Review of Quantitative Finance and Accounting, 19: 261–272, 2002

A Note on Forward Price and Forward Measure

REN-RAW CHEN*
FOM/SOB-NB, Rutgers University, Levin Bldg., Rockafeller Rd., Piscataway, NJ 08854
E-mail: rchen@rci.rutgers.edu; Tel.: (732) 445-4236; Fax: (732) 445-2333.

JING-ZHI HUANG
Penn State University, Smeal College of Business, University Park, PA 16802

Abstract. The forward measure is convenient in calculating various contingent claim prices under stochastic interest rates. We demonstrate that caution needs to be drawn when the forward measure is used to price contingent claims that involve multiple cash flows. We also derive partial different equations for the forward price to demonstrate how forward contracts can be used for dynamic hedging and how hedges can be conducted if the payoff of a contingent claim depends on the forward price.

Key words: forward measure, forward price, stochastic interest rates

JEL Classification: G13

1. Introduction

The forward measure pricing methodology (Jamshidian, 1987, 1989; and Geman, Karoui and Rochet, 1995) has been widely used in pricing securities when interest rates are stochastic. This technique provides great ease in deriving closed form solutions for various derivative contracts with European-style payoffs. However, for many financial contracts whose payoffs are dependent upon forward prices, the forward measure needs to be used with caution. In this paper, we demonstrate, using two popular interest rate contracts, how roll-over expectations under different forward measures should be used. We also derive an alternative partial differential equation (PDE) to the one in Jamshidian (1987) to demonstrate how the forward price can be interpreted alternatively. This alternative PDE is consistent with the interpretation that the forward price is an asset price, as opposed to the interpretation that the forward price is an index. In doing so, we effectively transfer an "index forward price" into an "forward asset price" that satisfies the Capital Asset Pricing Model, as argued in Cox, Ingersoll and Ross (1981, hereafter CIR).[1]

The theoretical behavior of the forward price was first derived and summarized by CIR. They also derived closed form solutions for forward and futures prices under a single factor

*Corresponding author.

square root process for the instantaneous interest rate. The futures price is dependent of the utility function of the single representative agent in the economy. The forward price, on the other hand, is independent of the agent's risk preference and the interest rate assumption. This is of course because the forward price can be derived from static (i.e., one-period) arbitrage which is independent of any assumption of the interest rate and of the utility function of the representative agent. Furthermore, the static arbitrage argument also implies that the forward price is independent of the number of factors that drive the economy.

Recently, the forward price is reinterpreted using the forward measure. It is shown that the forward price is an expectation of the terminal payoff of the underlying asset price, like the futures price which is a risk neutral expectation, except that the probability measure under which the expectation is taken is different from the risk neutral measure by an adjustment term. This methodology has been found very useful in calculating a variety of forward prices.[2]

We assume a single factor throughout the paper. Without any loss of generality, our analysis is based upon the forward price of a pure-discount bond. The paper is organized as follows. The next section reviews the forward measure and its use to derive interest rate sensitive derivatives. In Section 3, we discuss the use of forward measure. We show in two examples how the forward measure needs to be used with caution. We also derive partial differential equations that the forward price must satisfy. We show how the forward price can satisfy the Capital Asset Pricing Model with a "modified" discount rate. Finally, we summarize briefly the paper in Section 4.

2. A brief review of the forward measure

The forward measure technique is useful in computing contingent claim prices under stochastic interest rates. In this section, we briefly review the forward measure methodology. We limit ourselves to those key equations relevant to our analysis, while interested readers can find more detailed discussions in Rebonato (1998), Musiela and Rukowski (1998), or Hull (2000).

Define a T-maturity, \$1 face value, zero-coupon bond price as follows:

$$P(t, T) = \hat{E}_t \left[\exp\left(-\int_t^T r(u)\, du \right) \right] \tag{1}$$

where $\hat{E}_t[\cdot]$ is a conditional (on t) expectation taken under the risk neutral measure. The T-maturity forward price of an s-maturity pure-discount bond as follows:

$$\Psi(t, T, s) = \frac{P(t, s)}{P(t, T)} = \bar{E}_t^T [P(T, s)] \tag{2}$$

where $t < T < s$ and $\bar{E}_t^T[\cdot]$ is the conditional (on t) expectation taken under the T-maturity forward measure.

