Solution of an equation by Newton Iteration

You can learn the Newton solution procedure, by memorizing a simple formula, but I would like to explain how the method comes about to give you an alternative way of remembering it, and the way to generalize the method to the solution of a system of n equations with n unknowns (we'll get to that one later in the semester). Begin with a general form of an equation:

$$f(x) = 0 \qquad (1)$$

As we work through this execise, we will deal with the specific example:

$$f(x) = x^3 + x - 10 = 0 \qquad (2)$$

All iterative solution methods must begin with some guess x_0 for the value of x that solves the equation. The derivation of a solution method begins with an application of a Taylor Series expansion of the function about the point x_0.

$$f(x) = f(x_0) + \left.\frac{df}{dx}\right|_{x=x_0}(x - x_0) + \frac{1}{2}\left.\frac{d^2f}{dx^2}\right|_{x=x_0}(x - x_0)^2 + \ldots \qquad (3)$$

We continue by using this expanded equation to find the x such that f(x)=0. To accomplish this we make the assumption that x_0 is close enough to the final solution value of x that $(x-x_0)^2$, $(x-x_0)^3$, and higher powers of this difference are small enough to ignore. This leads to the equation:

$$0 = f(x_0) + \left.\frac{df}{dx}\right|_{x=x_0}(x - x_0) \qquad (4)$$

At this point we need to be honest, realize that x in the above equation is not the true solution of our original equation, and replace it with x_1 to designate the next approximation to the solution.

$$0 = f(x_0) + \left.\frac{df}{dx}\right|_{x=x_0}(x_1 - x_0) \qquad (5)$$

Solving this equation for x_1 we get:

$$x_1 = x_0 - \frac{f(x_0)}{\left.\frac{df}{dx}\right|_{x=x_0}} \tag{6}$$

For our example $f(x)=x^3+x-10$ we get:

$$x_1 = x_0 - \frac{x_0^3 + x_0 - 10}{3x_0^2 + 1} \tag{7}$$

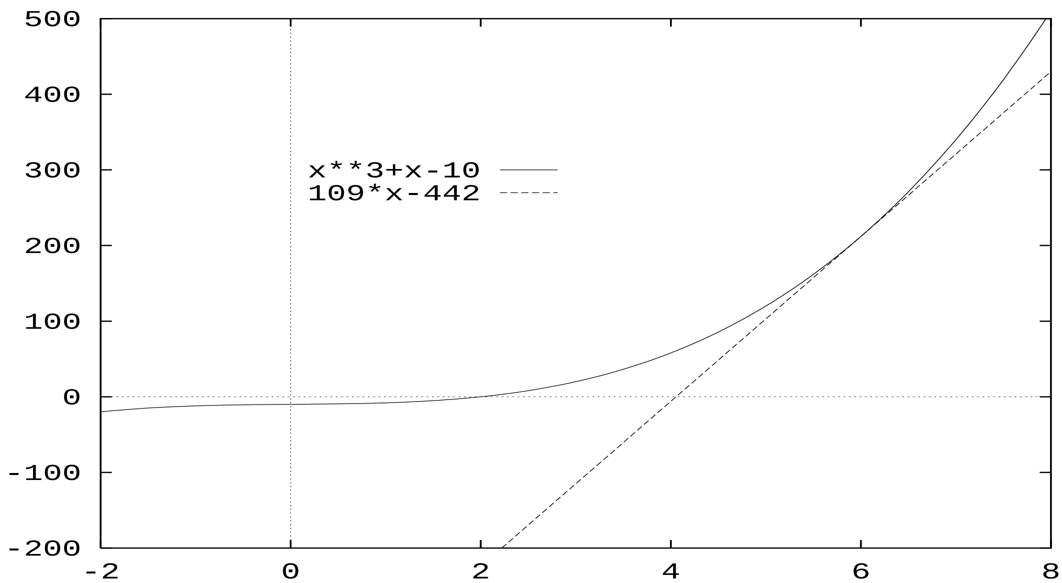

Figure 1

From a geometric point of view Equation (7) represents the intersection with the x axis of a line tangent to the curve y=f(x) at the point $x=x_0$. Figure 1 gives an example for $x_0=6$. The tangent to the curve at $x_0=6$ intercepts the x-axis at a value $x_1=4.055$, giving a starting point for another iteration.

The next approximation can be generated by doing the same Taylor expansion about the point x_1 and in general the ith approximation is:

$$x_i = x_{i-1} - \frac{f(x_{i-1})}{\left.\frac{df}{dx}\right|_{x=x_{i-1}}} \tag{8}$$