = x(t) - w(t) + u(t), \\
z(t) & = -x(t) + w(t) - u(t), \\
y(t) & = x(t) - w(t)
\end{align*}
\]
in mode 1 (i.e., when \( \theta^{(k)}(t) = 1 \), and
\[
\begin{align*}
x(t+1) & = x(t), \\
z(t) & = 0, \\
y(t) & = 0
\end{align*}
\]
in mode 2 (i.e., when \( \theta^{(k)}(t) = 2 \)). Let \( M \) be a nonnegative integer. If \( M < k \), then, because mode 2 is not stabilizable and because \( \mathcal{M}_M(\Theta^{(k)}) \) contains the switching path \( (2, \ldots, 2) \) that consists of mode 2 only, inequality (3.15) cannot be satisfied for all \( i = (i_0, \ldots, i_M) \in \mathcal{M}_M(\Theta^{(k)}) \). On the other hand, the cardinality of \( \mathcal{M}_M(\Theta^{(k)}) \) is equal to that of \( \mathcal{M}_k(\Theta^{(k)}) \) for all \( M > k \). Hence, to design an admissible synthesis for \( (\mathcal{T}, \Theta^{(k)}) \), it suffices to consider the single path length \( M = k \) in (3.15). It is readily verified that there indeed is an admissible finite-path-dependent controller for \( (\mathcal{T}, \Theta^{(k)}) \) for each \( k \). This shows that there does not exist a general upper bound on the path length \( M \) in Theorem 3.7.

In this particular example, it is easy to find solutions \( R_j, S_j > 0, j \in \mathcal{M}_M(\Theta^{(k)}) \), to (3.15) with \( M = k \), such that \( L = 0 \) (which leads to mode-dependent controllers) suffices for (3.13) to be feasible. This indicates that the existence of an admissible \( L \)-path controller synthesis does not necessarily lead to the feasibility of (3.15) with \( M = L \).

**Definition 3.9.** Let \( \gamma > 0 \). The system \( (\mathcal{G}, \Theta) \) is said to satisfy uniform disturbance attenuation level \( \gamma \) if there exists a \( \tilde{\gamma} \in (0, \gamma) \) such that, whenever \( x(t_0) = 0 \),
\[
\sum_{s=t_0}^t \|z(s)\|^2 \leq \tilde{\gamma}^2 \sum_{s=t_0}^t \|w(s)\|^2
\]
for \( t \geq t_0 \geq 0 \), for \( w \in L^2(\mathbb{R}^m) \), and for \( \theta \in \Theta \).

**Definition 3.10.** Let \( \gamma > 0 \). The controller \( (\mathcal{K}, \Theta_L) \) is said to be a \( \gamma \)-admissible (\( L \)-path-dependent) synthesis (of order \( nK \)), or to achieve uniform disturbance attenuation level \( \gamma \), for the system \( (\mathcal{T}, \Theta) \) if the closed-loop system \( (\mathcal{T}_K, \Theta_L) \) is uniformly exponentially stable and satisfies uniform disturbance attenuation level \( \gamma \).

Given a \( \gamma > 0 \), let
\[
\mathcal{G}^{(\gamma)} = \{(A_i, \gamma^{-1/2}B_i, \gamma^{-1/2}C_i, \gamma^{-1}D_i) : i = 1, \ldots, N\}.
\]
Then \( (\mathcal{G}, \Theta) \) satisfies uniform disturbance attenuation level \( \gamma \) if and only if \( (\mathcal{G}^{(\gamma)}, \Theta) \) is uniformly strictly contractive. Using this fact and Theorem 3.7, and applying the Schur complement formula to (3.15a) and (3.15b), we obtain the following.
COROLLARY 3.11. Let $T$ be as in (3.6), and let $\Theta \subset \Omega$ be nonempty. Let $\gamma > 0$. Suppose that $n_K \geq n$. There exists a $\gamma$-admissible finite-path-dependent synthesis of order $n_K$ for the system $(T, \Theta)$ if and only if there exist a nonnegative integer $M$ and an indexed family $\{(R_j, S_j) : j \in \mathcal{M}_M(\Theta)\}$ of pairs of symmetric positive definite matrices $R_j, S_j \in \mathbb{R}^{n \times n}$ such that

\begin{align}
N_{F,i_M} & \quad 0 \\
0 & \quad I \end{align}

(3.18a)

\begin{align}
A_{i_M} R_{i_M} \gamma & \quad A_{i_M} R_{i_M} \mathcal{C}_{1,i_M}^T \quad B_{1,i_M} \\
C_{1,i_M} R_{i_M} A_{i_M}^T & \quad C_{1,i_M} R_{i_M} \mathcal{C}_{1,i_M}^T + \gamma I \quad D_{11,i_M} \\
B_{1,i_M}^T & \quad -I \end{align}

\begin{pmatrix}
N_{F,i_M} & 0 \\
0 & I \end{pmatrix} < 0,

(3.18b)

\begin{align}
N_{G,i_M} & \quad 0 \\
0 & \quad I \end{align}

(3.18c)

\begin{align}
R_{i_-} & \quad I \\
I & \quad S_{i_-} \end{align}

\begin{pmatrix}
0 & \quad \mathcal{L}_{i_M} \quad 0 \\
\mathcal{L}_{i_M} & \quad -I \quad -I \\
0 & \quad -I \quad \mathcal{D}_{11,i_M} \end{pmatrix} \geq 0

