

# Pricing and Hedging American Options: A Recursive Integration Method

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*In this article, we present a new method for pricing and hedging American options along with an efficient implementation procedure. The proposed method is efficient and accurate in computing both option values and various option hedge parameters. We demonstrate the computational accuracy and efficiency of this numerical procedure in relation to other competing approaches. We also suggest how the method can be applied to the case of any American option for which a closed-form solution exists for the corresponding European option.*

A variety of financial products such as fixed-income derivatives, mortgage-backed securities, and corporate securities have early exercise or American-style features that significantly influence their valuation and hedging. Considerable interest exists, therefore, in both academic and practitioner circles, in methods of

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valuation and hedging of American-style options that are conceptually sound as well as efficient in their implementation.

It has been recognized early in the modern academic literature on options pricing that, in general, closed-form solutions to the problems of valuation and hedging of American-style options are very difficult, if not impossible, to achieve. Hence, many researchers have focused on numerical methods to solve the problems of valuation and hedging of such options. These methods can be divided into two broad categories.

The first approach is a solution to the integral equation, where the option value is written as the expected value, under the risk neutral probability measure, of the option payoffs. In the case of American-style options, the computation of the conditionally expected value on a given date is subject to the condition that the option has not been exercised previously. This approach is in the spirit of the risk neutral valuation approach suggested by Cox and Ross (1976) and formalized by Harrison and Kreps (1979) and others. It is often implemented in practice in the context of the binomial approach of Cox, Ross, and Rubinstein (1979). A variation on the integral equation approach is suggested by Geske and Johnson (1984), who provide a numerical approximation based on the analytical valuation formulae for options exercisable on discrete dates. Their approach uses the exact analytical expressions for options exercisable at one, two, three, and perhaps four dates, to derive an approximate price of an American-style option that is continuously exercisable.

The second approach is to directly solve the Black and Scholes (1973) partial differential equation, subject to the boundary conditions imposed by the possibility of early exercise. This approach is implemented by using numerical approximation methods. For instance, the finite-difference methods of Brennan and Schwartz (1977), Courtadon (1982), and Schwartz (1977) and the analytical approximation methods of Barone-Adesi and Whaley (1987) and MacMillan (1986) are applications of this approach.

Although both types of methods have been used extensively in the academic literature and in practice, they have some limitations. First, as pointed out by Omberg (1987a), some methods such as the binomial method and the Geske and Johnson (1984) extrapolation scheme may not yield uniform convergence in option prices. Second, many of these numerical methods use a perturbation scheme to compute the hedge parameters. Such a scheme requires fairly intensive computation to obtain accurate results. Third, in some cases, such as the explicit finite-difference scheme, the method may not always converge. Finally, it has been recognized that analytical approximations may have large pricing errors for long-dated options.

These difficulties, as well as the desire for a more elegant mathematical representation, have led to a search for an analytical framework for American-style options. In a sense, this is the continuous-time extension of the Geske and Johnson (1984) approach. This solution is discussed by Kim (1990), Carr, Jarrow, and Myneni (1992) and Jacka (1991). These authors obtain the analytical American option pricing formulae under the assumption of a lognormal diffusion for the underlying security price. Kim and Yu (1993) consider some alternative underlying price processes in the same framework.

Two facets of the literature on the analytical valuation of American-style options should be noted. First, much of the research has focused on *standard* American options on stocks such as simple put or call options. Increasingly there is a need to explore how these formulae can be extended to the cases of other American-style claims, such as exotic options. Second, none of the above studies on these formulae offers clear guidance on how to implement the analytical valuation formulae, in practice, for the efficient computation of American option values. This is problematic due to the fact that the early exercise boundary of an American option is implicitly defined by a complicated path integral and does not have any closed-form solution. Therefore, the determination of the early exercise boundary, a pivotal step in the implementation of these analytical valuation formulae, becomes a difficult task. For this reason, there is need for American-style option formulae that are analytically rigorous, and yet improve upon the computational simplicity and intuitive appeal of the existing numerical methods in the literature.

In this article, we provide a new unified framework for the valuation and hedging of American-style options that is based on a recursive computation of the early exercise boundary. This unified framework can be applied to value and hedge several interesting options: options on stocks with dividends, options on futures, and options on foreign exchange. In particular, we implement an efficient numerical procedure that combines a unified analytical valuation formula and the Geske and Johnson (1984) approximation method to compute option prices and hedge parameters. This procedure involves estimating the early exercise boundary at only a few points and then approximating the entire boundary using Richardson extrapolation, rather than computing the boundary point-wise. An attractive feature of this procedure is that it is robust for a general American-style option contract.

An additional important feature of our approach is that it permits the *direct* implementation of formulae to compute option values as well as option hedge parameters. Specifically, once the early exercise boundary is estimated, option values and hedge parameters can be

obtained analytically.<sup>1</sup> This feature distinguishes our approach from other existing approaches in the literature, for example, variations of the binomial and the finite-difference methods. These methods use a perturbation scheme to compute the option hedge parameters, and hence, reflect the approximation errors, once again, in the values of the hedge parameters. In our approach, in contrast, option prices and hedge parameters are calculated at the same time, given the early exercise boundary. As a result, the computational errors in option prices are not compounded in the calculation of hedge parameters.

It is useful to compare our approach to related work by Broadie and Detemple (1994), Carr and Faguet (1994), and Omberg (1987b). Omberg (1987b) uses an exponential function to approximate the early exercise boundary. Since he does not use the exact analytical valuation formula which imposes the optimal early exercise condition, the optimality and the convergence properties of his approach are somewhat unclear. Broadie and Detemple (1994) provide an upper bound on the value of American options using a lower bound for the early exercise boundary (i.e., an approximate exercise boundary).<sup>2</sup> However, their formula for computing option prices requires the use of parameters determined from regressions. Further, the implementation of their method for computing hedge parameters is not discussed in their paper. Carr and Faguet (1994) develop the “method of lines” based on approximations to the Black and Scholes (1973) partial differential equation (PDE). This involves discretization along either the time or the state-variable dimension, but not both. They obtain an (approximate) exercise boundary from this discretized PDE, and then calculate the option values and hedge parameters. Similar to Broadie and Detemple (1994), their method involves using some adjustment factors for short maturity option contracts. Thus, both Broadie and Detemple (1994) and Carr and Faguet (1994) would seem to require a recalibration of the regression coefficients and adjustment factors to be applicable to nonstandard options.

This article is organized as follows. In Section 1, we present the unified analytical valuation formulae for option prices and hedge ratios, for the case of American put options with a proportional cost of carrying the underlying security. Section 2 presents a numerical implementation procedure for the formulae. Section 3 applies this procedure to two American option pricing problems: the first, the case of stock options, and the second, the case of quanto options. We

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<sup>1</sup> This is in the same sense that the Black and Scholes formula is an analytical formula, although it involves numerical integration of the cumulative normal density function.

<sup>2</sup> They also develop a lower bound on the option value using a capped option written on the same underlying asset.

demonstrate the speed and the accuracy of our method by comparing its numerical results with those of existing methods in the literature. Section 4 concludes the article.