Equation (2) is interesting because it shows that the forward price is parallel to the futures price, they both are expectations of the future bond price. The standard expectation operations show that:

$$\begin{aligned}\hat{E}_t\left[\exp\left(-\int_t^T r(u)\,du\right)P(T,s)\right] &= \hat{\text{cov}}_t\left[\exp\left(-\int_t^T r(u)\,du\right), P(T,s)\right]\\ &\quad+\hat{E}_t\left[\exp\left(-\int_t^T r(u)\,du\right)\right]\hat{E}_t[P(T,s)]\\ &= \hat{\text{cov}}_t\left[\exp\left(-\int_t^T r(u)\,du\right), P(T,s)\right]\\ &\quad+P(t,T)\Phi(t,T,s)\end{aligned} \tag{3}$$

where $\Phi(t,T,s)=\hat{E}_t[P(T,s)]$ is a futures price. Combining this result with equation (2), we have:

$$\Psi(t,T,s)-\Phi(t,T,s)=\frac{\hat{\text{cov}}_t\left[\exp\left(-\int_t^T r(u)\,du\right), P(T,s)\right]}{P(t,T)} \tag{4}$$

It can be seen that the difference between the forward price and the futures price is only the covariance between the stochastic discount factor and the bond price at maturity. Since the covariance is always positive, the forward price is always greater than the futures price.

As a result, the difference between the forward expectation and the risk neutral expectation is very intuitive. Note that the difference between forward and futures is marking to market in the futures market. Marking to market differentiates the forward price from the futures price only under stochastic interest rate environment because there exist continuous cash flows from the futures contract that need to be reinvested at random interest rates. Consequently, if we incorporate the effect of the covariance between the interest rate and underlying asset in the forward measure, then the forward price should itself obey an expectation under the forward measure.

Equation (2) does not seem to be particularly useful because the same result of the forward price, a ratio of two bonds, can be easily obtained by static arbitrage. In other words, buying an s-maturity bond $P(t,s)$ is equivalent to buying a T-maturity bond $P(t,T)$ and rolling over to a forward price $\Psi(t,T,s)$. However, it is very useful in computing other contingent claim prices. For any (European) contingent claim, $C(t,T)$, with a terminal payoff of $X(T)$, the forward measure gives the following convenient result:

$$\begin{aligned}C(t,T) &= \hat{E}_t\left[\exp\left(-\int_t^T r(u)\,du\right)X(T)\right]\\ &= \hat{E}_t\left[\exp\left(-\int_t^T r(u)\,du\right)\right]\bar{E}_t^T[X(T)]\\ &= P(t,T)\bar{E}_t^T[X(T)]\end{aligned} \tag{5}$$

As a result, the forward measure can be used to derive a number of important derivative pricing results. For example, it can be used to compute instantaneous forward rates:

$$\begin{aligned} f(t,T) &= -\frac{d\ln P(t,T)}{dT} \\ &= -\frac{1}{P(t,T)}\hat{E}_t\left[\frac{d}{dT}\exp\left(-\int_t^T r(u)\,du\right)\right] \\ &= \frac{1}{P(t,T)}\hat{E}_t\left[\exp\left(-\int_t^T r(u)\,du\right)r(T)\right] \\ &= \bar{E}_t^T[r(T)] \end{aligned} \tag{6}$$

and discrete forward rates, $f_D(t,w,T)$:

$$\begin{aligned} f_D(t,w,T) &\equiv \frac{1}{\Psi(t,w,T)} - 1 \\ &= \frac{P(t,w)}{P(t,T)} - 1 \\ &= \frac{1}{P(t,T)}\hat{E}_t\left[\exp\left(-\int_t^T r(u)\,du\right)\frac{1}{P(w,T)}\right] - 1 \quad \text{where } t < w < T. \\ &= \bar{E}_t^T\left[\frac{1}{P(w,T)} - 1\right] \end{aligned} \tag{7}$$

The result stated by equation (7) is very useful since macroeconomists always wonder if the forward rate is a good predictor of the future spot rate. Here, it is proved theoretically that the forward rate can only predict the future spot rate under the forward measure. It is therefore a biased estimator of the future spot rate under the normal situation (i.e., original probability space). We can also use the forward measure to re-derive the bond option pricing results in CIR (1985) and Jamshidian (1989) and the stock option pricing result in Rabinovitch (1989).

