

The Method of Manufactured Solutions

A Summary by

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The Method of Manufactured Solutions (MMS) [1, 2] is a very simple process, although some of the algebra required for implementation can become complicated. You start with a partial differential equation or system of PDEs for which you want to verify the implementation of a discretized solution procedure. You choose a closed analytic form for the solution that you want for the final test problem. Next you substitute the analytic solution into the base PDEs to generate new or modified source terms in the equations. Finally, initial and boundary conditions for the test problem are obtained by evaluating the selected solution form at zero time and at whatever spatial locations constitute the boundary of the problem.. For a given code primary consideration for use of MMS is the level of difficulty involved in specifying or adding source terms to all PDEs involved in the solution.

As an example, suppose that we wish to verify a finite difference solution method for a one-dimensional transient conduction problem represented by the equation:

$$\frac{\partial T}{\partial t} - \alpha \frac{\partial^2 T}{\partial x^2} = \frac{q}{\rho c_p} \quad (1)$$

For this example a very simple functional form is chosen for the solution.

$$T(x,t) = 300 + (0.01 - x^2)t \quad (2)$$

Evaluating the differential operators gives:

$$\frac{\partial T}{\partial t} = 0.01 - x^2, \quad \frac{\partial^2 T}{\partial x^2} = -2t, \quad \frac{\partial T}{\partial t} - \alpha \frac{\partial^2 T}{\partial x^2} = 0.01 - x^2 + 2\alpha t \quad (3)$$

As a result the source term in the original model equation is specifically set as:

$$\frac{q}{\rho c_p} = 0.01 - x^2 + 2\alpha t \quad (4)$$

Looking at the original functional form, the initial conditions are $T(x,0)=300$, and boundary conditions for a 0.2 m thick metal slab would be $T(-0.1,t)=300$. and $T(0.1,t)=300$.

For most general purpose conduction solvers, the source term could be provided via tabulated input. However, complications can arise due to interpolation procedures applied to the input. For best results the source term should be installed as a function added to the program, or linked to the program via an interface provided to users by the code developers.

Verification testing of the code is very similar to a Richardson Extrapolation based mesh and time step sensitivity study. The error between the code and manufactured solution is followed for a sequence of mesh and a sequence of time step sizes. Any plot of error vs. mesh size (or time step) should trend clearly towards zero as the discretization approaches zero. In addition a fit of one of these curves to the equation

$$error = ah^p \quad (5)$$

provides a check of the order of accuracy quoted for the discrete approximation to the PDE.

For best results from a manufactured solution the following rules should be followed.

- The manufactured solution should be assembled using smooth analytic functions, i.e. trigonometric, exponential or polynomial functions. This ensures that the theoretical order of accuracy can be attained and also such functions are easy to differentiate.
- The solution should be general enough to exercise every term in the governing equation, including all dependent variables.
- The solution should have a sufficient number of non-trivial derivatives.
- The solution should not be a strongly varying function of space and time or have a singularity. This is accomplished by bounding a solution derivative by a relatively small constant.
- There is no requirement on physical realism or robustness. However, if the code contains assumptions, such as a positive solution or a positive equation term, make sure the manufactured solution satisfies those assumptions.

As part of a research project for the U.S. Nuclear Regulatory Commission, use of various higher order numerical methods was studied for a two-phase, three-fluid model (vapor, continuous liquid, entrained liquid). The following manufactured solution was used to check the implementation of the various difference methods.

$$\begin{aligned}
 \alpha_p &= [0.3 \sin(x^2 + \omega t) + 0.5], \quad p=v,e \\
 \alpha_l &= 1 - \alpha_v - \alpha_e \\
 u_p &= u_{p0} [\sin(x^2 + \omega t) + \varepsilon], \quad p=v,l,e \\
 h_p &= h_{p0} [\sin(x^2 + \omega t) + \varepsilon], \quad p=v,l \\
 P &= P_0 [\sin(x^2 + \omega t) + \varepsilon] \\
 \rho_p &= \text{const}, \quad p=v,l \\
 T &= T_0 [\sin(r^2 + z^2 + \omega t) + \varepsilon]
 \end{aligned} \tag{6}$$