\text{for all } i = (i_0, \ldots, i_M) \in \mathcal{M}_M(\Theta). Moreover, if (3.18) holds for all $i = (i_0, \ldots, i_M) \in \mathcal{M}_M(\Theta)$, then there exist a nonnegative integer $L$, matrices $K_{i_M-L, \ldots, i_M} \in \mathbb{R}^{(n_K + m_2) \times (n_K + l_2)}$ such that (3.13) holds for $(i_0, \ldots, i_M) \in \mathcal{M}_M(\Theta)$ with

\begin{equation}
H_{i_0, \ldots, i_M} = \begin{bmatrix}
-X_{i_0, \ldots, i_M}^{-1} & \mathcal{A}_{i_M} & \mathcal{B}_{i_M} & 0 \\
\mathcal{A}_{i_M}^T & -X_{i_0, \ldots, i_M}^{-1} & \mathcal{C}_{i_M} & 0 \\
\mathcal{B}_{i_M}^T & 0 & \mathcal{D}_{i_M} & -I \\
0 & \mathcal{C}_{i_M} & \mathcal{D}_{i_M} & -I \end{bmatrix},
\end{equation}

where the matrices $X_j$ are reconstructed via (3.16) for $j \in \mathcal{M}_M(\Theta)$; in particular, one may take $L = M$.

Example 4. Let

\begin{equation}
T = \{(0.3, 0, 0, 1, 1, 0, 1, 0), (3.0, 5, 1, 1, 0, 1, 0)\};
\end{equation}

let $\Theta$ be as in Example 1. Then the controlled system $(T, \Theta)$ has the representation

\begin{equation}
x(t+1) = 0.3x(t), \quad z(t) = x(t) + u(t), \quad y(t) = x(t)
\end{equation}

in mode 1, and

\begin{equation}
x(t+1) = 3x(t) + 0.5w(t) + u(t), \quad z(t) = x(t) + u(t), \quad y(t) = x(t)
\end{equation}

in mode 2. With $M = 0$, the system of linear matrix inequalities (3.18) is feasible for any $\gamma > 1$; it is easy to see that, if $\gamma = 1$, then inequalities (3.18) are not feasible for any $M \geq 0$. Solving the semidefinite program of minimizing $\gamma$ subject to (3.18), and applying Corollary 3.11, we obtain a mode-dependent controller $(K, \Theta_0)$ with

\begin{equation}
K_1 = \begin{bmatrix}
0 & 0 \\
0 & -1
\end{bmatrix}, \quad K_2 = \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}.
\end{equation}
The resulting controller \((K, \Theta_0)\) has the representation

\[
u(t) = \begin{cases} -y(t) & \text{if } \theta(t) = 1, \\
-3y(t) & \text{if } \theta(t) = 2. \end{cases}
\]

This controller is optimal in the sense that it achieves any uniform disturbance attenuation level greater than one, and that no finite-path-dependent linear dynamic output feedback controller achieves the uniform disturbance attenuation level equal to one. (The optimal controller is static in this case because \(C_{2,1} = C_{2,2} = 1\) and \(D_{21,1} = D_{21,2} = 0\) lead to perfect observation of the state.) The controller, however, is not optimal under the notion of path-by-path disturbance attenuation—see Example 7.

4. Control of Markovian jump linear systems. In Markovian jump linear systems, the switching sequence is modeled as a finite-state homogeneous Markov chain. Let \(G\) be as in (3.1). Let \(p = (p_i) \in \mathbb{R}^{1 \times N}\) be a row vector whose entries are nonnegative and sum to one; let \(P = (p_{ij}) \in \mathbb{R}^{N \times N}\) be a (row) stochastic matrix so that each row of \(P\) has nonnegative entries that sum to one. Then the discrete-time Markovian jump linear system, defined by the triple \((G, P, p)\), has the representation (3.2). Here, the switching sequence \(\theta\) is a realization of the Markov chain defined by the pair \((P, p)\), where \(P\) is the transition probability matrix and \(p\) the initial distribution. The state \(\theta(t)\) of the chain \((P, p)\) at time \(t\) defines the mode of \((G, P, p)\) at time \(t\); the distribution of the mode at time \(t\) is given by \(pP^t\). As in the previous section, let \(\Omega\) be the space of all infinite sequences in \(\{1, \ldots, N\}\). Let \(P\) be the unique consistent probability measure [39] on \(\Omega\) such that

\[
P\{ \theta(t + 1) = j | \theta(t) = i \} = p_{ij}, \quad P\{ \theta(0) = i \} = p_i
\]

for all \(i, j, t\).

**Definition 4.1.** The system \((G, P, p)\) is said to be almost surely uniformly (exponentially) stable if there exists a set \(\Theta \subset \Omega\) with \(P(\Theta) = 1\) such that the system \((G, \Theta)\) is uniformly exponentially stable.

**Definition 4.2.** The system \((G, P, p)\) is said to be almost surely uniformly (strictly) contractive if there exists a set \(\Theta \subset \Omega\) with \(P(\Theta) = 1\) such that the system \((G, \Theta)\) is uniformly strictly contractive.

