10.1 Introduction

Recall that a linear map, or its matrix representation, between finite-dimensional vector spaces are decomposed into Jordan blocks, which completely characterize the stability properties of the linear map. Similar things can be done to linear dynamical systems, or their state-space representations. Consider the linear time-invariant system

\[
\dot{x}(t) = Ax(t) + Bu(t),
\]
\[
y(t) = Cx(t) + Du(t),
\]

(10.1)

where \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{l \times n}, \) and \( D \in \mathbb{R}^{l \times m} \) are given matrices. If the system (10.1) is not controllable (i.e., not reachable), then it is possible to separate the controllable part of the system from the uncontrollable part. Similarly, if the system is not observable, then it is possible to separate the observable part from the unobservable part.

10.2 Standard Form for Uncontrollable Systems

Lemma 10.1 Let \( C \) be the controllability matrix of the pair \((A, B)\). If \( \text{rank} C = n_r \), then there exists a nonsingular matrix \( Q \in \mathbb{R}^{n \times n} \) such that

\[
Q^{-1}AQ = \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix} \quad \text{and} \quad Q^{-1}B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},
\]

(10.2)

where \( A_1 \in \mathbb{R}^{n_r \times n_r}, B_1 \in \mathbb{R}^{n_r \times m} \), and the pair \((A_1, B_1)\) is controllable.

Proof. Let \( \{v_1, \ldots, v_{n_r}\} \) be a basis for the controllable subspace \( \mathcal{R}(C) \). Since \( \mathcal{R}(C) \) is invariant under \( A \), there exists a unique matrix \( A_1 \in \mathbb{R}^{n_r \times n_r} \), whose columns are the coordinate representations of \( Av_1, \ldots, Av_{n_r} \) with respect to \( \{v_1, \ldots, v_{n_r}\} \), such that

\[
A \begin{bmatrix} v_1 & \cdots & v_{n_r} \end{bmatrix} = \begin{bmatrix} v_1 & \cdots & v_{n_r} \end{bmatrix} A_1.
\]

On the other hand, if an \( x \in \mathbb{R}^n \) is such that \( x = Bu \) for some \( u \in \mathbb{R}^m \), then

\[
x = [B A B \cdots A^{n-1}B][u^T \ 0 \cdots 0]^T,
\]

so \( \mathcal{R}(B) \subset \mathcal{R}(C) \). Thus there exists a unique \( B_1 \in \mathbb{R}^{n_r \times m} \), whose columns are the coordinate representations of the columns of \( B \) with respect to \( \{v_1, \ldots, v_{n_r}\} \), such that

\[
B = \begin{bmatrix} v_1 & \cdots & v_{n_r} \end{bmatrix} B_1.
\]
Now, choosing any linearly independent vectors $q_{n_r+1}, \ldots, q_n$ such that

$$Q = [v_1 \ \cdots \ v_{n_r} \ q_{n_r+1} \ \cdots \ q_n]$$

is nonsingular, we have

$$A [v_1 \ \cdots \ v_{n_r}] = Q [A_1 \ 0]$$

and

$$A [q_{n_r+1} \ \cdots \ q_n] = Q [A_{12} \ A_2]$$

for some $A_{12} \in \mathbb{R}^{n_r \times (n-n_r)}$ and $A_2 \in \mathbb{R}^{(n-n_r) \times (n-n_r)}$; similarly

$$B = Q [B_1 \ 0].$$

This leads to (10.2). It remains to show that $(A_1, B_1)$ is controllable. The controllability matrix $\hat{C}$ of the pair $(Q^{-1}A Q, Q^{-1}B)$ is given by

$$(10.3) \quad \hat{C} = \begin{bmatrix} B_1 & A_1 B_1 & \cdots & A_1^{n_r-1} B_1 \end{bmatrix} = Q^{-1}C.$$