where $\omega, u_{p0}, h_{p0}, P_0, T_0, \varepsilon$ are arbitrary constants, ε is small. The index p represents a corresponding field: vapor v , liquid l and entrained e . The constants are usually set to the value of one, but to avoid exceeding the thermodynamic bounds on certain variables and other code restrictions, they can be set to higher values. In this case the amplitude of the sine function should be increased as well, to better exercise the differential operators. Note that there are many other possibilities to manufacture different solutions than the ones given by Eq. 6. A steady state can be tested by setting $\omega = 0$.

In this test problem density was set constant because the necessary verification on spatial differences was driven by the functions chosen for field volume fractions and velocities. A separate verification test was constructed to be certain that densities were properly loaded into the discrete equations.

Below, the two-phase flow equations are modified by including a new source term to balance the terms obtained by differentiation of manufactured solutions. Notation in the source terms has shifted, with subscripts included in the source name, because the results were extracted directly from MathCad

Vapor mass conservation:

$$\frac{\partial(\alpha_v \rho_v)}{\partial t} = - \frac{\partial(\alpha_v \rho_v u_v)}{\partial x} + \Gamma''' + S'''_{vm} \tag{7}$$

$$\begin{aligned}
S_{vm} := & .3 \cdot \cos(x^2 + \omega \cdot t) \cdot \omega \cdot \rho_v \dots \\
& + .6 \cdot \cos(x^2 + \omega \cdot t) \cdot x \cdot \rho_v \cdot u_{v0} \cdot (\sin(x^2 + \omega \cdot t) + \varepsilon) \dots \\
& + 2 \cdot (.3 \cdot \sin(x^2 + \omega \cdot t) + .5) \cdot \rho_v \cdot u_{v0} \cdot \cos(x^2 + \omega \cdot t) \cdot x - \Gamma
\end{aligned} \tag{8}$$

Continuous liquid mass conservation:

$$\frac{\partial(\alpha_l \rho_l)}{\partial t} = -\frac{\partial(\alpha_l \rho_l u_l)}{\partial x} - (1 - \eta) \Gamma''' + S_D''' - S_E''' + S_{lm}''' \tag{9}$$

$$\begin{aligned}
S_{lm} := & -.6 \cdot \cos(x^2 + \omega \cdot t) \cdot \omega \cdot \rho_l - \dots \\
& +- 1.2 \cdot \cos(x^2 + \omega \cdot t) \cdot x \cdot \rho_l \cdot u_{l0} \cdot (\sin(x^2 + \omega \cdot t) + \varepsilon) - \dots \\
& +- 1.2 \cdot \sin(x^2 + \omega \cdot t) \cdot \rho_l \cdot u_{l0} \cdot \cos(x^2 + \omega \cdot t) \cdot x + (1 - \eta) \cdot \Gamma \dots \\
& + SE - SD
\end{aligned} \tag{10}$$

Entrained liquid mass conservation:

$$\frac{\partial(\alpha_e \rho_l)}{\partial t} = -\frac{\partial(\alpha_e \rho_l u_e)}{\partial x} - \eta \Gamma''' - S_D''' + S_E''' + S_{em}''' \tag{11}$$

$$\begin{aligned}
S_{em} := & .3 \cdot \cos(x^2 + \omega \cdot t) \cdot \omega \cdot \rho_l \dots \\
& + .6 \cdot \cos(x^2 + \omega \cdot t) \cdot x \cdot \rho_l \cdot u_{e0} \cdot (\sin(x^2 + \omega \cdot t) + \varepsilon) \dots \\
& + 2 \cdot (.3 \cdot \sin(x^2 + \omega \cdot t) + .5) \cdot \rho_l \cdot u_{e0} \cdot \cos(x^2 + \omega \cdot t) \cdot x \dots \\
& + \eta \cdot \Gamma + SD - SE
\end{aligned} \tag{12}$$