A switching sequence \(\theta\) in \(\{1, \ldots, N\}\) is said to be admissible with respect to \((P, p)\) if \(p_{\theta(0)} > 0\) and \(p_{\theta(t)\theta(t+1)} > 0\) for \(t \geq 0\). If we define

\[
\Theta(P, p) = \{ \theta \in \Omega : \theta \text{ is admissible with respect to } (P, p) \}
\]

and let

\[
\mathcal{M}_L(P, p) = \mathcal{M}_L(\Theta(P, p)), \quad \mathcal{M}_L^-(P, p) = \mathcal{M}_L^-(\Theta(P, p))
\]

for nonnegative integers \(L\), then we have \(P(\Theta(P, p)) = 1\); on the other hand, whenever \((i_0, \ldots, i_L) \in \mathcal{M}_L(P, p)\), we have that \(P(\theta \in \Omega : (i_0, \ldots, i_L) \in \mathcal{L}_L(\{\theta\})) > 0\) so that \(\mathcal{M}_L(P, p) \subset \mathcal{M}_L(\Theta)\) whenever \(\Theta \subset \Omega\) and \(P(\Theta) = 1\).

**Example 5.** Let \(N = 3\), and let \((P, p)\) be a Markov chain with

\[
1(P) = \begin{bmatrix} 1 & 1 & 1 \\
0 & 0 & 1 \\
0 & 0 & 1 \end{bmatrix}.
\]
If \( \mathbf{1}(p) = [1 \ 0 \ 0] \), then
\[
\mathcal{L}_0(\mathbf{P}, p) = \mathcal{M}_0(\mathbf{P}, p) = \{1, 2, 3\};
\]
\[
\mathcal{L}_1(\mathbf{P}, p) = \{01, 11, 12, 13, 23, 33\}, \quad \mathcal{M}_1(\mathbf{P}, p) = \{11, 12, 13, 23, 33\},
\]
and so on. However, if \( \mathbf{1}(p) = [0 \ 1 \ 1] \), then
\[
\mathcal{L}_0(\mathbf{P}, p) = \mathcal{M}_0(\mathbf{P}, p) = \{2, 3\};
\]
\[
\mathcal{L}_1(\mathbf{P}, p) = \{02, 03, 23, 33\}, \quad \mathcal{M}_1(\mathbf{P}, p) = \{02, 23, 33\},
\]
and so on.

**Theorem 4.3.** Let \( \mathcal{G} \) be as in (3.1); let \( (\mathbf{P}, p) \) be a Markov chain. The system \((\mathcal{G}, \mathbf{P}, p)\) is almost surely uniformly exponentially stable and almost surely uniformly strictly contractive if and only if there exist a nonnegative integer \( M \) and an indexed family \( \{\mathbf{X}_j : j \in \mathcal{M}_M(\mathbf{P}, p)\} \) of symmetric positive definite matrices \( \mathbf{X}_j \in \mathbb{R}^{n \times n} \) such that (3.4) holds for all \( M \)-paths \( \mathbf{i} = (i_0, \ldots, i_M) \in \mathcal{M}_M(\mathbf{P}, p) \).

**Proof.** The result is immediate from Theorem 3.3: sufficiency follows from \( P(\Theta(\mathbf{P}, p)) = 1 \), and necessity from the fact that \( \Theta \subseteq \Omega \) and \( P(\Theta) = 1 \) implies \( \mathcal{M}_L(\mathbf{P}, p) \subseteq \mathcal{M}_L(\Theta) \). \( \square \)

**Remark 9.** Define
\[
   n_i(0) = 1, \quad n_i(L + 1) = \sum_{(j : p_{ij} > 0)} n_j(L)
\]
for \( i \in \{1, \ldots, N\} \) and for \( L = 0, 1, \ldots \). Given a Markov chain \((\mathbf{P}, p)\), let
\[
   S(\mathbf{p}) = \{j : p_j > 0\}, \quad T(\mathbf{P}, p) = \{j : p_{ij}^{(k)} > 0 \text{ for some } (i, k)\},
\]
where \( \mathbf{P}^k = (p_{ij}^{(k)}) \) is the \( k \)-step transition probability matrix [24]. Then, in Theorem 4.3, the number of linear matrix inequalities (3.4) to solve simultaneously, with a fixed \( M \), is equal to the cardinality of \( \mathcal{M}_M(\mathbf{P}, p) \), which is precisely given by
\[
   \sum_{j \in S(\mathbf{p}) \setminus T(\mathbf{P}, p)} \sum_{k=0}^{M-1} n_j(k) + \sum_{j \in S(\mathbf{p}) \cup T(\mathbf{P}, p)} n_j(M).
\]
In particular, if \( \mathbf{P} \) is irreducible (i.e., if the directed graph of \( \mathbf{P} \) is strongly connected—see, e.g., [25]), then the cardinality of \( \mathcal{M}_M(\mathbf{P}, p) \) is equal to \( \sum_{j=1}^{N} n_j(M) \).

**Remark 10.** Theorem 4.3 implies that the Markovian jump linear system \((\mathcal{G}, \mathbf{P}, p)\) is almost surely uniformly stable and contractive if and only if the switched linear system \((\mathcal{G}, \Theta(\mathbf{P}, p))\) is uniformly stable and contractive. Therefore, Markovian jump linear systems can be treated as if they are switched linear systems. Moreover, since the set \( \Theta(\mathbf{P}, p) \) depends only on the sparsity patterns of \( \mathbf{P} \) and \( p \), the almost sure uniform stability and contractiveness of \((\mathcal{G}, \mathbf{P}, p)\) is robust against sparsity pattern-preserving deviations from \( \mathbf{P} \) and \( p \).