Here, the first equation in (10.3) implies that the rank of $\hat{C}$ is equal to the rank of

$$\begin{bmatrix} B_1 & A_1 B_1 & \cdots & A_1^{n_r-1} B_1 \end{bmatrix},$$

where $n \geq n_r$. By the Cayley-Hamilton theorem, however, the columns of this matrix are linear combinations of the columns of the controllability matrix of $(A_1, B_1)$ given by

$$C_1 = \begin{bmatrix} B_1 & A_1 B_1 & \cdots & A_1^{n_r-1} B_1 \end{bmatrix} \in \mathbb{R}^{n_r \times mn_r},$$

whose rank is at most $n_r$. On the other hand, the second equation in (10.3) says that $\text{rank} \hat{C} = \text{rank} C = n_r$. Therefore, we have $\text{rank} C_1 = n_r$, and hence the pair $(A_1, B_1)$ is controllable. \Box

If $Q$ is as in Lemma 10.1, then the change of state variables given by $Q \dot{x}(t) = x(t)$ results in an equivalent system of the form

$$(10.4) \quad \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + Du(t),$$

where $\dot{x}(t) = [\dot{x}_1(t)^T \ \dot{x}_2(t)^T]^T$ with $\dot{x}_1(t) \in \mathbb{R}^{n_r}$ and $\dot{x}_2(t) \in \mathbb{R}^{n-n_r}$, and where $(A_1, B_1)$ is controllable. This form of state-space representations is called the standard form for uncontrollable systems. The eigenvalues of $A_1$ (resp. $A_2$) are called controllable eigenvalues (resp. uncontrollable eigenvalues) of the system (10.1). Recall that the inverse Laplace transform of $(sI - A)^{-1}$ yields

$$e^{At} = \sum_{i=1}^{p} \sum_{k=0}^{m_i-1} A_{ik} t^k e^{\lambda_i t}$$

with

$$A_{ik} = \frac{1}{k! (m_i - 1 - k)} \lim_{s \to \lambda_i} \left\{ \left[ (s - \lambda_i)^{m_i} (sI - A)^{-1} \right]^{m_i - 1 - k} \right\},$$

where $\lambda_1, \ldots, \lambda_p$ are distinct eigenvalues of $A$ and $m_1, \ldots, m_p$ are their algebraic multiplicities, and where the terms $t^k e^{\lambda_i t}$ are called the modes of the system (10.1). In particular, the modes of the system corresponding to controllable eigenvalues (resp. uncontrollable eigenvalues) are called controllable modes (resp. uncontrollable modes) of the system.
Lemma 10.2 The input-output description of the system (10.1) is determined solely by its controllable part; that is, if (10.1) and (10.4) are equivalent, then the transfer function matrix of (10.1) is given by

\[ \hat{H}(s) = C(sI - A)^{-1}B + D = C_1(sI - A_1)^{-1}B_1 + D. \]

Proof. If \( Q \) is as in Lemma 10.1, then we have

\[ e^{Q^{-1}AQ^r} = Q^{-1}e^{A^rQ} \quad \text{and} \quad e^{[A_1 \ A_{12}]^r} = \begin{bmatrix} e^{A_1^r} & * \\ 0 & e^{A_{22}^r} \end{bmatrix}, \]

where the symbol * denotes the block that we do not care. Hence

\[ Ce^{A^r}B = (CQ)(Q^{-1}e^{A^r}Q)(Q^{-1}B) = [C_1 \ C_2] \begin{bmatrix} e^{A_1^r} & * \\ 0 & e^{A_{22}^r} \end{bmatrix} [B_1 \ 0] = C_1e^{A_1^r}B_1, \]

from which the result follows. \( \square \)