## 1. A Unified Valuation Formula for American Put Options

In this section, we present a unified analytical valuation formula for American options on an underlying security that has an instantaneous, proportional cost of carry at a rate  $b$ . As will be clear later, depending on the characterization of  $b$  and its value, we can have many commonly traded financial contracts as special cases. For expositional purposes, we choose American put options for analysis, although our method can be applied to almost any American option for which an equivalent European-style valuation formula exists in closed form.

Assume that the capital markets are frictionless and arbitrage-free with continuous trading possibilities. Let  $S_t$  denote the underlying security price at time  $t$  and  $K$  be the exercise price of an American put option expiring at time  $T$ . Assume also that the price process  $\{S_t; t \geq 0\}$  follows a lognormal diffusion with constant return volatility  $\sigma$  and expected return  $\mu$ :<sup>3</sup>

$$dS_t = \mu S_t dt + \sigma S_t dZ_t, \quad (1)$$

where  $\{Z_t; t \geq 0\}$  is a one-dimensional standard Brownian motion (defined on some probability space).

As in McKean (1965) and Merton (1973), we define the American put value as the solution to a free boundary problem. We assume that the early exercise boundary for this American put is well-defined, unique, and has a continuous sample path. Let  $\mathcal{B} \equiv \{B_t; B_t \geq 0, t \in [0, T]\}$  denote the optimal early exercise boundary of the American put, and  $P(S_t, t)$  denote the put value at time  $t$ . Function  $P(S_t, t)$  is  $C^{1,1}$  on  $[B_t, \infty) \times [0, T]$  and  $C^{2,1}$  on  $(B_t, \infty) \times [0, T]$ . Then, the arbitrage-free put price,  $P(S_t, t)$ , satisfies the fundamental Black and Scholes (1973) PDE

$$\frac{\sigma^2 S_t^2}{2} \frac{\partial^2 P}{\partial S^2} + b S_t \frac{\partial P}{\partial S} - rP + \frac{\partial P}{\partial t} = 0, \quad (2)$$

subject to the following boundary conditions:

$$P(S_T, T) = \max[0, K - S_T], \quad (3)$$

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<sup>3</sup> We discuss and illustrate our numerical procedures for the lognormal diffusion process, although the method can be applied to other diffusion processes using the formulae derived by Kim and Yu (1993).

$$\lim_{S_t \uparrow \infty} P(S_t, t) = 0, \tag{4}$$

$$\lim_{S_t \downarrow B_t} P(S_t, t) = K - B_t, \tag{5}$$

$$\lim_{S_t \downarrow B_t} \frac{\partial P(S_t, t)}{\partial S_t} = -1, \tag{6}$$

where  $r > 0$  denotes the continuously compounded interest rate and is assumed constant. Equation (5) is the “value matching” condition and Equation (6) is the “super-contact” condition, and they are jointly referred to as the “smooth pasting” conditions. These conditions ensure that the premature exercise of the put option on the early exercise boundary,  $\mathcal{B}$ , will be optimal and self-financing.

Let  $\Psi(S_t; S_0)$  be the risk neutral transitional density function of the underlying security price. We can then derive the following expression for the price of the American put:<sup>4</sup>

$$P_0 = p_0 + rK \int_0^T e^{-rt} \int_0^{B_t} \Psi(S_t; S_0) dS_t dt - \alpha \int_0^T e^{-rt} \int_0^{B_t} S_t \Psi(S_t; S_0) dS_t dt, \tag{7}$$

where  $\alpha = r - b$ ,  $P_0 \equiv P(S_0, 0)$  and

$$p_0 \equiv p(S_0, 0) = e^{-rT} \int_0^K (K - S_T) \Psi(S_T; S_0) dS_T \tag{8}$$

is the price of a European put. The ex-expiration date early exercise boundary is given by

$$B_T = \min\left(K, \frac{r}{\alpha}K\right),$$

independent of the underlying security price distribution.<sup>5</sup>

Intuitively, the analytical valuation formula in Equation (7) gives the sources of value of an American put. The first term, the European put price,  $p_0$ , is the value of the guaranteed payoffs from the put option. The last two terms represent the early exercise premium of the American put. More specifically, the second term of Equation (7) is the present value of the benefits from prematurely exercising the American put through the interest gained by receiving the exercise

<sup>4</sup> This is a direct extension of the results in Carr, Jarrow, and Myneni (1992) and Kim and Yu (1993).

<sup>5</sup> The simple derivation of the ex-expiration date early exercise boundary,  $B_T$ , is as follows. First,  $B_T \leq K$ . In addition, since holding an asset with a cost of carry,  $b$ , is equivalent to holding a stock with a constant dividend ratio  $\alpha \equiv r - b$ , the early exercise decision at the ex-expiration date instant  $T - dt$  is justified if  $(rK - \alpha B_T)dt \geq 0$ , implying  $B_T \leq rK/\alpha$ . Therefore,  $B_T = \min(K, rK/\alpha)$ .

proceeds early. The last term of Equation (7) captures the costs associated with the early exercise decision through the loss of insurance value, that is, the costs of taking a short position on the underlying security following exercising the put option early. Alternatively, these values can be linked to the expected payoffs from a dynamic trading strategy, suggested by Carr, Jarrow, and Myneni (1992), that converts an American put into an otherwise identical European put.

Under the assumed lognormal security price process in Equation (1), the pricing formula in Equation (7) becomes

$$P_0 = p_0 + \int_0^T \left[ rKe^{-rt}N(-d_2(S_0, B_t, t)) - \alpha S_0 e^{-\alpha t}N(-d_1(S_0, B_t, t)) \right] dt, \quad (9)$$

where  $p_0$  is the Black and Scholes (1973) and Merton (1973) European put option price:

$$p_0 = Ke^{-rT}N(-d_2(S_0, K, T)) - S_0 e^{-\alpha T}N(-d_1(S_0, K, T)), \quad (10)$$

where  $N(\cdot)$  is the standard normal distribution function, and

$$d_1(x, y, t) \equiv \frac{\ln(x/y) + (b + \sigma^2/2)t}{\sigma\sqrt{t}},$$

$$d_2(x, y, t) \equiv d_1(x, y, t) - \sigma\sqrt{t}.$$

It follows from Equations (5) and (9) that the time  $t$  optimal point on the early exercise boundary,  $B_t$ , satisfies the following integral equation:

$$K - B_t = Ke^{-r\tau_1}N(-d_2(B_t, K, \tau_1)) - B_t e^{-\alpha\tau_1}N(-d_1(B_t, K, \tau_1)) + \int_t^T \left[ rKe^{-r\tau_2}N(-d_2(B_t, B_s, \tau_2)) - \alpha B_t e^{-\alpha\tau_2}N(-d_1(B_t, B_s, \tau_2)) \right] ds, \quad (11)$$

where  $\tau_1 = T - t$  and  $\tau_2 = s - t$ . In the above equation, the first two terms on the right-hand side represent the value of a European put with a forward price equal to  $B_t$ . The last two terms represent the gain due to interest earned and the loss due to the insurance value foregone through early exercise, respectively, which are reflected in the increased value of an American option over an otherwise identical European option.