**Remark 11.** When only the almost sure uniform stability is considered, it is immediate from Corollary 3.4 that it suffices to consider the “irreducible parts” of the Markov chain \((\mathbf{P}, p)\); see [30].

Let \( \mathcal{T} \) be as in (3.6), and let \((\mathbf{P}, p)\) be a Markov chain. Then the triple \((\mathcal{T}, \mathbf{P}, p)\) defines the controlled Markovian jump linear system described by (3.7), where \( \Theta \) is a realization of \((\mathbf{P}, p)\). As in the previous section, we make the standard assumption that the state \( \theta(t) \) of the chain \((\mathbf{P}, p)\) is perfectly observed at each time instant \( t \); we consider all finite-path-dependent controllers.
Fix a nonnegative integer $L$. Let

$$
\mathcal{K} = \{(A_{K,i}, B_{K,i}, C_{K,i}, D_{K,i}) : i \in \mathcal{L}_L(\Theta(p, p))\}.
$$

Then the pair $(\mathcal{K}, \Theta(p, p)_L)$ defines an $L$-path-dependent controller, whose representation is given by (3.9). Label the $L$-paths in $\mathcal{L}_L(\Theta(p, p))$ in dictionary order from 1 to $N_L$, where $N_L$ is the cardinality of $\mathcal{L}_L(\Theta(p, p))$. Define $P_L = (q_{ij}) \in \mathbb{R}^{N_L \times N_L}$ as follows: Whenever $(i_0, \ldots, i_L)$ and $(j_0, \ldots, j_L)$ are $L$-paths labeled $i$ and $j$, respectively, set $q_{ij} = p_{i_0 j_0}$ if $(i_0, \ldots, i_L)_+ = (j_0, \ldots, j_L)_-$; otherwise, set $q_{ij} = 0$. Also, define a row vector $p_L = (q_i) \in \mathbb{R}^{N_L}$ as follows: Whenever $(i_0, \ldots, i_L)$ is an $L$-path labeled $i$, set $q_i = p_{i_0}$ if $(i_0, \ldots, i_L)_- = (0, \ldots, 0)$; otherwise, set $q_i = 0$. Then the pair $(P_L, p_L)$ defines the $L$-path Markov chain generated by $(P, p)$. Consequently, the closed-loop system, given by the triple $(\mathcal{T}_\mathcal{K}, P_L, p_L)$ with

$$
\mathcal{T}_\mathcal{K} = \{(\tilde{A}_i, \tilde{B}_i, \tilde{C}_i, \tilde{D}_i) : i \in \mathcal{L}_L(\Theta(p, p))\},
$$

is a Markovian jump system whose representation is of the form (3.11) for each realization $\theta_L$ of $(P_L, p_L)$.

**Definition 4.4.** Let $\gamma > 0$. The system $(\mathcal{G}, P, p)$ is said to satisfy almost sure uniform disturbance attenuation level $\gamma$ if there exists a set $\Theta \subset \Omega$ with $P(\Theta) = 1$ such that the system $(\mathcal{G}, \Theta)$ satisfies uniform disturbance attenuation level $\gamma$.

**Definition 4.5.** Let $\gamma > 0$. The controller $(\mathcal{K}, \Theta(p, p)_L)$ is said to be a $\gamma$-admissible $(L$-path-dependent) synthesis of order $n_\mathcal{K}$, or to achieve almost sure uniform disturbance attenuation level $\gamma$, for the system $(\mathcal{T}, P, p)$ if the closed-loop system $(\mathcal{T}_\mathcal{K}, P_L, p_L)$ satisfies almost sure uniform disturbance attenuation level $\gamma$.

**Theorem 4.6.** Let $\mathcal{T}$ be as in (3.6); let $(P, p)$ be a Markov chain. Let $\gamma > 0$. Suppose that $n_\mathcal{K} \geq n$. There exists a $\gamma$-admissible finite-path-dependent synthesis of order $n_\mathcal{K}$ for the system $(\mathcal{T}, P, p)$ if and only if there exist a nonnegative integer $M$ and an indexed family $\{R_j, S_j : j \in \mathcal{M}_\mathcal{M}(P, p)\}$ of pairs of symmetric positive definite matrices $R_j, S_j \in \mathbb{R}^{n \times n}$ such that (3.18) holds for all $L$-paths $i = (i_0, \ldots, i_M) \in \mathcal{M}_\mathcal{M}(P, p)$. Moreover, if (3.18) holds for all $i = (i_0, \ldots, i_M) \in \mathcal{M}_\mathcal{M}(P, p)$, then there exist a nonnegative integer $L$ and matrices $K_{(i_M, i_{M-1}, i_{M-2}, \ldots, i_0)} \in \mathbb{R}^{(n_K + m_P) \times (n_K + m_P)}$ such that (3.13), with (3.19), holds for $(i_0, \ldots, i_M) \in \mathcal{M}_\mathcal{M}(P, p)$, where the matrices $X_j$ are given by (3.16) for $j \in \mathcal{M}_\mathcal{M}(P, p)$; in particular, one may take $L = M$.