As a simple example, consider \( A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \) and \( C = \begin{bmatrix} 1 & 2 \end{bmatrix}. \) Then \( C = \begin{bmatrix} 1 & 1 \end{bmatrix}, \) so the reachable subspace \( \mathcal{R}(C) = \text{span}\{v_1\} \) with \( v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \) Choose \( v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \) and let \( Q = [v_1 \ v_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \) to obtain \( Q^{-1}AQ = \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \) \( Q^{-1}B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \) and \( CQ = [c_1 \ c_2] = \begin{bmatrix} 1 & 2 \end{bmatrix}. \) Thus, among the two eigenvalues (namely, 1 and 0) of the system, the eigenvalue 1 is controllable and the eigenvalue 0 is uncontrollable. Indeed, the transfer function of the system is

\[ H(s) = C(sI - A)^{-1}B = \begin{bmatrix} 1 & 2s \\ s - 1 & s(s - 1) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ s - 1 \end{bmatrix} = C_1(sI - A_1)^{-1}B_1. \]

Since \( \text{rank } O = \text{rank } \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = 2, \) both eigenvalues are observable. However, the uncontrollable eigenvalue 0 is never excited by the input, and hence does not appear as a system pole.

10.3 Standard Form for Unobservable Systems

Lemma 10.3 Let \( O \) be the observability matrix of the pair \((A, C)\). If \( \text{rank } O = n_o \), then there exists a nonsingular matrix \( Q \in \mathbb{R}^{n \times n} \) such that

\[ Q^{-1}AQ = \begin{bmatrix} A_1 & 0 \\ A_2 & A_2 \end{bmatrix} \quad \text{and} \quad CQ = [C_1 \ 0], \quad \text{(10.5)} \]

where \( A_1 \in \mathbb{R}^{n_o \times n_o}, C_1 \in \mathbb{R}^{l \times n_o} \), and the pair \((A_1, C_1)\) is observable.

Proof. Let \{\( v_{n_o+1}, \ldots, v_n \)\} be a basis for the unobservable subspace \( \mathcal{N}(O) \). Since \( \mathcal{N}(O) \) is invariant under \( A \), there exists a unique matrix \( A_2 \in \mathbb{R}^{(n-n_o) \times (n-n_o)} \), whose columns are the coordinate representations of \( Av_{n_o+1}, \ldots, Av_n \) with respect to \{\( v_{n_o+1}, \ldots, v_n \)\}, such that

\[ A \begin{bmatrix} v_{n_o+1} & \cdots & v_n \end{bmatrix} = \begin{bmatrix} v_{n_o+1} & \cdots & v_n \end{bmatrix} A_2. \]

On the other hand, if \( Ox = 0 \), then \( Cx = 0 \), so \( \mathcal{N}(O) \subset \mathcal{N}(C) \). Thus we have

\[ C \begin{bmatrix} v_{n_o+1} & \cdots & v_n \end{bmatrix} = 0. \]

Choose \( q_1, \ldots, q_{n_o} \) be linearly independent vectors such that

\[ Q = [q_1 \ \cdots \ q_{n_o} \ v_{n_o+1} \ \cdots \ v_n] \]
is nonsingular. Then we have
\[
\begin{bmatrix}
A_1 & \cdots & A_{n_o} \\
A_{21} & \cdots & A_{2n_o}
\end{bmatrix} = Q
\begin{bmatrix}
A_1 \\
A_2
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
\mathbf{v}_{n_o+1} & \cdots & \mathbf{v}_n
\end{bmatrix} = Q
\begin{bmatrix}
0 \\
A_2
\end{bmatrix}
\]
for some \( A_1 \in \mathbb{R}^{n_o \times n_o} \) and \( A_{21} \in \mathbb{R}^{(n-n_o) \times n_o} \). Similarly, \( CQ = [C_1 \ 0] \) for some \( C_1 \in \mathbb{R}^{t \times n_o} \). This gives (10.5). Moreover, the observability matrix \( \mathcal{O} \) of the pair \( (Q^{-1}AQ, CQ) \) is given by
\[
\mathcal{O} = \begin{bmatrix}
C_1^T & A_1^T C_1^T & \cdots & (A_1^{n-1})^T C_1^T
\end{bmatrix}^T = \mathcal{O}Q.
\]
Here, the first equation, along with the Cayley-Hamilton theorem, implies that \( \text{rank} \mathcal{O} \) is equal to the rank of the observability matrix \( \mathcal{O}_1 = [C_1^T \ A_1^T C_1^T \ \cdots \ (A_1^{n-1})^T C_1^T]^T \in \mathbb{R}^{n_o \times n_o} \) of \( (A_1, C_1) \), which is at most \( n_o \); however, the second equation gives \( \text{rank} \mathcal{O} = n_o \). Therefore, we have \( \text{rank} \mathcal{O}_1 = n_o \), and hence \( (A_1, C_1) \) is observable. \( \square \)