Since the cost of carry,  $b$ , can be specified in quite a general fashion, Equation (9) unifies the American put valuation formulae for a wide range of American option valuation problems. Some straightforward examples are standard stock options for which the cost of carry is the

riskless interest rate,  $r$ , less the constant dividend ratio,  $\beta$ , that is,  $b \equiv r - \beta$ ; futures options analyzed in Black (1976), in which  $b \equiv 0$ ; commodities options where there is a convenience yield,  $b \equiv r - \gamma$ , where  $\gamma$  is the convenience yield; and options on foreign currencies analyzed in Garman and Kohlhagen (1983) and Grabbe (1983), in which the cost of carrying the foreign currency is the domestic riskless interest rate,  $r$ , less the foreign riskless interest rate,  $r^*$ , that is,  $b \equiv r - r^*$ .

Clearly, once the early exercise boundary  $\mathcal{B}$  is determined, the American put option value can be easily computed using Equation (9). The valuation formula also offers a simple way to calculate the hedge parameters. Differentiating Equation (9) yields the following explicit comparative statics results:

$$\text{delta: } \frac{\partial P(S, t)}{\partial S_t} = -e^{-\alpha\tau_1} N(-d_1(S_t, K, \tau_1)) - \int_t^T e^{-\alpha\tau_2} \left[ \frac{rK - \alpha B_s}{B_s \alpha \sqrt{\tau_2}} n(d_1(S_t, B_s, \tau_2)) + \alpha N(-d_1(S_t, B_s, \tau_2)) \right] ds, \quad (12)$$

$$\text{gamma: } \frac{\partial^2 P(S, t)}{\partial S_t^2} = \frac{e^{-\alpha\tau_1}}{S_t \sigma \sqrt{\tau_1}} n(d_1(S_t, K, \tau_1)) + \frac{1}{\sigma^2 S_t} \int_t^T \frac{e^{-\alpha\tau_2}}{\tau_2} n(d_1(S_t, B_s, \tau_2)) \times \left[ \frac{rK}{B_s} d_1(S_t, B_s, \tau_2) - \alpha d_2(S_t, B_s, \tau_2) \right] ds, \quad (13)$$

$$\text{vega: } \frac{\partial P(S, t)}{\partial \sigma} = S_t \sqrt{\tau_1} n(d_1(S_t, K, \tau_1)) e^{-\alpha\tau_1} + S_t \int_t^T e^{-\alpha\tau_2} n(d_1(S_t, B_s, \tau_2)) \times \left[ \frac{rK d_1(S_t, B_s, \tau_2) - \alpha B_s d_2(S_t, B_s, \tau_2)}{\sigma B_s} \right] ds, \quad (14)$$

$$\frac{\partial P(S, t)}{\partial K} = \frac{1}{K} \left[ P(S, t) - \frac{\partial P(S, t)}{\partial S_t} S_t \right], \quad (15)$$

$$\text{theta: } \frac{\partial P(S, t)}{\partial \tau_1} = \frac{\sigma^2 S_t^2}{2} \frac{\partial^2 P(S, t)}{\partial S_t^2} + b S_t \frac{\partial P(S, t)}{\partial S_t} - r P(S, t), \quad (16)$$

where  $n(\cdot)$  is the standard normal density function.<sup>6</sup> Similar to the

<sup>6</sup> In practice, we need to compute the delta, gamma, and vega only, since the other two comparative statics parameters are explicit functions of these values.

option pricing formula in Equation (9), these hedge parameters also have an intuitive explanation. One can see that all three parameters—delta, gamma, and vega—are equal to their European counterparts plus a term that depends on the early exercise premium. Moreover, given the early exercise boundary, Equations (12) through (16) provide a *direct* approach to calculate the hedge parameters, while other existing approximation methods based on numerical computation of option hedge ratios have to rely on some scheme of perturbing the option valuation formula.

It is instructive to discuss the reasons why the analytical formulae presented above have some desirable properties in terms of both efficiency and accuracy. Once the early exercise boundary is specified, one can analytically obtain the option values and hedge parameters using Equation (9) and Equations (12) through (16). In contrast, in methods using a perturbation scheme, the option values have to be recomputed for other values of the input variables. Hence, the analytical method is likely to be more efficient compared to the alternatives. In terms of accuracy, the direct computation of option values and hedge parameters restricts the error to that from approximating the early exercise boundary. The competing methods that perturb the option values may compound the approximation error, since they require repetitive calculation of option values, that is, the perturbation methods may enhance error propagation.

## 2. Implementation

Implementation of the valuation formula in Equation (9) and the hedge ratio formulae of Equations (12) through (16) requires the early exercise boundary as an input. Once the optimal exercise boundary is obtained, calculations of the option prices and sensitivity parameters are straightforward. In the following, we first use a recursive scheme to calculate the exercise boundary and then our approach, which is based on the Richardson extrapolation method, to calculate the option prices and hedge parameters.

### 2.1 Optimal exercise boundary

The optimal exercise boundary  $\mathcal{B}$  has no known analytical solution. An approximate exercise boundary, however, can be obtained numerically. A straightforward algorithm is to compute the boundary recursively using Equation (11).<sup>7</sup> Namely, we divide the interval  $[0, T]$  into  $n$  subintervals  $t_i$  for  $i = 0, 1, 2, \dots, n$ , with  $t_0 = 0$  and  $t_n = T$ .

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<sup>7</sup> This is in the spirit of Kim (1990).

This discretization yields, from Equation (11),  $n$  implicit integral equations defining the optimal exercise points,  $B_{t_0}, B_{t_1}, \dots, B_{t_{n-1}}$ . Given the boundary condition on expiration that  $B_{t_n} = \min(K, rK/\alpha)$ ,  $\mathcal{B}_n \equiv \{B_{t_i}, 0 \leq i \leq n-1\}$  can be computed recursively using the  $n$  integral equations. For a large enough  $n$ , this algorithm produces a good approximation of the optimal exercise boundary. Given the discrete optimal boundary  $\mathcal{B}_n$ , the option price and hedge parameters can then be easily calculated from Equation (9) and Equations (12) through (16) using a numerical integration scheme.

This recursive algorithm is conceptually simple and easy to implement, especially for the valuation problem of short-term American options, where only a few points on the boundary need to be calculated. However, this recursive method can be computationally intensive when an option has a long time to expiration or where the exercise price is a function of time.<sup>8</sup> This is because, in these cases, a fairly large number of points on the boundary may be needed in order to obtain a good approximation of the boundary itself. It would, therefore, be useful to rapidly compute the option value and hedge ratios without approximating the whole early exercise boundary. This calls for a more efficient implementation of the recursive method.

## 2.2 Accelerated recursive method

In the following, we suggest an approach that uses the Richardson extrapolation method to accelerate the recursive method mentioned earlier. Our method is in the spirit of Geske and Johnson (1984), except that we estimate the early exercise boundary first and then calculate the option values and hedge ratios directly. In other words, we calculate option values based on only a few points on the optimal boundary and then extrapolate the correct option value. This extrapolation scheme gains efficiency without sacrificing much accuracy.

For instance, consider a three-point Richardson extrapolation scheme. The Geske and Johnson (1984) formula for extrapolating the American put value  $P_0$  in this case is as follows:

$$\hat{P}_0 = (P_1 - 8P_2 + 9P_3)/2, \quad (17)$$

where  $P_i$ ,  $i = 1, 2, 3$ , is the price of an  $i$ -times exercisable option, and  $\hat{P}_0$  denotes an estimate of  $P_0$ .