Proof. The result immediately follows from Corollary 3.11.

**Remark 12.** The existence of finite-path-dependent controller syntheses achieving an almost sure uniform disturbance attenuation level for the Markovian jump linear system $(\mathcal{T}, P, p)$ is robust against sparsity pattern–preserving deviations from $P$ and $p$. The conservatism associated with this robustness property can be reduced via path-by-path disturbance attenuation—see the examples in the next section.

**Example 6.** Let the Markov chain $(P, p)$ have

$$
\begin{align*}
1(P) &= \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad 1(p) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}.
\end{align*}
$$

Then we have

$$
\mathcal{M}_0(P, p) = \{1, 2, 3\}, \quad \mathcal{M}_1(P, p) = \{12, 13, 23, 31\},
$$

$$
\mathcal{M}_2(P, p) = \{123, 131, 231, 312, 313\}.
$$
and so on. Let
\[
\mathcal{T} = \{(0.5, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0), (3, 0, 0, 0, 0, 0, 0, 0, 0, 0), (2, 1, 1, 1, 1, 1, 1, 1, 1, 1)\}
\]
so that the controlled Markovian jump linear system \((\mathcal{T}, P, p)\) has the representation
\[
x(t+1) = 0.5x(t), \quad z(t) = 0, \quad y(t) = 0;
\]
\[
x(t+1) = 3x(t), \quad z(t) = 0, \quad y(t) = 0;
\]
or
\[
x(t+1) = 2x(t) + w(t) + u(t),
\]
\[
z(t) = x(t) + w(t) + u(t),
\]
\[
y(t) = x(t) + w(t),
\]
depending on whether the mode at time \(t\) is 1, 2, or 3, respectively. With \(M = 0\), the system of linear matrix inequalities (3.18), over \(i_0 \in \mathcal{M}_0(P, p)\), is not feasible for any \(\gamma > 0\). However, with \(M = 1\), the semidefinite program of minimizing \(\gamma\) subject to (3.18) over \((i_0, i_1) \in \mathcal{M}_1(P, p)\) leads to \(\gamma = 0.834\); the same is true for all \(M > 1\).

Setting \(L = M = 1\) in (3.13) with (3.19), we obtain
\[
(4.2) \quad K_{(1,2)} = K_{(3,1)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad K_{(1,3)} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \quad K_{(2,3)} = \begin{bmatrix} 0 & 0.0000 \\ 0 & -1.5555 \end{bmatrix}.
\]
The resulting one-path-dependent controller is optimal (up to the third digit below the decimal point) in the sense that no finite-path-dependent linear dynamic output feedback controller achieves the disturbance attenuation level 0.833. This controller is applied as follows: For \(t = 0\), use \(K_{(1,3)}\) if \(\theta(0) = 1\), use \(K_{(1,2)}\) if \(\theta(0) = 2\), and use either \(K_{(1,3)}\) or \(K_{(2,3)}\) if \(\theta(0) = 3\); for \(t > 0\), use \(K_{(\theta(t-1), \theta(t))}\).

It turns out that setting \(L = 0\) and \(M = 1\) in (3.13) with (3.19) also works and results in a mode-dependent optimal controller with
\[
(4.3) \quad K_1 = K_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad K_3 = \begin{bmatrix} 0 & 0.0000 \\ 0 & -1.5555 \end{bmatrix}.
\]
This mode-dependent controller, however, is not optimal under a path-by-path disturbance attenuation criterion—see Example 8.

5. **Path-by-path disturbance attenuation.** In this section, we formulate a refined disturbance attenuation problem for switched linear systems. The result holds for Markovian jump linear systems as well. We introduce the notion of path-by-path disturbance attenuation that improves upon the uniform disturbance attenuation performance presented in previous sections. It turns out that, under the notion of path-by-path disturbance attenuation, finite-path-dependent controllers can outperform mode-dependent ones—see Examples 7 and 8. We shall use the same notation as in section 3.

**Definition 5.1.** If there exist a nonnegative integer \(M\), a positive integer \(n_K\), an indexed family \(\Gamma = \{\gamma_i; i \in \mathcal{M}(\Theta)\}\) of positive numbers \(\gamma_i\), and an indexed family \(\{R_j, S_j; j \in \mathcal{M}(\Theta)\}\) of pairs of symmetric positive definite matrices \(R_j, S_j \in \mathbb{R}^{n \times n}\) such that...
A nonnegative integer $L$ is said to be $M$-attained for all $(L,M)$ for the system $\Gamma$. Path-by-path attenuation and path-by-path disturbance attenuation is that, while a single variable $M$ achieves path-by-path disturbance attenuation level $\gamma$, a path-dependent controller can achieve a disturbance attenuation level $\gamma_M(\Theta)$.

Step 2. Reconstruct $\mathcal{X}_j$ from $(R_j, S_j)$ via (5.16) for all $j \in \mathcal{M}_M(\Theta)$.

Step 3. Solve (3.13) with (5.2) for all $(i_0, \ldots, i_M) \in \mathcal{M}_M(\Theta)$, and obtain a nonnegative integer $L \leq M$ and matrices $K_{(i_M-L, \ldots, i_M)}$.