3. The use of forward measure

In this section, we present two examples in which the applications of the forward measure are not so straightforward. For the ease of exposition, we assume a one-factor term structure model while the results hold in a multi-factor environment. We assume that the instantaneous interest rate follows the following stochastic differential equation:

$$dr(t) = \hat{\mu}(r,t)\,dt + \sigma(r,t)\,d\hat{W}(t) \tag{8}$$

where $\hat{W}(t)$ is the standard Wiener process defined in the risk-neutral probability space, and $\hat{\mu}(r,t)$ and $\hat{\sigma}(r,t)$ are the state- and time-dependent drift and diffusion for the process respectively.

A T-maturity forward measure can then be defined as:

$$dr = \left[\hat{\mu}(r,t) - \sigma(r,t)^2 \frac{P_r(t,T)}{P(t,T)}\right] dt + \sigma(r,t)\, d\bar{W}^T \tag{9}$$

in which a Girsonav transformation of the following is performed:

$$d\hat{W}(t) = d\bar{W}^T(t) + \sigma^2(r,t)\frac{P_r(t,T)}{P(t,T)}\, dt \tag{10}$$

where $\bar{W}^T(t)$ is the Wiener process defined under the T-maturity forward measure.[3] Under this forward measure, we can compute the forward price of an s-maturity pure-discount bond as shown in equation (2).

3.1. Roll-over forward measures

Despite of the usefulness of this forward measure, a number of popular contracts need "roll-over" forward measures. For example, the Treasury bond futures contracts traded on the Chicago Board of Trade (CBOT) and the Eurodollar futures contracts traded on the Chicago Mercantile Exchange (CME) are in fact roll-over forward contracts since marking to market takes place daily rather than continuously. Assuming no delivery options in these futures contracts, the future price at $T - 1$ (one day before the delivery date T) is in fact a forward price for one day. Hence the futures price at $T - 1$ should be computed by the forward pricing formula of equation (2). At $T - 2$ (two days before the delivery date), the futures price should be the forward price of the forward price at $T - 1$. The procedure repeats until the current time t is reached.[4] As a comparison, one can compute the forward price today, t, for the delivery at T without marking to market. As shown in Chen (1992b), the two prices, a one-time forward measure and a series of revolving forward measures, are not identical. A similar result can be observed in discrete term structure models such as the Ho-Lee model (1986) and the Black-Derman-Toy model (1989) where the expected value calculated using their binomial probabilities is the futures price under discrete marking to market. The same conclusion can be drawn.

Another example is a popular contract in credit derivative markets—forward asset swaps. Asset swaps differ from interest rate swaps because of their high credit risks. The credit swap rates in asset swaps can vary dramatically due to various cash flow arrangements upon default. Forward asset swaps are forward contracts on asset swaps. These contracts allow investors to lock in a specific asset swap rate. If we assume independence between the credit risk and the interest rate risk and the forward contract is default-free, then we can model the forward contract as a simple forward contract on an interest rate swap. Since the swap rate itself is a linear combination of forward rates, the forward swap rate is a forward price on a forward price.

In both of the above examples, we need to compute the forward price of a forward price. In the simplest case where we compute the forward price of a forward price on a pure-discount bond, we shall see that:

$$
\begin{aligned}
\bar{E}_t^w[\Psi(w,T,s)] &= \hat{E}_t\left[\Psi(w,T,s)\frac{1}{P(t,w)}\exp\left(-\int_t^w r(u)\,du\right)\right]\\
&= \hat{E}_t\left[\hat{E}_w\left[P(T,s)\frac{1}{P(w,T)}\exp\left(-\int_w^T r(u)\,du\right)\right]\right.\\
&\quad \left.\times\frac{1}{P(t,w)}\exp\left(-\int_t^w r(u)\,du\right)\right]\\
&= \hat{E}_t\left[P(T,s)\frac{1}{P(w,T)P(t,w)}\exp\left(-\int_t^T r(u)\,du\right)\right]\\
&\neq \hat{E}_t\left[P(T,s)\frac{1}{P(t,T)}\exp\left(-\int_t^T r(u)\,du\right)\right]\\
&= \Psi(t,T,s)
\end{aligned}
\tag{11}
$$

It is seen that the forward expectation of the forward price is not equal to the forward price which is itself an expectation: $\Psi(t,T,s) = \bar{E}_t^T[P(T,s)]$. The difference results from the two forward measures, one on the LHS and one at the last line of the RHS are taken to a different future points in time, w and T respectively. This is an interesting contrast the risk neutral measure which is maturity independent. In the discrete marking to market case mentioned above, today's futures price is a repeated use of equation (11). In the forward asset swap case, a linear combination of discrete forward rates defined in equation (7) needs to be used in conjunction with equation (11). In either case, roll-over forward measures are used.