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<sup>8</sup> The case of a nonconstant exercise price (i.e., an exercise schedule) merits closer attention even for short-term options, since the computational procedure has to adequately capture all the points where there is a change in the exercise price so as to be computationally accurate and efficient.

It follows from Equation (9) that<sup>9</sup>

$$\begin{aligned}
 P_1 &= p_0, \\
 P_2 &= p_0 + \frac{rKT}{2} e^{-\frac{rT}{2}} N(-d_2(S_0, B_{\frac{T}{2}}, T/2)) \\
 &\quad - \frac{\alpha S_0 T}{2} e^{-\frac{\alpha T}{2}} N(-d_1(S_0, B_{\frac{T}{2}}, T/2)), \\
 P_3 &= p_0 + \frac{rKT}{3} \left[ e^{-\frac{rT}{3}} N(-d_2(S_0, B_{\frac{T}{3}}, T/3)) \right. \\
 &\quad \left. + e^{-\frac{2rT}{3}} N(-d_2(S_0, B_{\frac{2T}{3}}, 2T/3)) \right] \\
 &\quad - \frac{\alpha S_0 T}{3} \\
 &\quad \times \left[ e^{-\frac{\alpha T}{3}} N(-d_1(S_0, B_{\frac{T}{3}}, T/3)) + e^{-\frac{2\alpha T}{3}} N(-d_1(S_0, B_{\frac{2T}{3}}, 2T/3)) \right].
 \end{aligned}$$

A special case of the above expressions applies to the case of a non-dividend paying stock, where  $\alpha = 0$ :

$$P_1 = p_0, \quad (18)$$

$$P_2 = p_0 + \frac{rKT}{2} e^{-\frac{rT}{2}} N(-d_2(S_0, B_{\frac{T}{2}}, T/2)), \quad (19)$$

$$\begin{aligned}
 P_3 &= p_0 + \frac{rKT}{3} \\
 &\quad \times \left[ e^{-\frac{rT}{3}} N(-d_2(S_0, B_{\frac{T}{3}}, T/3)) + e^{-\frac{2rT}{3}} N(-d_2(S_0, B_{\frac{2T}{3}}, 2T/3)) \right].
 \end{aligned} \quad (20)$$

One can see that only three boundary points— $B_{\frac{T}{3}}$ ,  $B_{\frac{T}{2}}$ , and  $B_{\frac{2T}{3}}$ —are needed to calculate  $\{P_i\}$  and consequently  $\hat{P}_0$ .

Given  $B_{\frac{T}{3}}$ ,  $B_{\frac{T}{2}}$ , and  $B_{\frac{2T}{3}}$ , the Richardson extrapolation method can also be used to calculate the option hedge ratios such as the delta, gamma, and vega. For example, the delta value can be extrapolated as follows:

$$\hat{\Delta}_0 = (\Delta_1 - 8\Delta_2 + 9\Delta_3)/2, \quad (21)$$

where, for an option on a non-dividend-paying stock,

$$\Delta_1 = -N(-d_1(S_0, K, T)), \quad (22)$$

$$\begin{aligned}
 \Delta_2 &= -N(-d_1(S_0, K, T)) \\
 &\quad - \frac{rK\sqrt{T/2}}{\sigma S_0} e^{-\frac{rT}{2}} n(d_2(S_0, B_{\frac{T}{2}}, T/2)),
 \end{aligned} \quad (23)$$

<sup>9</sup> These approximations are made by replacing the integration in Equation (9) with a simple sum over exercisable points.

$$\begin{aligned} \Delta_3 = & -N(-d_1(S_0, K, T)) - \frac{rK\sqrt{T/3}}{\sigma S_0} e^{-\frac{rT}{3}} n(d_2(S_0, B_{\frac{T}{3}}, T/3)) \\ & - \frac{rK\sqrt{2T/3}}{2\sigma S_0} e^{-\frac{2rT}{3}} n(d_2(S_0, B_{\frac{2T}{3}}, 2T/3)). \end{aligned} \quad (24)$$

### 2.3 Comparison with the Geske and Johnson method

As mentioned earlier, the accelerated recursive method is similar in some ways to the Geske and Johnson (1984) method. Although both methods employ Richardson extrapolation, they use different formulae for computing  $\{P_i, i = 1, \dots, n\}$ .<sup>10</sup> In the Geske and Johnson framework, the exact option price is the limit ( $n \uparrow \infty$ ) of the price of an option exercisable on  $n$  dates. As Geske and Johnson showed, the price of such an option,  $P_n$ , is a sum over several multivariate cumulative normal density functions. Specifically, for a given  $n$ ,  $P_n$  involves two univariate integrals (from a European option value), two bivariate integrals, two trivariate integrals, ..., and two  $n$ -variate integrals [see Equation (5) of Geske and Johnson (1984)]. It is known that the computation of a multivariate normal integral is time intensive for large  $n$ . Consequently, the Geske and Johnson method has been implemented only for  $n \leq 4$ . On the other hand, as shown in Equation (9),  $P_n$  in the recursive method involves only the *univariate* normal integral. For example, in the case of options on non-dividend-paying stocks,  $P_n$  involves  $n$  univariate normal integrals, even under the crudest integration scheme, using a step function. As a result, the recursive method can be easily implemented, in principle, for any value of  $n$  without too much computational difficulty.

Due to its ease of implementation, the recursive method can improve the convergence of the extrapolation. As Omberg (1987a) pointed out, the arithmetic sequence of  $n = \{1, 2, 3, \dots\}$  used by Geske and Johnson in their extrapolation may not yield uniform convergence in  $\hat{P}_0$ , whereas a geometric sequence of  $n = \{1, 2, 4, \dots\}$  would give such convergence. However, it is not feasible to implement the Geske and Johnson method using a geometric sequence of  $n$  due to the substantial increase in computational cost beyond  $n = 3$ . On the other hand, one can see from Equation (9) that the recursive method can be implemented easily using a geometric sequence and can, therefore, improve the convergence in  $\hat{P}_0$ .

The difference between the two methods in the dimensionality of

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<sup>10</sup> Other similar methods are Breen (1991), Bunch and Johnson (1992), Carr and Faguet (1994), and Ho, Stapleton, and Subrahmanyam (1994), which use the binomial method, a maximization scheme, the method of lines, and an exponential approximation, respectively, to calculate  $\{P_i\}$ , and in turn,  $\hat{P}_0$ .

the integrals involved also indicates that the recursive method can improve the efficiency of the Geske and Johnson method. To illustrate this, consider a three-point (an arithmetic sequence of  $n$ ) Richardson extrapolation scheme. In this case, computation of  $\hat{P}_0$  using the Geske and Johnson method would require calculations of six single integrals, four double integrals, and two triple integrals, whereas using the recursive method would require only five single integrals [compare Equations (18) through (21)].<sup>11</sup> Although it is difficult to say precisely how much more time it takes to calculate a multinormal integral than a single one, we can obtain a rough estimate. The most common algorithm to calculate a univariate integral is Hill's (1973), which approximates the integral by a summation of four terms. A commonly used algorithm to calculate a bivariate integral is the one by Drezner (1978), which uses a Gauss quadrature method. Typically, this method approximates the double integral by a sum of 9 to 25 terms. Schervish's (1984) algorithm for a multinormal integral also uses a Gauss quadrature method. In this method, calculation of an  $n$  integral ( $n \geq 3$ ) needs computation of at least two  $(n - 1)$  dimensional integrals. It would not, therefore, be unreasonable to conclude from these facts that the recursive method is likely to be much more efficient. Furthermore, the larger the  $n$  value, the more efficient is the recursive method compared to the Geske and Johnson method, which involves multinormal integrals.<sup>12</sup>

As far as the accuracy is concerned, the recursive method and the Geske and Johnson method should be similar, given that both use the same extrapolation scheme, *provided* that  $\{P_n\}$  values calculated in the two methods are equally accurate. Under a three-point extrapolation scheme, the truncation error of using  $\hat{P}_0$  defined in Equation (17) is  $O(b^3)$  where  $b = T/3$ .<sup>13</sup> However, given the relative ease of computation, the recursive method can be implemented under a higher-

<sup>11</sup> In the Geske and Johnson method,  $P_1$  involves the calculation of two single integrals,  $P_2$  the calculation of two single and two double integrals, and  $P_3$  the calculation of two single, two double, and two triple integrals.