The path-by-path optimal controller $(K, \Theta_L)$ resulting from Algorithm 5.2 shall be said to be $M$-path Pareto optimal because of the following property: If $(K, \Theta_L)$ achieves path-by-path disturbance attenuation level $\{\gamma_i; i \in \mathcal{M}_M(\Theta)\}$, then no $M$-path-dependent controller can achieve a disturbance attenuation level $\{\tilde{\gamma}_i; i \in \mathcal{M}_M(\Theta)\}$ such that $\tilde{\gamma}_i \leq \gamma_i$ for all $i \in \mathcal{M}_M(\Theta)$ and such that $\tilde{\gamma}_i < \gamma_i$ for some $i$.
In general, different sets of weights $\lambda_i$, $i \in \mathcal{M}_M(\Theta)$, result in different interpretations of optimality—see Examples 7 and 8—and hence different Pareto optimal path-by-path disturbance attenuation levels—see Example 9.

Example 7. Let us revisit the controlled switched linear system $(\mathcal{T}, \Theta)$ considered in Example 4. In this example, we set $\gamma = 10^3$ and $\lambda_i = 1$, $i \in \mathcal{M}_M(\Theta)$, for Algorithm 5.2. Running Algorithm 5.2 with $M = 0$, it turns out that the mode-dependent controller obtained in Example 4 achieves any path-by-path disturbance attenuation level $\{\gamma_i\}$ satisfying $\gamma_1 > 0$ and $\gamma_2 > 1$. In particular, this mode-dependent controller is $\gamma_1$-admissible for $(\mathcal{T}, (1,1,\ldots))$ and $\gamma_2$-admissible for $(\mathcal{T}, ((1,2,1,2,\ldots),(2,2,\ldots)))$ whenever $\gamma_1 > 0$ and $\gamma_2 > 1$. Running Algorithm 5.2 with $M = 1$ leads to $\gamma_{(1,1)} = \gamma_{(2,1)} = 0$ and $\gamma_{(1,2)} = 0.795$, but $\gamma_{(2,2)} = 1.16$, which is greater than one.

However, running Algorithm 5.2 with $M = 2$ leads to a one-path-dependent controller ($L = 1$) with

$$K_{(0,1)} = K_{(1,1)} = K_{(2,1)} = \begin{bmatrix} 0 & 0 \\
0 & -1 \end{bmatrix}, \quad K_{(1,2)} = \begin{bmatrix} 0 & 0 \\
0 & -1 \end{bmatrix}, \quad K_{(2,2)} = \begin{bmatrix} 0 & 0 \\
0 & -3 \end{bmatrix}. \quad \text{(5.3)}$$

This controller is two-path Pareto optimal and achieves any path-by-path disturbance attenuation level $\{\gamma_i: i \in \mathcal{M}_2(\Theta)\}$ such that $\gamma_i > 1$ for $i = (2,2,2)$, and $\gamma_i > 0$ otherwise. In particular, this one-path-dependent controller is $\gamma_1$-admissible for $(\mathcal{T}, ((1,1,\ldots),(1,2,1,2,\ldots)))$ and $\gamma_2$-admissible for $(\mathcal{T}, (2,2,\ldots))$ whenever $\gamma_1 > 0$ and $\gamma_2 > 1$. Clearly, this is an improvement over the path-by-path performance of the mode-dependent controller obtained in Example 4 with $M = 0$. In fact, it is not difficult to see that, in this example, no mode-dependent controller can achieve a path-by-path disturbance attenuation level $\{\gamma_i: i \in \mathcal{M}_2(\Theta)\}$ such that, say, $\gamma_i = 0.01$ for $i \neq (2,2,2)$.

Example 8. This example revisits the controlled Markovian jump linear system $(\mathcal{T}, \mathcal{P}, p)$ considered in Example 6. Since $1(\mathcal{P}) = 1$, the Markov chain $(\mathcal{P}, p)$ is irreducible and aperiodic [25]. Let $\pi = [\pi_1 \pi_2 \pi_3]$ be the unique steady-state distribution of the chain. For each nonnegative integer $M$, let

$$\lambda_{(i_0,\ldots,i_M)} = \pi_{i_0}p_{i_0i_1}\cdots p_{i_{M-1}i_M}, \quad (i_0,\ldots,i_M) \in \mathcal{M}_M(\mathcal{P}, p). \quad \text{(5.4)}$$

Then Algorithm 5.2 minimizes the “average steady-state disturbance attenuation level over $M$-paths,” where each $M$-path $(i_0,\ldots,i_M)$ is interpreted as the mode $i_M$ preceded by the path $(i_0,\ldots,i_{M-1})$. With $\gamma = \infty$ and $M = 1$, we obtain the one-path-dependent controller ($L = 1$) given by (4.2). This controller achieves the path-by-path disturbance attenuation level $\{\gamma_i: i \in \mathcal{M}_1(\mathcal{P}, p)\}$ where, e.g., $\gamma_{(1,2)} = \gamma_{(1,3)} = \gamma_{(3,1)} = 0.001$ and $\gamma_{(2,3)} = 0.834$; the weighted sum of these disturbance attenuation levels $\gamma_i$ is $0.001 + 0.833\pi_2$. On the other hand, the mode-dependent controller given by (4.3) achieves the path-by-path disturbance attenuation level $\{\gamma_i: i \in \mathcal{M}_1(\mathcal{P}, p)\}$ where, e.g., $\gamma_{(1,2)} = \gamma_{(3,1)} = 0.001$ and $\gamma_{(1,3)} = \gamma_{(2,3)} = 0.834$. Clearly, the one-path-dependent controller performs better than the mode-dependent controller in terms of average steady-state performance. In fact, it is not difficult to see that, for some chain $(\mathcal{P}, p)$ with (4.1), there does not exist a mode-dependent controller that achieves the average steady-state performance level $0.001 + 0.833\pi_2$.