If \( Q \) is as in Lemma 10.3, then the change of state variables given by \( Q\hat{x}(t) = x(t) \) results in an equivalent system of the form
\[
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t)
\end{bmatrix} = \begin{bmatrix}
A_1 & 0 \\
A_{21} & A_2
\end{bmatrix} \begin{bmatrix}
\hat{x}_1(t) \\
\hat{x}_2(t)
\end{bmatrix} + \begin{bmatrix}
B_1 \\
B_2
\end{bmatrix} u(t),
\]
\[
y(t) = \begin{bmatrix}
C_1 & 0
\end{bmatrix} \begin{bmatrix}
\hat{x}_1(t) \\
\hat{x}_2(t)
\end{bmatrix} + Du(t),
\]
(10.6)
\]
where \( \hat{x}(t) = [\hat{x}_1(t)^T \ \hat{x}_2(t)^T]^T \) with \( \hat{x}_1(t) \in \mathbb{R}^{n_o} \) and \( \hat{x}_2(t) \in \mathbb{R}^{n-n_o} \), and where \( (A_1, C_1) \) is observable. This form of state-space representations is called the standard form for unobservable systems. The eigenvalues of \( A_1 \) (resp. \( A_2 \)) and the corresponding modes are called observable eigenvalues (resp. unobservable eigenvalues) and observable modes (resp. unobservable modes) of the system (10.1).

\textbf{Lemma 10.4} \quad \text{The input-output description of the system (10.1) is determined solely by its observable part. That is, if (10.1) and (10.6) are equivalent, then the transfer function matrix of (10.1) is given by}
\[
\hat{H}(s) = C(sI - A)^{-1}B + D = C_1(sI - A_1)^{-1}B_1 + D.
\]
\textit{Proof.} The proof parallels that of Lemma 10.2. \( \square \)

For example, suppose \( A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \), \( B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \), and \( C = \begin{bmatrix} 1 & 1 \end{bmatrix} \). Since \( \mathcal{O} = \begin{bmatrix} 1 & 1 \end{bmatrix} \), the unobservable subspace \( \mathcal{N}(\mathcal{O}) = \text{span}\{\mathbf{v}_2\} \) with \( \mathbf{v}_2 = [\ 1 \ \ 0 \ -1 \]^T \). Choose a \( \mathbf{v}_1 = [1 \ 0] \), and let \( Q = [\mathbf{v}_1 \ \mathbf{v}_2] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 \end{bmatrix} \) to obtain \( Q^{-1}A Q = \begin{bmatrix} A_1 & 0 \\ A_{21} & A_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \), \( Q^{-1}B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \), and \( CQ = [C_1 \ 0] = [1 \ 0] \). Thus, among the two eigenvalues of the system, the eigenvalue 1 is observable and the eigenvalue 0 is unobservable. Indeed, the transfer function of the system is
\[
H(s) = C(sI - A)^{-1}B = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s+1 & \frac{s(s-1)}{s} \\ \frac{1}{s} & \frac{1}{s-1} \end{bmatrix} = \frac{2}{s-1} = C_1(sI - A_1)^{-1}B_1.
\]
Since \( \text{rank} C = \text{rank} [\begin{bmatrix} 1 & 2 \end{bmatrix}^T] = 2 \), both eigenvalues are controllable. However, the unobservable eigenvalue 0 is never observed from the system output, and hence does not appear as a system pole.
10.4 Kalman Canonical Form of LTI Systems

The following theorem is called the Kalman decomposition theorem. The theorem unifies the decomposition lemmas for uncontrollable and unobservable linear time-invariant systems.