<sup>12</sup> Selby and Hodges (1987) present an identity relating a sum of nested multinormal integrals to a single multidimensional integral. Schroder (1989) improves Curnow and Dunnett's (1962) reduction formula for high dimensional normal integrals. However, the Selby and Hodges method still involves computing high dimensional integrals. The Schroder method has a significant advantage over the Curnow and Dunnett formula only when the dimension is  $n > 5$ , and involves at least 200 univariate integrals for  $n \geq 5$ . As a result, a four-point extrapolation with these new reduction schemes is still slow, as noted by Bunch and Johnson (1992).

<sup>13</sup> This result is based on the assumption that the estimate  $\hat{P}_0$  of the true price  $P_0$  has the following structure:  $\hat{P}_0 = P_0 + \sum_{i=1}^m a_i b^i$ , where  $a_i$  are constants independent of  $P_0$ ,  $m (\geq 1)$  is an integer, and the series does not necessarily converge. Notice that the Richardson extrapolation method is valid only if this assumption is satisfied [see, e.g., Dahlquist and Bjorck (1974)]. However, it is not clear that the Taylor expansion argument used in Geske and Johnson (1984) can indeed justify this assumption. Results from our attempt to answer this question numerically are inconclusive.

order extrapolation, say a five-point extrapolation [whose truncation error is  $O(b^5)$ ], to reduce the truncation error, that is, to improve the accuracy.<sup>14</sup>

### 3. Applications

In this section, we illustrate the application of the analytical valuation formula in Equation (9) and its implementation using the accelerated recursive method to two common American option pricing problems: (1) American options on stocks and (2) American quanto options. It should be emphasized that this method can be applied to other American-style contracts, such as futures options, Asian options, barrier options, and look-back options, since a closed-form solution for the corresponding European-style option exists.

#### 3.1 Stock options

Assume that the underlying security is a common stock without a dividend payout. This case is chosen for ease of comparison with results in previous papers. Then, the American put price is obtained from Equation (9) by letting  $b \equiv r$ :<sup>15</sup>

$$P_0 = p_0 + \int_0^T rK e^{-rt} N(-d_2(S_0, B_t, t)) dt, \quad (25)$$

where  $p_0$  is the Black and Scholes European put price:

$$p_0 = K e^{-rT} N(-d_2(S_0, K, T)) - S_0 N(-d_1(S_0, K, T)). \quad (26)$$

The integral equation defining the early exercise boundary  $B_t$  is obtained from Equation (11):

$$K - B_t = p(B_t, t) + rK \int_t^T e^{-r(s-t)} N(-d_2(B_t, B_s, s-t)) ds, \quad (27)$$

with the boundary condition  $B_T = K$ . The formulae for option hedge parameters can be obtained from Equations (12) through (16) by letting  $b \equiv r$ .

To test the accuracy and speed of the (accelerated) recursive method, we compare the results of option prices and hedge parameters calculated from the recursive method with those from three widely used numerical methods: the finite difference method, the binomial

<sup>14</sup> An improvement in the accuracy may also be achieved at a lower-order extrapolation using a more sophisticated integration scheme, say Simpson's rule.

<sup>15</sup> This formula was obtained by Kim (1990) and Carr, Jarrow, and Myneni (1992).

method, and Breen's (1991) accelerated binomial method.<sup>16</sup> We also report the results from Geske and Johnson (1984) as a basis for comparison. The accuracy is measured by the deviation from a benchmark, which is chosen to be results from the binomial method with  $N = 10,000$  time steps.<sup>17</sup> The speed is measured by the CPU time required to compute option prices or hedge parameters for a given set of contracts. The set of contracts is chosen to be the one in Table I of Geske and Johnson (1984).

Table 1 reports the results of option prices from the five alternative methods. Column 4 shows the numerical results from the binomial method with  $N = 10,000$  time steps (the benchmark). Column 5, reported in Table I of Geske and Johnson (1984), is included here for comparison. Columns 6 through 9 show the results from the binomial method with  $N = 150$  time steps, the accelerated binomial method with  $N = 150$  time steps, the finite difference method with 200 steps in both time and underlying state variables, and the recursive method with a four-point extrapolation, respectively. The accuracy of a particular method is measured by its root of the mean squared error (RMSE), as shown in the second-to-last row. The CPU time in seconds is shown in the last row. Note that the CPU time in column 5 is absent because the corresponding results are taken directly from Geske and Johnson (1984).

Table 1 indicates that the RMSE of the recursive method has the same order of magnitude as that of the binomial method with  $N = 150$ , in spite of the fact that the former method is implemented with a step-function discretization.<sup>18</sup> One can also see that the amount of CPU time using the recursive method is significantly less than that using the other three numerical methods listed in the table. In particular, the recursive method is about *five times* faster than the accelerated binomial method, and more accurate as well.

Table 2 reports the results of deltas for the same contracts as shown in Table 1. As before, the results in column 4 are taken as the benchmark. Again, column 5, reported in Table I of Geske and Johnson (1984), is included here for comparison. One can see from the table

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<sup>16</sup> It would be interesting to include more methods for comparison, for instance, Carr and Faguet (1994) and Broadie and Detemple (1994). However, as mentioned before, the implementation of these two methods involves using some regression coefficients and adjustment factors, whose values are determined from data sets that are different from ours. See these two papers for a comparison of their methods against other methods.

<sup>17</sup> Broadie and Detemple (1994) also use option values from the binomial method with  $N = 10,000$  time steps as their benchmark. Carr and Faguet (1994) use the average of values from  $N = 1,000$  and  $N = 1,001$  as their benchmark.

<sup>18</sup> One alternative measure is the root of the mean squared *relative* error. In fact, the recursive method is more accurate under this measure than under the RMSE measure.