Example 9. We are to balance the Pendubot [40], a two-link planar robot with revolute joints and actuation at the shoulder, subject to random but bounded delays in the feedback loop from the relative angular position sensor to the actuator. We
use the following linearized model (of order 4) borrowed from [43]:

\[
x(t + 1) = Ax(t) + B_1w(t) + B_2u(t),
\]

\[
z(t) = C_1x(t) + D_{11}w(t) + D_{12}u(t),
\]

\[
y(t) = C_2x(t) + D_{21}w(t);
\]

\[
A = \begin{bmatrix}
0.9992 & 0.0050 & 0.0003 & 0.0000 \\
-0.3369 & 0.9992 & 0.1242 & 0.0003 \\
0.0008 & 0.0000 & 1.0007 & 0.0050 \\
0.3263 & 0.0008 & 0.2786 & 1.0007
\end{bmatrix}, \quad B_1 = \begin{bmatrix}
-0.0001 \\
-0.0232 \\
0.0012 \\
0.4742
\end{bmatrix}, \quad B_2 = \begin{bmatrix}
0.0006 \\
0.2243 \\
-0.0001 \\
-0.0232
\end{bmatrix},
\]

\[
C_1 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad D_{11} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}, \quad D_{12} = \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}, \quad C_2 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}, \quad D_{21} = \begin{bmatrix}
0
\end{bmatrix}.
\]

The random delays in the feedback loop are modeled as a Markov chain such that the amount of delay at time \( t \) is given by \( \theta(t) - 1 \) if \( \theta(t) \) is the state of the chain at time \( t \). If the maximum possible amount of delay is \( N \), then the augmented state

\[
\hat{x}(t) = [x(t)^T \quad y(t-1)^T \quad \cdots \quad y(t-N+1)^T]^T
\]

and delayed measurement \( \hat{y}(t) \) yield the Markovian jump linear system (of order 14)

\[
\dot{\hat{x}}(t) = A_{\theta(t)}\hat{x}(t) + B_{1,\theta(t)}w(t) + B_{2,\theta(t)}u(t),
\]

\[
z(t) = C_{1,\theta(t)}\hat{x}(t) + D_{11,\theta(t)}w(t) + D_{12,\theta(t)}u(t),
\]

\[
\dot{\hat{y}}(t) = C_{2,\theta(t)}\hat{x}(t) + D_{21,\theta(t)}w(t),
\]

where

\[
A_i = \begin{bmatrix}
A & 0 & \cdots & 0 & 0 \\
C_2 & 0 & \cdots & 0 & 0 \\
0 & I & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & I & 0
\end{bmatrix}, \quad B_{1,i} = \begin{bmatrix}
B_1 \\
D_{21} \\
0 \\
\vdots \\
0
\end{bmatrix}, \quad B_{2,i} = \begin{bmatrix}
B_2 \\
0 \\
\vdots \\
0
\end{bmatrix},
\]

\[
C_{1,i} = \begin{bmatrix}
C_1 & 0 & \cdots & 0 \\
0 & C_2 & \cdots & 0 \\
0 & 0 & \cdots & I
\end{bmatrix}, \quad D_{11,i} = D_{11}, \quad D_{12,i} = D_{12}
\]

for \( i = 1, \ldots, N \), and

\[
\begin{bmatrix}
C_{2,1} \\
C_{2,2} \\
\vdots \\
C_{2,N}
\end{bmatrix} = \begin{bmatrix}
C_2 & 0 & \cdots & 0 \\
0 & I & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{bmatrix}, \quad \begin{bmatrix}
D_{21,1} \\
D_{21,2} \\
\vdots \\
D_{21,N}
\end{bmatrix} = \begin{bmatrix}
D_{21} \\
0 \\
\vdots \\
0
\end{bmatrix}.
\]

The measurements are time-stamped, so the controller perfectly observes \( \theta(t) \) at each time \( t \). Let \( N = 6 \). Let the transition probability matrix \( P \) of the Markov chain be

\[
P = \begin{bmatrix}
0.3 & 0.7 & 0 & 0 & 0 & 0 \\
0.2 & 0.2 & 0.6 & 0 & 0 & 0 \\
0.2 & 0.2 & 0.2 & 0.4 & 0 & 0 \\
0.2 & 0.2 & 0.2 & 0.2 & 0 & 0 \\
0.1 & 0.1 & 0.1 & 0.3 & 0.3 & 0.3 \\
0.1 & 0.1 & 0.1 & 0.3 & 0.3 & 0.3
\end{bmatrix}.
\]
where the fact that the controller can always use the most recent measurement is accounted for [42]; the initial distribution \(p\) is assumed to be equal to the unique steady-state distribution

\[
\pi = \begin{bmatrix}
0.2142 & 0.2999 & 0.2699 & 0.1440 & 0.0504 & 0.0216
\end{bmatrix}.
\]