**Theorem 10.5** Let $\mathcal{C}$ and $\mathcal{O}$ be the controllability and observability matrices of the triple $(A, B, C)$. If $
abla C = n_r$, $
abla O = n_o$, and $\nabla R(\mathcal{C}) \cap \nabla N(\mathcal{O}) = n_o$, then there exists a nonsingular matrix $Q \in \mathbb{R}^{n \times n}$ such that

$$Q^{-1}AQ = \begin{bmatrix} A_{11} & 0 & A_{13} & 0 \\ A_{21} & A_{22} & A_{23} & A_{24} \\ 0 & 0 & A_{33} & 0 \\ 0 & 0 & A_{43} & A_{44} \end{bmatrix}, \quad Q^{-1}B = \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix}, \quad CQ = \begin{bmatrix} C_1 & 0 & C_3 & 0 \end{bmatrix}, \quad (10.7)$$

where $A_{11} \in \mathbb{R}^{(n_r - n_o) \times (n_r - n_o)}$, $A_{22} \in \mathbb{R}^{n_o \times n_o}$, $A_{33} \in \mathbb{R}^{(n_o - (n_r - n_o)) \times (n_o - (n_r - n_o))}$, and $A_{44} \in \mathbb{R}^{(n - n_o - n_r - n_o) \times (n - n_o - n_r - n_o)}$, and such that the following hold:

(a) The pair $(A_c, B_c)$ with

$$A_c = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \quad \text{and} \quad B_c = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

is controllable.

(b) The pair $(A_o, C_o)$ with

$$A_o = \begin{bmatrix} A_{11} & A_{13} \\ 0 & A_{33} \end{bmatrix} \quad \text{and} \quad C_o = \begin{bmatrix} C_1 & C_3 \end{bmatrix}$$

is observable.

(c) The triple $(A_{11}, B_1, C_1)$ is controllable and observable.

**Proof.** Choose a basis $\{v_1, \ldots, v_{n_r}\}$ for $\nabla R(\mathcal{C})$ such that $\{v_{n_r - n_o + 1}, \ldots, v_{n_r}\}$ is a basis for $\nabla R(\mathcal{C}) \cap \nabla N(\mathcal{O})$. Choose vectors $\hat{\mathbf{v}}_{n_o + n_r - 1}, \ldots, \hat{\mathbf{v}}_{n_r}$ such that

$$\{v_{n_r - n_o + 1}, \ldots, v_{n_r}, \hat{\mathbf{v}}_{n_o + n_r - 1}, \ldots, \hat{\mathbf{v}}_{n_r}\}$$

is a basis for $\nabla N(\mathcal{O})$. Finally, choose $q_{n_r + 1}, \ldots, q_{n_o + n_r}$ such that

$$Q = \begin{bmatrix} v_1 & \cdots & v_{n_r} & q_{n_r + 1} & \cdots & q_{n_o + n_r - 1} & \hat{\mathbf{v}}_{n_o + n_r - 1} & \cdots & \hat{\mathbf{v}}_{n_r} \end{bmatrix}$$

is nonsingular. Then, since the first $n_r$ columns of $Q$ span $\nabla R(\mathcal{C})$, (the proof of) Lemma 10.1 implies that $Q^{-1}AQ$, $Q^{-1}B$, and $CQ$ are of the following partitioned forms:

$$Q^{-1}AQ = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ 0 & 0 & A_{33} & A_{34} \\ 0 & 0 & A_{43} & A_{44} \end{bmatrix}, \quad Q^{-1}B = \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix}, \quad CQ = \begin{bmatrix} C_1 & C_2 & C_3 & C_4 \end{bmatrix}. \quad (10.8)$$