**Table 1**  
**Values of American put options on stocks using different numerical methods ( $S = \$40, r = 4.88\%$ )**

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
$K$	$\sigma$	$T$ (yr)	Binomial ( $N = 10,000$ )	Geske and Johnson	Binomial ( $N = 150$ )	Accelerated binomial	Finite difference	Recursive method
35	0.2	0.0833	0.0062	0.0062	0.0061	0.0061	0.0278	0.0062
35	0.2	0.3333	0.2004	0.1999	0.1995	0.1994	0.2382	0.2004
35	0.2	0.5833	0.4328	0.4321	0.4340	0.4331	0.4624	0.4337
40	0.2	0.0833	0.8522	0.8528	0.8512	0.8517	0.9874	0.8543
40	0.2	0.3333	1.5798	1.5807	1.5783	1.5752	1.6244	1.5873
40	0.2	0.5833	1.9904	1.9905	1.9886	1.9856	2.0177	1.9987
45	0.2	0.0833	5.0000	4.9985	5.0000	4.9200	5.0052	5.0020
45	0.2	0.3333	5.0883	5.0951	5.0886	4.9253	5.1327	5.0954
45	0.2	0.5833	5.2670	5.2719	5.2677	5.2844	5.2699	5.2631
35	0.3	0.0833	0.0774	0.0744	0.0775	0.0772	0.1216	0.0775
35	0.3	0.3333	0.6975	0.6969	0.6993	0.6977	0.7300	0.6978
35	0.3	0.5833	1.2198	1.2194	1.2239	1.2218	1.2407	1.2233
40	0.3	0.0833	1.3099	1.3100	1.3083	1.3095	1.3860	1.3116
40	0.3	0.3333	2.4825	2.4817	2.4799	2.4781	2.5068	2.4919
40	0.3	0.5833	3.1696	3.1733	3.1665	3.1622	3.1819	3.1842
45	0.3	0.0833	5.0597	5.0599	5.0600	5.0632	5.1016	5.0604
45	0.3	0.3333	5.7056	5.7012	5.7065	5.6978	5.7193	5.6970
45	0.3	0.5833	6.2436	6.2365	6.2448	6.2395	6.2477	6.2303
35	0.4	0.0833	0.2466	0.2466	0.2454	0.2456	0.2949	0.2467
35	0.4	0.3333	1.3460	1.3450	1.3505	1.3481	1.3696	1.3468
35	0.4	0.5833	2.1549	2.1568	2.1602	2.1569	2.1676	2.1603
40	0.4	0.0833	1.7681	1.7679	1.7658	1.7674	1.8198	1.7694
40	0.4	0.3333	3.3874	3.3632	3.3835	3.3863	3.4011	3.3970
40	0.4	0.5833	4.3526	4.3556	4.3480	4.3426	4.3567	4.3699
45	0.4	0.0833	5.2868	5.2855	5.2875	5.2863	5.3289	5.2853
45	0.4	0.3333	6.5099	6.5093	6.5103	6.5054	6.5147	6.5128
45	0.4	0.5833	7.3830	7.3831	7.3897	7.3785	7.3792	7.3865
	RMSE		0.0000	5.3383e-03	2.6380e-03	3.5302e-02	4.1041e-02	6.7047e-03
	GPU time (sec)				1.1766e+00	7.8330e-01	6.4664e+00	1.3333e-01

Columns 1-3 represent the values of the parameters,  $K$  (strike price),  $\sigma$  (volatility), and  $T$  (the time to expiration), respectively. Column 4 shows the numerical results of option values from the binomial method with  $N = 10,000$  time steps (the benchmark). Column 5 is as reported in Table I of Geske and Johnson (1984). Columns 6-9 show the results from the binomial method with  $N = 150$  time steps, the accelerated binomial method with 150 time steps, the finite difference method with 200 steps in both time and the underlying state variable, and the recursive method with  $n = 4$  (a four-point extrapolation), respectively. The second-to-last row shows the root of the mean squared errors (RMSE), a measure of deviation from the benchmark. The CPU time is the time required on a SPARC-10 machine to compute the option values for all 27 contracts listed in the table.

**Table 2**  
**Hedge ratios (deltas) of American put options on stocks using different numerical methods ( $S = \$40, r = 4.88\%$ )**

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
$K$	$\sigma$	$T$ (yr)	Binomial ( $N = 10,000$ )	Geske and Johnson	Binomial ( $N = 150$ )	Accelerated binomial	Finite difference	Recursive method
35	0.2	0.0833	-0.0080	-0.008	-0.0081	-0.0081	-0.0522	-0.0080
35	0.2	0.3333	-0.0901	-0.090	-0.0911	-0.0913	-0.1337	-0.0900
35	0.2	0.5833	-0.1338	-0.134	-0.1355	-0.1352	-0.1692	-0.1351
40	0.2	0.0833	-0.4693	-0.470	-0.4698	-0.4690	-0.5738	-0.4712
40	0.2	0.3333	-0.4435	-0.443	-0.4443	-0.4425	-0.5083	-0.4416
40	0.2	0.5833	-0.4287	-0.427	-0.4299	-0.4297	-0.4821	-0.4228
45	0.2	0.0833	-1.0000	-1.000	-1.0000	-0.9867	-0.9617	-0.9993
45	0.2	0.3333	-0.8812	-0.888	-0.8803	-0.7439	-0.8897	-0.8964
45	0.2	0.5833	-0.7948	-0.805	-0.7944	-0.6271	-0.8355	-0.8112
35	0.3	0.0833	-0.0516	-0.052	-0.0526	-0.0524	-0.1041	-0.0516
35	0.3	0.3333	-0.1741	-0.174	-0.1758	-0.1758	-0.2067	-0.1744
35	0.3	0.5833	-0.2126	-0.213	-0.2140	-0.2140	-0.2372	-0.2145
40	0.3	0.0833	-0.4695	-0.470	-0.4699	-0.4697	-0.5437	-0.4710
40	0.3	0.3333	-0.4420	-0.442	-0.4429	-0.4409	-0.4810	-0.4429
40	0.3	0.5833	-0.4256	-0.425	-0.4268	-0.4253	-0.4544	-0.4243
45	0.3	0.0833	-0.9233	-0.926	-0.9220	-0.8762	-0.9111	-0.9293
45	0.3	0.3333	-0.7266	-0.726	-0.7260	-0.7302	-0.7516	-0.7229
45	0.3	0.5833	-0.6520	-0.651	-0.6520	-0.6576	-0.6776	-0.6459
35	0.4	0.0833	-0.1062	-0.106	-0.1073	-0.1072	-0.1526	-0.1062
35	0.4	0.3333	-0.2260	-0.226	-0.2277	-0.2278	-0.2502	-0.2265
35	0.4	0.5833	-0.2539	-0.254	-0.2557	-0.2553	-0.2714	-0.2558
40	0.4	0.0833	-0.4668	-0.467	-0.4673	-0.4673	-0.5236	-0.4679
40	0.4	0.3333	-0.4360	-0.437	-0.4370	-0.4362	-0.4626	-0.4376
40	0.4	0.5833	-0.4173	-0.418	-0.4186	-0.4162	-0.4360	-0.4177
45	0.4	0.0833	-0.8363	-0.835	-0.8350	-0.8364	-0.8478	-0.8341
45	0.4	0.3333	-0.6475	-0.646	-0.6474	-0.6479	-0.6678	-0.6424
45	0.4	0.5833	-0.5819	-0.580	-0.5819	-0.5833	-0.5982	-0.5753
		RMSE	0.0000	2.5289e-03	1.0593e-03	4.2777e-02	4.2216e-02	5.1702e-03
		CPU time (sec)			3.5315e+00	2.3332e+00	6.4664e+00	1.3333e-01

Columns 1–3 represent the values of the parameters,  $K$  (strike price),  $\sigma$  (volatility), and  $T$  (the time to expiration), respectively. Column 4 shows the numerical results of option values from the binomial method with  $N = 10,000$  time steps (the benchmark). Column 5 is as reported in Table 1 of Geske and Johnson (1984). Columns 6–9 show the results from the binomial method with  $N = 150$  time steps, the accelerated binomial method with 150 time steps, the finite difference method with 200 steps in both time and the underlying state variable, and the recursive method with  $n = 4$  (a four-point extrapolation), respectively. The second-to-last row shows the root of the mean squared errors (RMSE), a measure of deviation from the benchmark. The CPU time is the time required on a SPARC-10 machine to compute the delta values for all 27 contracts listed in the table.