Vapor energy conservation:

$$\frac{\partial(\alpha_v \rho_v h_v)}{\partial t} = -\frac{\partial(\alpha_v \rho_v h_v u_v)}{\partial x} + \Gamma''' h_g + q_{iv} + Q_{wv} + \alpha_v \frac{\partial P}{\partial t} + S_{ve}''' \tag{13}$$

$$\begin{aligned}
Sve := & .3 \cdot \cos(x^2 + \omega \cdot t) \cdot \omega \cdot \rho v \cdot hv0 \cdot (\sin(x^2 + \omega \cdot t) + \varepsilon) \dots \\
& + (.3 \cdot \sin(x^2 + \omega \cdot t) + .5) \cdot \rho v \cdot hv0 \cdot \cos(x^2 + \omega \cdot t) \cdot \omega \dots \\
& + .6 \cdot \cos(x^2 + \omega \cdot t) \cdot x \cdot \rho v \cdot uv0 \cdot (\sin(x^2 + \omega \cdot t) + \varepsilon)^2 \cdot hv0 \dots \\
& + 4 \cdot (.3 \cdot \sin(x^2 + \omega \cdot t) + .5) \cdot \rho v \cdot uv0 \cdot (\sin(x^2 + \omega \cdot t) + \varepsilon) \cdot \dots \\
& + \dots \cdot hv0 \cdot \cos(x^2 + \omega \cdot t) \cdot x - (.3 \cdot \sin(x^2 + \omega \cdot t) + .5) \cdot P0 \cdot \dots \\
& + \dots \cdot \cos(x^2 + \omega \cdot t) \cdot \omega - qiv - \Gamma \cdot hg - Qwv
\end{aligned} \tag{14}$$

Liquid energy conservation:

$$\begin{aligned}
\frac{\partial((1-\alpha_v)\rho_l h_l)}{\partial t} = & - \frac{\partial(\alpha_l u_l + \alpha_e u_e)\rho_l h_l}{\partial x} - \Gamma''' h_f + q_{il} + Q_{wl} + \\
& + (1-\alpha_v) \frac{\partial P}{\partial t} + S'''_{le}
\end{aligned} \tag{15}$$

$$\begin{aligned}
Sle := & .6 \cdot \cos(x^2 + \omega \cdot t) \cdot \omega \cdot \rho l \cdot hl0 \cdot (\sin(x^2 + \omega \cdot t) + \varepsilon) \dots \\
& + (1 + .6 \cdot \sin(x^2 + \omega \cdot t)) \cdot \rho l \cdot hl0 \cdot \cos(x^2 + \omega \cdot t) \cdot \omega \dots \\
& + \left[\begin{aligned} & -1.2 \cdot \cos(x^2 + \omega \cdot t) \cdot x \cdot ul0 \cdot (\sin(x^2 + \omega \cdot t) + \varepsilon) - \dots \\ & + -1.2 \cdot \sin(x^2 + \omega \cdot t) \cdot ul0 \cdot \cos(x^2 + \omega \cdot t) \cdot x \dots \\ & + .6 \cdot \cos(x^2 + \omega \cdot t) \cdot x \cdot ue0 \cdot (\sin(x^2 + \omega \cdot t) + \varepsilon) \dots \\ & + 2 \cdot (.3 \cdot \sin(x^2 + \omega \cdot t) + .5) \cdot ue0 \cdot \cos(x^2 + \omega \cdot t) \cdot x \end{aligned} \right] \cdot \rho l \cdot hl0 \cdot \dots \\
& + \dots \cdot \left[\sin(x^2 + \omega \cdot t) + \varepsilon \right] \dots \\
& + 2 \cdot \left[\begin{aligned} & -.6 \cdot \sin(x^2 + \omega \cdot t) \cdot ul0 \cdot (\sin(x^2 + \omega \cdot t) + \varepsilon) \dots \\ & + (.3 \cdot \sin(x^2 + \omega \cdot t) + .5) \cdot ue0 \cdot (\sin(x^2 + \omega \cdot t) + \varepsilon) \end{aligned} \right] \cdot \rho l \cdot hl0 \cdot \dots \\
& + \dots \cdot \cos(x^2 + \omega \cdot t) \cdot x \dots \\
& + \left[\dots - (.5 - .3 \cdot \sin(x^2 + \omega \cdot t)) \cdot P0 \cdot \cos(x^2 + \omega \cdot t) \cdot \omega - qil + \Gamma \cdot hf \right] - Qwl
\end{aligned} \tag{16}$$