Let

$$T = \begin{bmatrix} I_{n_r - n_o} & 0 & 0 & 0 \\ 0 & 0 & I_{n_r - n_o} & 0 \\ 0 & I_{n_o - (n_r - n_o)} & 0 & 0 \\ 0 & 0 & 0 & I_{(n - n_o) - n_r - n_o} \end{bmatrix},$$
where $I_k$ are the $k$-by-$k$ identity matrices. Then

$$ QT = \begin{bmatrix} v_1 & \cdots & v_{n_r-n_r^0} & q_{n_r+1} & \cdots & q_{n_0+n_r} & v_{n_r-n_r^0+1} & \cdots & v_{n_r} & \hat{v}_{n_0+n_r} & \cdots & \hat{v}_n \end{bmatrix}, $$

where the last $n-n_0$ columns of $QT$ span $\mathcal{N}(O)$. Thus (the proof of) Lemma 10.3, along with the fact that $T^{-1} = T^T$, implies that $Q^{-1}A = Q^{-1}B$, and $CQ$ are of the following partitioned forms:

$$ Q^{-1}A = T(QT)^{-1}A(QT)T^{-1} = T \begin{bmatrix} F_{11} & F_{12} & 0 & 0 \\ F_{21} & F_{22} & 0 & 0 \\ F_{31} & F_{32} & F_{33} & F_{34} \\ F_{41} & F_{42} & F_{43} & F_{44} \end{bmatrix}, $$

$$ Q^{-1}B = T(QT)^{-1}B = T \begin{bmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \end{bmatrix}, $$

$$ CQ = C(QT)T^{-1} \begin{bmatrix} H_1 & H_2 & 0 & 0 \end{bmatrix}, $$

Comparing these equations with those in (10.8), we conclude that (10.7) holds. The pair $(A, B)$ is controllable by Lemma 10.1, and the pair $(A_o, C_o)$ is observable by Lemma 10.3. It follows from

$$ \text{rank} \begin{bmatrix} B_c & A_cB_c & \cdots & A_{n_r}^{-1}B_c \end{bmatrix} = \text{rank} \begin{bmatrix} B_1 & A_{11}B_1 & \cdots & A_{11}^{n_r-1}B_1 \end{bmatrix} = n_r, $$

along with the Cayley-Hamilton theorem, that

$$ \text{rank} \begin{bmatrix} B_1 & A_{11}B_1 & \cdots & A_{11}^{n_r-1}B_1 \end{bmatrix} = \text{rank} \begin{bmatrix} B_1 & A_{11}B_1 & \cdots & A_{11}^{n_r-n_r^0-1}B_1 \end{bmatrix} = n_r - n_r^0. $$

Thus the triple $(A_{11}, B_1, C_1)$ is controllable. Since $(A_c, B_c, [C_1 0])$ is already in the standard form for unobservable systems, we conclude that $(A_{11}, B_1, C_1)$ is observable as well.

If $A_{11}, A_{22}, A_{33},$ and $A_{44}$ are as in Theorem 10.5, then we have

$$ \det(sI - A) = \det(sI - A_{11}) \det(sI - A_{22}) \det(sI - A_{33}) \det(sI - A_{44}). $$

The eigenvalues of $A_{11}$ are controllable and observable, those of $A_{22}$ are controllable and unobservable, those of $A_{33}$ are uncontrollable and observable, and those of $A_{44}$ are uncontrollable and unobservable.

**Theorem 10.6** The input-output description of the system (10.1) is determined solely by its controllable and observable part. That is, if $Q$ is as in (10.7), then the transfer function matrix of (10.1) is given by

$$ \tilde{H}(s) = C(sI - A)^{-1}B + D = C_1(sI - A_{11})^{-1}B_1 + D. $$

**Proof.** The result is an immediate consequence of Lemmas 10.2 and 10.4. □

For example, consider

$$ A = \begin{bmatrix} -1 & -3 & -2 \\ 1 & 3 & 3 \\ 1 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 \\ 1 & 1 \\ 0 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}. $$