**Table 3**  
**Convexity parameter (gammas) of American put options on stocks using different numerical methods** ( $S = \$40$ ;  $r = 4.88\%$ )

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
$K$	$\sigma$	$T$ (yr)	Binomial ( $N = 10,000$ )	Binomial ( $N = 150$ )	Accelerated binomial	Finite difference	Recursive method
35	0.2	0.0833	0.0095	0.0096	0.0095	0.0273	0.0095
35	0.2	0.3333	0.0357	0.0358	0.0359	0.0434	0.0358
35	0.2	0.5833	0.0364	0.0364	0.0361	0.0412	0.0374
40	0.2	0.0833	0.1776	0.1772	0.1757	0.1638	0.1759
40	0.2	0.3333	0.0923	0.0922	0.0928	0.0942	0.0870
40	0.2	0.5833	0.0719	0.0718	0.0736	0.0748	0.0669
45	0.2	0.0833	0.0000	0.0000	0.0132	0.0106	0.0010
45	0.2	0.3333	0.0827	0.0829	0.3140	0.0541	0.0804
45	0.2	0.5833	0.0787	0.0785	-0.2757	0.0755	0.0860
35	0.3	0.0833	0.0306	0.0308	0.0307	0.0422	0.0306
35	0.3	0.3333	0.0376	0.0376	0.0376	0.0409	0.0381
35	0.3	0.5833	0.0326	0.0325	0.0322	0.0344	0.0332
40	0.3	0.0833	0.1170	0.1168	0.1160	0.1160	0.1173
40	0.3	0.3333	0.0597	0.0596	0.0592	0.0601	0.0579
40	0.3	0.5833	0.0459	0.0458	0.0461	0.0462	0.0434
45	0.3	0.0833	0.0578	0.0581	-0.1188	0.0393	0.0615
45	0.3	0.3333	0.0572	0.0571	0.0599	0.0519	0.0614
45	0.3	0.5833	0.0485	0.0484	0.0509	0.0471	0.0517
35	0.4	0.0833	0.0398	0.0399	0.0399	0.0469	0.0398
35	0.4	0.3333	0.0330	0.0329	0.0329	0.0345	0.0334
35	0.4	0.5833	0.0269	0.0268	0.0266	0.0276	0.0272
40	0.4	0.0833	0.0872	0.0871	0.0866	0.0869	0.0877
40	0.4	0.3333	0.0441	0.0440	0.0437	0.0441	0.0434
40	0.4	0.5833	0.0336	0.0336	0.0333	0.0337	0.0324
45	0.4	0.0833	0.0590	0.0591	0.0608	0.0459	0.0612
45	0.4	0.3333	0.0439	0.0438	0.0450	0.0417	0.0431
45	0.4	0.5833	0.0355	0.0354	0.0365	0.0346	0.0347
RMSE			0.0000	1.4332e-04	8.8301e-02	9.2263e-03	2.5499e-03
CPU time (sec)				3.5332e+00	2.3332e+00	6.4664e+00	1.3333e-01

Columns 1–3 represent the values of the parameters,  $K$  (strike price),  $\sigma$  (volatility), and  $T$  (the time to expiration), respectively. Columns 4–8 show the numerical results of gamma values from the binomial method with  $N = 10,000$  time steps (the benchmark), with  $N = 150$  time steps, the accelerated binomial method with 150 time steps, the finite difference method with 200 steps in both time and the underlying state variable, and the recursive method with  $n = 4$  (a four-point extrapolation), respectively. The second-to-last row shows the root of the mean squared errors (RMSE), a measure of deviation from the benchmark. The CPU time is the time required on a SPARC-10 machine to compute the gamma values for all 27 contracts listed in the table.

that the recursive method again produces much smaller RMSE than the accelerated binomial and the finite difference methods do. It is worth noticing that it takes almost the same amount of CPU time for the recursive method (and also the finite difference method) to compute deltas as option prices, whereas it takes *twice* as much CPU time to compute deltas as option prices for both the binomial and the accelerated binomial methods. This indicates that the recursive method is particularly efficient in calculating hedge parameters.

Table 3 reports the results of gammas for the same contracts as

shown in Table 1.<sup>19</sup> Again, the RMSE of the recursive method is much smaller than that of the accelerated binomial method. Also, the speed of computation is faster by at least one *order of magnitude* compared with the alternatives. It is apparent that the relative computational merit of the recursive method increases as we get to *higher-order* hedge parameters.<sup>20</sup>

### 3.2 American quanto options

We now consider the application of our method to nonstandard options, in particular, to quanto options. We do this to illustrate the use of the method for any case where a closed-form solution exists for the corresponding European-style option. Quanto options are options based on two (or more) underlying state variables, usually the foreign exchange rate and an asset price. Examples of these options available to U.S. investors are Nikkei index warrants and currency-protected warrants on foreign treasury bonds, warrants on cross-currency rates denominated in dollars, as well as a range of over-the-counter products. Some basic features and valuation results on these instruments are described in, among others, Derman, Karasinski, and Wecker (1990) and Dravid, Richardson, and Sun (1993; henceforth DRS). These papers examine the issues of hedging, sensitivity analysis, and parameter estimations of these contracts. In this section, we focus on the valuation of American-style put warrants using our alternative approach.<sup>21</sup> The analysis of the American-style call warrants can be analogously developed.

Let  $S_t^*$  be the yen price at time  $t$  of the underlying Nikkei index with an instantaneous dividend payout rate  $\delta_t$ . Assume that  $Y_t$  denotes the spot exchange rate specified in units of dollars (or cents) per yen. Define the dollar price of the Nikkei index at  $t$  by  $S_t \equiv S_t^* Y_t$ . Denote by  $r_t$  and  $r_t^*$  the instantaneous U.S. and Japanese riskless interest rates, respectively. Following DRS, we have, under the risk neutral

<sup>19</sup> The Geske and Johnson method is left out in Table 3 because no results about gammas for the same contracts are reported in Geske and Johnson (1984).

<sup>20</sup> Notice that two of the gamma values from the accelerated binomial method in column 6 are negative and that another one with  $K = 45$ ,  $\sigma = 0.2$ , and  $T = 0.3333$ , though positive, deviates significantly from the corresponding benchmark value. This may be due to the fact that the accelerated binomial method is an approximation of the Geske and Johnson formula (also an approximation). This calls for caution when one uses the accelerated binomial method to compute hedge parameters.