3.2. Partial differential equations

Although the forward measure methodology introduced in the previous section is very convenient in deriving pricing results, it is also important to understand the dynamics of the forward price. This is important because through the dynamics investors can trade and hedge contingent claims whose payoffs are dependent upon forward prices.

Forward prices are not the values of forward contracts. Rather, they are prices to be paid to acquire underlying assets (like the strike price of an option). Hence, they are similar to an index. CIR (1981) show that forward prices do not satisfy the Capital Asset Pricing Model. In this section, we derive two partial different equations (PDE) that the forward price has to satisfy through forward price dynamics. One uses the forward measure defined above. This PDE is based upon the interpretation that the forward price is an index. The other one interprets the forward price as an asset. We derive an implied discount rate at which the forward price satisfies the Capital Asset Pricing Model.

Applying Ito's lemma to equation (8), we can write the dynamics of the forward price, $d\Psi(t, T, s)$, as:

$$d\Psi = \left[\frac{1}{2}\sigma^2(r,t)\Psi_{rr} + \hat{\mu}(r,t)\Psi_r + \Psi_t\right]dt + \Psi_r\sigma(r,t)\,d\hat{W} \tag{12}$$

The underlying bond, $P(t, s)$, can be used to hedge this innovation in the forward price, which has the following dynamics:

$$\begin{aligned} dP &= \left[\frac{1}{2}\sigma^2(r,t)P_{rr} + \hat{\mu}(r,t)P_r + P_t\right]dt + P_r\sigma(r,t)\,d\hat{W} \\ &= rP\,dt + P_r\sigma(r,t)\,d\hat{W} \end{aligned} \tag{13}$$

The portfolio that contains one long bond and h short forward contracts of the following:

$$V = P(t,s) - h\Psi(t,T,s) \tag{14}$$

is a different from that of the futures because the forward contract does not generate $d\Psi$ but $(d\Psi)P(t + dt, T)$ over time. Hence, the dynamics of equation (14) is:

$$\begin{aligned} dV &= dP(t,s) - hP(t+dt,T)\,\Delta \\ &= dP(t,s) - h(P(t,T) + dP(t,T))\,\Delta \\ &= rP(t,s)\,dt + P_r(t,s)\sigma\,d\hat{W} - hP(t,T)\Psi_r\sigma(r,t)\,d\hat{W} \\ &\quad - h\Psi_r\sigma(r,t)P_r(t,T)\sigma(r,t) + hP(t,T)\left[\frac{1}{2}\sigma^2(r,t)\Psi_{rr} + \hat{\mu}(r,t)\Psi_r + \Psi_t\right]dt \end{aligned} \tag{15}$$

where $\Delta = d\Psi(t, T, s)$. The hedge ratio, h, is:

$$h = \frac{P_r(t,s)}{P(t,T)\Psi_r} \tag{16}$$

This gives a PDE of the following:

$$\frac{\sigma^2(r,t)}{2}\Psi_{rr} + \left[\hat{\mu}(r,t) - \sigma^2(r,t)\frac{P_r(t,T)}{P(t,T)}\right]\Psi_r + \Psi_t = 0 \tag{17}$$

This is a Kolmogorov backward equation under the process for the interest rate precisely specified by equation (9) and hence the solution to the forward price specified by equation (2) is verified by the solution to the Kolmogorov backward equation.

In addition to the above PDE that is derived under the forward measure, we can interpret the forward price as an asset price and show that the forward price should also satisfy the following PDE:[5]

$$\frac{\sigma^2(r,t)}{2}\Psi_{rr} + \hat{\mu}(r,t)\Psi_r + \Psi_t = \Psi\left\{\left[\frac{P_r(t,T)}{P(t,T)}\sigma(r,t)\right]^2 - \frac{P_r(t,s)}{P(t,s)}\frac{P_r(t,T)}{P(t,T)}\sigma(r,t)^2\right\}$$
$$= \hat{r}\Psi \tag{18}$$