The controllability and observability matrices are

$$ C = \begin{bmatrix} -1 & 0 & -2 & -1 & -4 & -1 \\ 1 & 1 & 2 & 0 & 4 & 2 \\ 0 & -1 & 0 & 1 & 0 & -1 \end{bmatrix}, \quad O = \begin{bmatrix} 1 & 2 & 0 & 1 & 1 & 2 \\ 1 & 2 & 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 & 0 & 1 \end{bmatrix}. $$
The controllable (or reachable) subspace \( R_r = \mathcal{R}(\mathbf{C}) \), the unobservable subspace \( R_o = \mathcal{N}(\mathbf{O}) \), and their intersection \( R_{r_o} = \mathcal{R}(\mathbf{C}) \cap \mathcal{N}(\mathbf{O}) \) are

\[
R_r = \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}, \quad R_o = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}, \quad \text{and} \quad R_{r_o} = \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\},
\]

respectively. Then

\[
\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{q}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \mathbf{Q} = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{q}_3] = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}.
\]

This \( \mathbf{Q} \) gives

\[
\mathbf{Q}^{-1} \mathbf{A} \mathbf{Q} = \begin{bmatrix} -1 & 0 & -1 \\ -1 & 2 & -7 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{Q}^{-1} \mathbf{B} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{C} \mathbf{Q} = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 0 & 5 \end{bmatrix}.
\]

Among the three eigenvalues of \( \mathbf{A} \), the eigenvalue \(-1\) is both controllable and observable, the eigenvalue \(2\) is controllable but unobservable, and the eigenvalue \(1\) is observable but uncontrollable. Indeed, the transfer function matrix shows a single pole at \(s = -1\):

\[
\tilde{\mathbf{H}}(s) = \mathbf{C}(s \mathbf{I} - \mathbf{A})^{-1} \mathbf{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} (s - (-1))^{-1} \begin{bmatrix} 0 & 1 \\ 0 & 1/(s + 1) \end{bmatrix}.
\]

### 10.5 Popov-Belevitch-Hautus (PBH) Tests

We know the system (10.1) is controllable if and only if its controllability matrix has full row rank. Similarly, the system is observable if and only if its observability matrix has full rank. The decomposition results presented above reveal the structural properties of linear systems, and provide additional tests for controllability and observability. These tests facilitate further insights into the structural properties of linear systems, and will play an important role in controller synthesis problems.

**Theorem 10.7** (PBH Eigenvector Tests)

(a) *The pair \((\mathbf{A}, \mathbf{B})\) is controllable if and only if no eigenvector \(\mathbf{v}\) of \(\mathbf{A}^T\) satisfies \(\mathbf{B}^T \mathbf{v} = 0\).*

(b) *The pair \((\mathbf{A}, \mathbf{C})\) is observable if and only if no eigenvector \(\mathbf{v}\) of \(\mathbf{A}\) satisfies \(\mathbf{C} \mathbf{v} = 0\).*

**Proof.** Suppose there exists an eigenvector \(\mathbf{v} \in \mathbb{C}^n\) of \(\mathbf{A}^T\) such that \(\mathbf{B}^T \mathbf{v} = 0\). Let \(\lambda \in \mathbb{C}\) be the eigenvalue associated with \(\mathbf{v}\). Then we have \(\mathbf{v}^T \mathbf{B} = 0, \mathbf{v}^T \mathbf{A} \mathbf{B} = \lambda \mathbf{v}^T \mathbf{B} = 0, \mathbf{v}^T \mathbf{A}^2 \mathbf{B} = \lambda^2 \mathbf{v}^T \mathbf{A} \mathbf{B} = 0, \ldots\), and so \(\mathbf{v}^T \mathbf{C} = 0\), where \(\mathbf{C}\) is the controllability matrix of \((\mathbf{A}, \mathbf{B})\). That is, \(\Re(\mathbf{v})^T \mathbf{C} = \Im(\mathbf{v})^T \mathbf{C} = 0\). Since \(\mathbf{v} \neq 0\), we have \(\Re(\mathbf{v}) \neq 0\) or \(\Im(\mathbf{v}) \neq 0\), so rank \(\mathbf{C} < n\). This proves that the pair \((\mathbf{A}, \mathbf{B})\) being controllable implies \(\mathbf{B}^T \mathbf{v} \neq 0\) for any eigenvector \(\mathbf{v}\) of \(\mathbf{A}^T\). Conversely, suppose \((\mathbf{A}, \mathbf{B})\) is not controllable, so that there exists a nonsingular \(\mathbf{Q}\) satisfying (10.2), where \(\mathbf{A}_2 \in \mathbb{R}^{(n-n_r) \times (n-n_r)}\) with \(n_r < n\). Choose an eigenvalue \(\lambda\) of \(\mathbf{A}_2^T\), and let \(\mathbf{v}\) be the corresponding eigenvector of \(\mathbf{A}_2^T\). Then putting \(\mathbf{v} = (\mathbf{Q}^{-1})^T [0 \ \mathbf{v}]^T \in \mathbb{R}^n\) gives