<sup>21</sup> In this article, we present the valuation framework in terms of the Nikkei index put warrants. The same intuition and valuation methodology can be applied to the valuation of other types of contingent claims on a foreign stock index, as well as the valuation of dollar-denominated cross-currency warrants, such as warrants on the yen/DM cross-currency rate.

probability measure,

$$dS_t = (r_t - \delta_t)S_t dt + \sigma_{s^*} S_t dZ_t^{(s^*)} + \sigma_y S_t dZ_t^{(y)}, \quad (28)$$

$$dY_t = (r_t - r_t^*)Y_t dt + \sigma_y Y_t dZ_t^{(y)}, \quad (29)$$

$$dS_t^* = r_f S_t^* dt + \sigma_{s^*} S_t^* dZ_t^{(s^*)}, \quad (30)$$

where  $\sigma_{s^*}$  and  $\sigma_y$  are the instantaneous volatilities of the Nikkei index and the exchange rate,  $\{Z_t^{(s^*)}; t \geq 0\}$  and  $\{Z_t^{(y)}; t \geq 0\}$  are two standard one-dimensional Brownian motions with a correlation coefficient  $\rho$ , and  $r_f \equiv r_t^* - \delta_t - \rho\sigma_{s^*}\sigma_y$ .

Consider a put option whose yen payoffs can be converted into U.S. currency at a prespecified exchange rate  $\bar{Y}$ .<sup>22</sup> With a fixed exchange rate, the European put price depends on the fixed exchange rate and the parameters of the exchange rate dynamics, but is independent of the current level of the exchange rate. This is because the U.S. investor has to continuously rebalance his position in the Nikkei index through the foreign currency market, by converting dollars into yen, and *vice versa*, in order to hedge this option (since the Nikkei index is not directly traded in dollars). Equivalently, it can be said that simply by using the Nikkei index (denoted in yen) and riskless Japanese bonds, the U.S. investor completely hedges the European put option, but is exposed to the foreign exchange risk, which can be hedged over time by continuously rebalancing the positions in the Nikkei index through the foreign currency market. Consequently, by using riskless Japanese bonds and the Nikkei index, the U.S. investor can also hedge/replicate the American put, but in terms of yen. This essentially makes the pricing of fixed exchange rate American Nikkei puts a one-dimensional problem. Namely, the problem is equivalent to pricing an American put option whose underlying asset price process is described by Equation (30), and hence can be recast in the framework developed in Section 1.

Substituting  $\alpha = r - r_f$  and  $\sigma = \sigma_{s^*}$  into Equations (9) through (11), we have that the value of a fixed exchange rate American put (in yen) with a strike price  $K^*$  is

$$P_0/\bar{Y} = p_0 + \int_0^T \left[ rK^* e^{-rs} N(-x_2(S_0^*, B_s, s)) - (r - r_f)S_s^* e^{-(r-r_f)s} N(-x_1(S_0^*, B_s, s)) \right] ds, \quad (31)$$

and that the value of the corresponding European put option in yen is

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<sup>22</sup> Our method also applies to some variations on the basic quanto structure analyzed here. An example is the flexible exchange rate case. [See DRS for details.]

(see DRS)

$$p_0 = K^* e^{-rT} N(-x_2(S_0^*, K^*, T)) - S_0^* e^{-(r-r_f)T} N(-x_1(S_0^*, K^*, T)), \quad (32)$$

where

$$x_1(x, y, t) \equiv \frac{\ln(x/y) + (r_f + \sigma_{s^*}^2/2)t}{\sigma_{s^*} \sqrt{t}},$$

$$x_2(x, y, t) \equiv x_1(x, y, t) - \sigma_{s^*} \sqrt{t},$$

and  $B_t$  is the early exercise boundary at time  $t$ , in terms of the Nikkei index. The value of  $B_t$  is determined by

$$K^* - B_t = p_1 + \int_t^T [rK^* e^{-r(s-t)} N(-x_2(B_t, B_s, s-t)) - (r-r_f)B_t e^{-(r-r_f)(s-t)} N(-x_1(B_t, B_s, s-t))] ds, \quad (33)$$

where

$$p_1 = K^* e^{-r(T-t)} N(-x_2(B_t^*, K^*, T-t)) - S_t^* e^{-(r-r_f)(T-t)} N(-x_1(B_t^*, K^*, T-t)),$$

and  $x_1$  and  $x_2$  are as defined in Equation (32). The formulae for hedge ratios can be obtained in a straightforward manner from Equation (31), and will not be presented here in the interest of brevity. Once the exercise boundary is computed using methods analogous to those described in Section 2, the values and hedge parameters of options exercisable at discrete points can be obtained from Equation (31). The use of Richardson extrapolation then yields the estimates of the values and hedge parameters of the American-style quanto options.<sup>23</sup>

It should be noted that the pricing formula in Equation (31) can also be obtained directly using the arguments underlying Equation (7) and the fact that a closed-form solution exists for the corresponding European-style option. Further, this scheme also applies to those non-standard American-style pricing problems that cannot be recast in the framework of the general pricing formula in Equation (9).

The above analysis illustrates the general principles to be applied in implementing the recursive method to nonstandard American-style options. Given that a closed-form solution exists for the corresponding European-style option, one can set up the value-matching condition as in Equation (33) above. One can also write down the discrete time analog of Equation (31) for the  $i$ -times exercisable option,  $P_i$ ,

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<sup>23</sup> Numerical results are available from the authors upon request.

$i = 1, 2, 3 \dots$ , and the corresponding ones for the hedge parameters. Richardson extrapolation can then be used to obtain the values of the American-style option and the hedge parameters.

#### **4. Conclusions**

This article has presented a method of recursive implementation of analytical formulae, taking the early exercise boundary as given, for the value and hedge parameters of an American-style option. The early exercise boundary is computed by using Richardson extrapolation based on options exercisable on (one, two, three, etc.) discrete dates. The method can be used for *any* American-style option for which a closed-form solution exists for the equivalent European-style option. A unified analytical formula was derived for American-style put options on an underlying security with a cost of carry. This formula can be applied to a range of standard American-style options: options on stocks with dividends, options on futures, options on foreign exchange, and quanto options. The method was implemented for American put options on non-dividend-paying stocks, for ease of comparison with other methods in the literature.

The method presented in this article has three attractive features. First, since its implementation is based on using an analytical formula for both option values and hedge parameters, the latter are computed directly, rather than by perturbation of the option pricing formula. Second, as a result, the computation is both efficient and accurate, since the analytical formula involves only univariate integrals. The improvement in efficiency over competing methods is especially advantageous in the management of options books in practice, since the implementation of dynamic trading strategies involves the computation of values and hedge ratios for large numbers of options on a continuous basis. Third, the analytical formulae presented here permit an intuitive decomposition of the value and the hedge ratios into three terms: that of a corresponding European-style option, the gain in the time value of money, and the loss in the insurance value from early exercise.

Several modifications and extensions of the methods proposed in this article can be explored in further research. First, the accuracy and efficiency of the method can be improved, particularly for long-term options, by exploring the possibility of unequal spacing of the discrete points or by explicitly taking into account the slope of the exercise boundary between the discrete points. Second, the method can be applied to the valuation of other American options such as those with two or more sources of uncertainty, for instance, mortgage-backed securities, options on bonds, foreign exchange, or real assets

in capital budgeting applications. In particular, the applications to exotic options in these situations can be studied in greater detail. Third, the valuation of American options can be studied in the context of alternative stochastic processes for the underlying asset. We leave these directions of enquiry for later work.

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