This PDE shows that the forward price should earn, in the risk neutral world, an interest rate $\hat{r}$ that is the difference between the variance of the discount factor and the covariance between the discount factor and the underlying spot. It is clear that when the variance is equal to the covariance, the forward price is equal to the futures price, since the futures price satisfies the same PDE with zero on the right hand side. One example would be deterministic interest rates. In such a case, both terms are zero and equation (18) will become the futures PDE. CIR point out that this risk neutral return for the forward price is negative and therefore the forward price should always be more than the futures price. The solution to this PDE is a Kac functional and identical to the solution to equation (17):

$$\Psi(t,T,s) = \hat{E}_t\left[\exp\left(-\int_t^T \hat{r}(u)\,du\right)P(T,s)\right] \tag{19}$$

Recognizing that the forward price earns $\hat{r}$ in the risk neutral world, the martingale process for the normalized forward price can be written as, i.e.:

$$\begin{aligned}\Psi(t,T,s) &= \hat{E}_t\left[\exp\left(-\int_t^T \hat{r}(u)\,du\right)P(T,s)\right]\\ &= \hat{E}_t\left[\exp\left(-\int_t^w \hat{r}(u)\,du\right)\hat{E}_w\left[\exp\left(-\int_w^T \hat{r}(u)\,du\right)P(T,s)\right]\right]\\ &= \hat{E}_t\left[\exp\left(-\int_t^w \hat{r}(u)\,du\right)\Psi(w,T,s)\right]\end{aligned} \tag{20}$$

This result is equivalent to CIR's proposition 8 where the forward price (which is not an asset price) is translated into an asset price. Like CIR, we argue that the forward price is equivalent to the value of an asset that earns $\hat{r}$ in the risk neutral world.

Note that equations (17) and (18) use the same boundary condition. That is, at maturity, the forward price is equal to the bond price. Therefore, the two PDE's need to equal each other. Eliminating terms, we get:

$$\frac{P_r(t,T)}{P(t,T)}\Psi_r = \Psi\left\{\left[\frac{P_r(t,T)}{P(t,T)}\right]^2 - \frac{P_r(t,s)}{P(t,s)}\frac{P_r(t,T)}{P(t,T)}\right\} \tag{21}$$

This equality can be verified easily. Equation (21) gives an alternative derivation for equation (17). We can transfer the risk neutral return of a certain asset into the drift of the underlying process under which the asset earns 0 return so that an expectation can be established.

4. Summary

In this paper, we reiterate the convenience of the using the forward measure in pricing interest rate sensitive contingent claims. We demonstrate in two examples that caution needs to be drawn when the forward measure is used to price contingent claims that involve multiple cash flows. For some contracts whose payoffs are dependent upon forward prices (e.g., swaptions), the dynamic behavior of the forward price is important. In view of this, we derive two partial differential equations (PDE), one under the forward measure and the other under the risk neutral measure. The PDE under the forward measure has the similarity to the one satisfied by the futures price under the risk neutral measure while the one under the risk measure interprets the forward price as a traded asset.

Appendix

Partial differential equation

We first establish the following partials:

$$\begin{aligned}
\Psi_r &= P_r(t,s)P(t,T)^{-1} - P(t,s)P(t,T)^{-2}P_r(t,T)\\
\Psi_{rr} &= P_{rr}(t,s)P(t,T)^{-1} - 2P_r(t,s)P(t,T)^{-2}P_r(t,T)\\
&\quad + 2P(t,s)P(t,T)^{-3}P_r(t,T)^2 - P(t,s)P(t,T)^{-2}P_{rr}(t,T)\\
\Psi_t &= P_t(t,s)P(t,T)^{-1} - P(t,s)P(t,T)^{-2}P_t(t,T)
\end{aligned} \tag{A.1}$$

Applying risk neutral coefficients for the backward equation, we have:

$$\begin{aligned}
&\frac{\sigma^2(r,t)}{2}\Psi_{rr} + \hat{\mu}(r,t)\Psi_r + \Psi_t\\
&= \frac{\sigma^2}{2}\begin{bmatrix} P_{rr}(t,s)P(t,T)^{-1}\\ \quad -2P_r(t,s)P(t,T)^{-2}P_r(t,T)\\ \quad +2P(t,s)P(t,T)^{-3}P_r(t,T)^2\\ \quad -P(t,s)P(t,T)^{-2}P_{rr}(t,T)\end{bmatrix} + \hat{\mu}\begin{bmatrix} P_r(t,s)P(t,T)^{-1}\\ \quad -P(t,s)P(t,T)^{-2}P_r(t,T)\end{bmatrix}\\
&\quad + \begin{bmatrix} P_t(t,s)P(t,T)^{-1}\\ \quad -P(t,s)P(t,T)^{-2}P_t(t,T)\end{bmatrix}\\
&= rP(t,s)P(t,T)^{-1} - rP(t,s)P(t,T)^{-2}P(t,T) + \sigma^2\begin{Bmatrix} -P_r(t,s)P(t,T)^{-2}P_r(t,T)\\ +P(t,s)P(t,T)^{-3}P_r(t,T)^2\end{Bmatrix}\\
&= 0 + \sigma^2\left\{-\frac{P_r(t,s)}{P(t,s)}\frac{P(t,s)}{P(t,T)}\frac{P_r(t,T)}{P(t,T)} + \frac{P(t,s)}{P(t,T)}\left[\frac{P_r(t,T)}{P(t,T)}\right]^2\right\}\\
&= \Psi\left\{\left[\frac{P_r(t,T)}{P(t,T)}\sigma\right]^2 - \frac{P_r(t,s)}{P(t,s)}\sigma\frac{P_r(t,T)}{P(t,T)}\sigma\right\}
\end{aligned} \tag{A.2}$$

Forward measure

Forward measure is a probability measure under which the expectation gives rise to the forward price. The bond price has to satisfy the law of iterative expectations under the risk neutral probability space:

$$P(t,s) = \hat{E}_t[\Lambda(t,T)P(T,s)] \tag{A.3}$$

where

$$\Lambda(t,T) = \exp\left(-\int_t^T r(u)\,du\right).$$

(A.3) can be separated into a product of two expectations using the Radon-Nikodym derivative:

$$\begin{aligned} P(t,s) &= \hat{E}_t[\Lambda(t,T)]\bar{E}_t[P(T,s)] \\ &= P(t,T)\bar{E}_t[P(T,s)] \end{aligned} \tag{A.4}$$

Given that the forward price is the ratio between two bonds, the forward measure should surely give the forward price. Now the problem is to find the Radon-Nikodym derivative that can help the Girsanov transformation. Define the derivative as:

$$\eta(t,T) = \frac{\Lambda(t,T)}{P(t,T)} = \frac{d\bar{\wp}}{d\hat{\wp}} \tag{A.5}$$

Also, directly from equation (13), we have

$$\begin{aligned} 0 &= \ln P(T,T) \\ &= \ln P(t,T) + \int_t^T r(w)\,dw + \int_t^T \sigma(r,w)\frac{P_r(w,T)}{P(w,T)}\,d\hat{W}(w) \\ &\quad - \int_t^T \frac{1}{2}\left(\sigma(r,w)\frac{P_r(w,T)}{P(w,T)}\right)^2 dw \end{aligned} \tag{A.6}$$

As a result,

$$\begin{aligned} \eta(t,T) = \frac{\Lambda(t,T)}{P(t,T)} &= \exp\left[\int_t^T \sigma(r,w)\frac{P_r(w,T)}{P(w,T)}\,d\hat{W}(w)\right. \\ &\quad \left. - \int_t^T \frac{1}{2}\left(\sigma(r,w)\frac{P_r(w,T)}{P(w,T)}\right)^2 dw\right] \end{aligned} \tag{A.7}$$

This implies the Girsanov transformation of the following:

$$\hat{W}(t) = \bar{W}(t) + \int_t^T \sigma(r, w)\frac{P_r(w, T)}{P(w, T)}\,dw \tag{A.8}$$

and this completes the proof. A more general version of the above result is to let the terminal date be an arbitrary choice. Then, the Radon-Nikodym derivative is:

$$\eta(t, u) = \frac{\Lambda(t, u)P(u, T)}{P(t, T)} = \frac{\Lambda(t, T)P(u, T)}{\Lambda(u, T)P(t, T)} = \frac{\Lambda(t, T)}{P(t, T)}\cdot\frac{P(u, T)}{\Lambda(u, T)} \tag{A.9}$$

From the derivation above, we know that:

$$\begin{aligned}\ln\Lambda(t, T) - \ln P(t, T) = &\int_t^T \sigma(r, w)P_r(w, T)/P(w, T)\,d\hat{W}\\ &- \int_t^T \frac{1}{2}[\sigma(r, w)P_r(w, T)/P(w, T)]^2\,dw\end{aligned} \tag{A.10}$$

this implies

$$\begin{aligned}\ln\Lambda(u, T) - \ln P(u, T) = &\int_u^T \sigma^2(r, w)P_r(w, T)/P(w, T)\,d\hat{W}\\ &- \int_u^T \frac{1}{2}[\sigma^2(r, w)P_r(w, T)/P(w, T)]^2\,dw\end{aligned} \tag{A.11}$$

