Linearization and the Newton Iteration
for more than one equation

Consider the case of two equations in two unknowns:

$f(x,y) = x^2 + y = 0$
$g(x,y) = x - y^2 = 0$

Expand each function about an initial guess for the solution of x_0, y_0 .

$$f\ (x,y)\ =\ f\ (x_0,y_0)\ +\ \frac{\partial f}{\partial x}\Big|_{x_0\ y_0}\ \Delta x\ +\ \frac{\partial f}{\partial y}\Big|_{x_0\ y_0}\ \Delta y\ +\ \frac{1}{2}\ \frac{\partial^2 f}{\partial x^2}\ \Delta x^2$$
$$+\ \frac{\partial^2 f}{\partial y dx}\ \Delta x \Delta y\ +\ \frac{1}{2}\ \frac{\partial^2 f}{\partial y^2}\ \Delta y^2\ +\ \ldots$$

$$g\ (x,y)\ =\ g\ (x_0,y_0)\ +\ \frac{\partial g}{\partial x}\Big|_{x_0\ y_0}\ \Delta x\ +\ \frac{\partial g}{\partial y}\Big|_{x_0\ y_0}\ \Delta y\ +\ \frac{1}{2}\ \frac{\partial^2 g}{\partial x^2}\ \Delta x^2$$
$$+\ \frac{\partial^2 g}{\partial y dx}\ \Delta x \Delta y\ +\ \frac{1}{2}\ \frac{\partial^2 g}{\partial y^2}\ \Delta y^2\ +\ \ldots$$

save only terms of order Δx, Δy, or lower

$$f(x,y)\ =\ f\ (x_0,y_0)\ +\ \frac{\partial f}{\partial x}\Big|_{x_0,y_0}\ \Delta x\ +\ \frac{\partial f}{\partial y}\Big|_{x_0,y_0}\ \Delta y$$

$$g\ (x,y)\ =\ g(x_0,y_0)\ +\ \frac{\partial f}{\partial x}\Big|_{x_0,y_0}\ \Delta X\ +\ \frac{\partial f}{\partial y}\Big|_{x_0,y_0}\ \Delta y$$

Assume f(x,y) = g (x,y) = 0

Solve for Δx, Δy

$$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}\ =\ -\begin{pmatrix} f\ (x_o,y_o) \\ g\ (x_o,y_o) \end{pmatrix}$$

For our example, the chosen forms of f and g yield a Jacobian Matrix

$$J\ =\ \begin{pmatrix} 2x_o & 1 \\ 1 & -2y_o \end{pmatrix}$$

pick $x_o = \frac{3}{2}$, $y_o = -\frac{3}{2}$

Substituting into J, f, and g we get:

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = - \begin{pmatrix} 3/4 \\ -3/4 \end{pmatrix}$$

Multiplying the first row by 3 and subtracting the second row gives:

$$8\,\Delta x = -\frac{12}{4} = -3$$

or

$$\Delta x = -\frac{3}{8}$$

Substituting this value back into the first row gives:

$$\Delta y = -\frac{3}{4} - 3\Delta x = \frac{3}{8}$$

New guesses for the solution then are

x = 9 / 8

y = - (9 / 8)

The actual solution is x=1, y=-1, so we are getting fairly close fairly quickly. To get closer we next pick x_0=9/8 and y_0=-9/8, and loop back to the step where we substitute these values into J, f, and g.