\[
\mathbf{A}^T \mathbf{v} = (\mathbf{Q}^{-1})^T (\mathbf{Q}^{-1} \mathbf{A} \mathbf{Q})^T [0 \ \mathbf{v}] = (\mathbf{Q}^{-1})^T \begin{bmatrix} \mathbf{A}_2^T & \mathbf{0}^T \\ \mathbf{A}_2^T & \mathbf{A}_2^T \end{bmatrix} [0 \ \mathbf{v}] = (\mathbf{Q}^{-1})^T [0 \ \lambda \mathbf{v}] = \lambda \mathbf{v}.
\]
and
\[
\begin{bmatrix}
B^T v = B^T (Q^{-1})^T \begin{bmatrix} 0 \\
v \end{bmatrix} = (Q^{-1}B)^T \begin{bmatrix} 0 \\
v \end{bmatrix} = \begin{bmatrix} B^T & 0 \end{bmatrix} \begin{bmatrix} 0 \\
v \end{bmatrix} = 0.
\end{bmatrix}
\]

This shows that the pair \((A, B)\) is controllable whenever no eigenvector \(v\) of \(A^T\) satisfies \(B^Tv = 0\), and hence that part (a) of the theorem holds. The proof of part (b) is similar. \(\square\)

**Corollary 10.8** (PBH Rank Tests)

(a) The pair \((A, B)\) is controllable if and only if
\[
\text{rank } \begin{bmatrix} \lambda I - A & B \end{bmatrix} = n
\]
for all eigenvalues \(\lambda\) of \(A\) (and hence for all \(\lambda \in \mathbb{C}\)).

(b) A number \(\lambda \in \mathbb{C}\) is an uncontrollable eigenvalue of \(A\) if and only if
\[
\text{rank } \begin{bmatrix} \lambda I - A & B \end{bmatrix} < n.
\]

(c) The pair \((A, C)\) is observable if and only if
\[
\text{rank } \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n
\]
for all eigenvalues \(\lambda\) of \(A\) (and hence for all \(\lambda \in \mathbb{C}\)).

(d) A number \(\lambda \in \mathbb{C}\) is an unobservable eigenvalue of \(A\) if and only if
\[
\text{rank } \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} < n.
\]

### 10.6 Kalman Decomposition of Discrete-Time Systems

For discrete-time LTI systems of the form
\[
\begin{align*}
\dot{x}(t+1) &= Ax(t) + Bu(t), \quad t = 0, 1, \ldots; \\
y(t) &= Cx(t) + Du(t), \quad t = 0, 1, \ldots,
\end{align*}
\]
even though reachability implies controllability, controllability does not imply reachability. Similarly, observability implies constructibility, but the converse does not necessarily hold. Thus, we will speak of the reachability (as well as observability) of discrete-time systems. Otherwise, all the theorems in Sections 10.4 and 10.5, with “controllability” replaced by “reachability” and the “Laplace transform” \(\hat{H}(s)\) by the “z-transform” \(\hat{H}(z)\), carry over to discrete-time LTI systems without further change.

### References