The two together gives:

$$\begin{aligned}\eta(t, u) = \exp\Bigg[&\int_t^u \sigma(r, w)P_r(w, T)/P(w, T)\,d\hat{W}\\ &- \int_t^u \frac{1}{2}[\sigma(r, w)P_r(w, T)/P(w, T)]^2\,dw\Bigg]\end{aligned} \tag{A.12}$$

Acknowledgment

We would like to thank an anonymous referee for valuable suggestions. All errors are ours.

Notes

1. In their proposition 11, CIR show that the futures price satisfies the CAPM but not the forward price.
2. See, for example, Jamshidian (1987, 1989), Longstaff (1990), Chen (1992a, b), Chen and Scott (1992), Geman, Karoui and Rochet (1995) and Schroder (1999). Detailed discussions of the forward measure can be found in Musiela and Rukowski (1998).

3. Note that $\sigma^2(r, t)P_r(t, T)/P(t, T) = E[(dr)(dP/P)]$ is the instantaneous covariance between the discount factor and the underlying interest rate. This corresponds to equation (4).
4. Exact details can be found in Chen (1992b) for the Vasicek term structure model.
5. The derivation is given in an appendix.
6. This is checked with the Vasicek model.

References

Black, F., E. Derman and W. Toy, "A One-Factor Model of Interest Rates and Its Application to Treasury Bond Options." *Financial Analysts Journal* 46, 33–39, (1990).

Chen, R. and L. Scott, "Pricing Interest Rate Options in a Two Factor CIR Model of the Term Structure." *Review of Financial Studies* 5(4), 613–636, (1992).

Chen, R., "Exact Solutions for Futures and European Futures Options on Pure Discount Bonds." *Journal of Financial and Quantitative Analysis* 27(1), March, 97–107, (1992a).

Chen, R., "A New Look at Interest Rate Futures Contracts." *Journal of Futures Markets,* October, 539–548, (1992b).

Cheng, S., "On the Feasibility of Arbitrage-Based Option Pricing when Stochastic Bond Price Processes are Involved." *Journal of Economic Theory* 53, 185–198, (1991).

Cox, J., J. Ingersoll and S. Ross, "A Theory of the Term Structure of Interest Rates." *Econometrica* 53(2), 385–408, (1985).

Cox, J., J. Ingersoll and S. Ross, "The Relation Between Forward Prices and Futures Prices." *Journal of Financial Economics* 321–346, (1981).

Geman, H., N. El Karoui and J. Rochet, "Changes of Numeraire, Changes of Probability Measure and Option Pricing." *Journal of Applied Probability* 32, 443–458, (1995).

Ho, T. and S. Lee, "Term Structure Movements and Pricing Interest Rate Contingent Claims." *Journal of Finance,* 1011–1029, (1986).

Hull, Options, *Futures, Options, and Other Derivatives,* 4th edition, Prentice Hall, 2000.

Jamshidian, F., "Pricing of Contingent Claims in the One-Factor Term Structure Model." Working paper, Merrill Lynch Capital Markets, 1987.

Jamshidian, F., "An Exact Bond Option Formula." *Journal of Finance,* 205–209, (1989).

Longstaff, F., "The Valuation of Options on Yields." *Journal of Financial Economics* 26(1), 97–122, (1990).

Musiela, M. and M. Rutkowski, *Martingale Methods in Financial Modeling,* Springer-Verlag, 1998.

Rabinovitch, R., "Pricing Stock and Bond Options when the Default-Free Rate is Stochastic." *The Journal of Financial and Quantitative Analysis* 24(4), 447–457, (1989).

Rebonato, R., *Interest-Rate Option Models: Understanding, Analyzing and Using Models for Exotic Interest-Rate Options,* Wiley Series in Financial Engineering, 2nd edition, 1998.

Schroder, M., "Changes of Numeraire for Pricing Futures, Forwards, and Options." *Review of Financial Studies* 12, 1143–1163, (1999).

Vasicek, O., "An Equilibrium Characterization of The Term Structure." *Journal of Financial Economics,* 177–188, (1977).