The semidefinite program minimizing the uniform disturbance attenuation level \(\gamma\) subject to (3.18) over \(i = (i_0, \ldots, i_M) \in \mathcal{M}_M(\mathbf{P}, p)\) yields approximately \(\gamma = 11.1\) for \(M = 0\) and \(\gamma = 10.6\) for \(M \geq 1\). Now running Algorithm 5.2 with \(\gamma = 100\), \(M = 1\), and \(\lambda_i\) as in (5.3) yields \(\gamma_i, i \in \mathcal{M}_1(\mathbf{P}, p)\), such that \(\sum_{i \in \mathcal{M}_1(\mathbf{P}, p)} \lambda_i \gamma_i = 9.96\), where, if \(\Lambda = (\lambda(i, j))\) and \(\Gamma = (\gamma(i, j))\), then

\[
\Lambda = \begin{bmatrix}
0.0643 & 0.1499 & 0 & 0 & 0 & 0 \\
0.0600 & 0.0600 & 0.1799 & 0 & 0 & 0 \\
0.0540 & 0.0540 & 0.0540 & 0.1080 & 0 & 0 \\
0.0228 & 0.0228 & 0.0228 & 0.0228 & 0.0228 & 0 \\
0.0050 & 0.0050 & 0.0050 & 0.0050 & 0.0151 & 0.0151 \\
0.0022 & 0.0022 & 0.0022 & 0.0022 & 0.0065 & 0.0065 \\
\end{bmatrix},
\]

\[
\Gamma = \begin{bmatrix}
9.52 & 9.56 & 0 & 0 & 0 & 0 \\
9.48 & 9.52 & 9.39 & 0 & 0 & 0 \\
9.61 & 9.65 & 9.53 & 9.24 & 0 & 0 \\
10.1 & 9.99 & 9.87 & 9.57 & 10.1 & 0 \\
17.7 & 17.7 & 17.6 & 17.1 & 10.4 & 10.9 \\
27.9 & 28.0 & 27.7 & 27.2 & 18.9 & 11.5 \\
\end{bmatrix}.
\]

On the other hand, running Algorithm 5.2 with \(\gamma = 100\), \(M = 1\), and \(\lambda_i = 1\), \(i \in \mathcal{M}_1(\mathbf{P}, p)\), yields the disturbance attenuation levels \(\tilde{\gamma}_i, i \in \mathcal{M}_1(\mathbf{P}, p)\), such that \(\frac{1}{N_1} \sum_{i \in \mathcal{M}_1(\mathbf{P}, p)} \tilde{\gamma}_i = 10.3\), where \(N_1\) is the cardinality of \(\mathcal{M}_1(\mathbf{P}, p)\) and equals 26. If \(\tilde{\Gamma} = (\tilde{\gamma}(i, j))\), then

\[
\tilde{\Gamma} = \begin{bmatrix}
13.3 & 16.9 & 0 & 0 & 0 & 0 \\
9.90 & 12.9 & 17.7 & 0 & 0 & 0 \\
7.05 & 8.86 & 12.4 & 16.9 & 0 & 0 \\
5.74 & 6.75 & 8.84 & 12.0 & 16.0 & 0 \\
5.11 & 5.80 & 7.05 & 9.09 & 11.9 & 13.5 \\
5.32 & 5.68 & 6.75 & 8.61 & 11.2 & 12.1 \\
\end{bmatrix}.
\]

The two sets of disturbance attenuation levels given by \(\Gamma\) and \(\tilde{\Gamma}\) are very different from each other, yet they are both one-path Pareto optimal. □

6. Conclusion. This paper dealt with switched linear systems and Markovian jump linear systems in the discrete-time domain and developed complete conditions for (almost sure) uniform disturbance attenuation and (almost sure) path-by-path disturbance attenuation. These conditions naturally give rise to finite-path-dependent controllers and admit semidefinite programming algorithms for optimal dynamic output feedback controller synthesis. Limitations of these algorithms include that, in the worst case, the computational complexity grows exponentially in the number \(M\) of past modes that the optimal controller recalls, and that nonexistence of an admissible controller synthesis is not guaranteed to be correctly determined after a finite amount of computation. These limitations are due to the problem’s nature and considered
unavoidable; nevertheless, they will not pose difficulties in most cases because $M$ is usually very small.

There is at least one conceptually important question unanswered in this paper. This question is whether mode-dependent controllers perform as well as finite-path-dependent controllers as long as uniform disturbance attenuation is concerned; a more specific question is whether, given a Markovian jump linear system, there exists a stabilizing mode-dependent controller whenever a finite-path-dependent controller can stabilize the system. Under the notion of path-by-path disturbance attenuation, however, we showed that finite-path-dependent controllers can outperform mode-dependent ones.

The optimal disturbance attenuation example in Example 9 showed that our results provide a new contribution to the study of networked control systems. The proposed controller synthesis techniques are expected to be applicable to other time-delay systems. Related to switched and Markovian jump linear systems are linear parameter-varying systems, where the common design approaches for gain scheduling [5, 2, 3] are similar to those for mode-dependent controllers. A possible future research direction is to investigate if the approach of finite-path-dependent controller synthesis applies to the control of linear parameter-varying systems.

REFERENCES