Vapor momentum conservation:

$$\begin{aligned}
\alpha_v \rho_v \frac{\partial u_v}{\partial t} = & -\alpha_v \rho_v u_v \frac{\partial u_v}{\partial x} - \alpha_v \rho_v g - \alpha_v \frac{\partial P}{\partial x} - \\
& - \tau_{wv} - \tau_{i,vl} - \tau_{i,ve} - \Gamma'''_E (u_v - (1-\eta)u_l - \eta u_e) + S'''_{vmo}
\end{aligned} \tag{17}$$

$$\begin{aligned}
S_{vmo} := & (.3 \cdot \sin(x^2 + \omega \cdot t) + .5) \cdot \rho_v \cdot u_v \omega \cos(x^2 + \omega \cdot t) \cdot \omega \dots \\
& + 2 \cdot (.3 \cdot \sin(x^2 + \omega \cdot t) + .5) \cdot \rho_v \cdot u_v \omega^2 \cdot (\sin(x^2 + \omega \cdot t) + \varepsilon) \cdot \dots \\
& + \dots \cdot \cos(x^2 + \omega \cdot t) \cdot x + (.3 \cdot \sin(x^2 + \omega \cdot t) + .5) \cdot \rho_v \cdot g \dots \\
& + 2 \cdot (.3 \cdot \sin(x^2 + \omega \cdot t) + .5) \cdot P_0 \cdot \cos(x^2 + \omega \cdot t) \cdot x \dots \\
& + \tau_{wv} + \tau_{ivl} + \tau_{ive} \dots \\
& + \Gamma E \cdot \left[\begin{array}{l} u_v \omega \cdot (\sin(x^2 + \omega \cdot t) + \varepsilon) - \dots \\ + (1 - \eta) \cdot u_l \omega \cdot (\sin(x^2 + \omega \cdot t) + \varepsilon) - \dots \\ + \eta \cdot u_e \omega \cdot (\sin(x^2 + \omega \cdot t) + \varepsilon) \end{array} \right]
\end{aligned} \tag{18}$$

Continuous liquid momentum conservation:

$$\begin{aligned}
\alpha_l \rho_l \frac{\partial u_l}{\partial t} = & -\alpha_l \rho_l u_l \frac{\partial u_l}{\partial x} - \alpha_l \rho_l g - \alpha_l \frac{\partial P}{\partial x} - \\
& - \tau_{wl} + \tau_{i,vl} + (1 - \eta) \Gamma_C''(u_v - u_l) + S_D''(u_e - u_l) + S_{lmo}'''
\end{aligned} \tag{19}$$

$$\begin{aligned}
S_{lmo} := & -.6 \cdot \sin(x^2 + \omega \cdot t) \cdot \rho_l \cdot u_l \omega \cos(x^2 + \omega \cdot t) \cdot \omega - \dots \\
& + -1.2 \cdot \sin(x^2 + \omega \cdot t) \cdot \rho_l \cdot u_l \omega^2 \cdot (\sin(x^2 + \omega \cdot t) + \varepsilon) \cdot \dots \\
& + \dots \cdot \cos(x^2 + \omega \cdot t) \cdot x - .6 \cdot \sin(x^2 + \omega \cdot t) \cdot \rho_l \cdot g - \dots \\
& + -1.2 \cdot \sin(x^2 + \omega \cdot t) \cdot P_0 \cdot \cos(x^2 + \omega \cdot t) \cdot x \cdot (-1 + \eta) \cdot \dots \\
& + \dots \cdot \Gamma C \cdot \left[u_v \omega \cdot (\sin(x^2 + \omega \cdot t) + \varepsilon) - u_l \omega \cdot (\sin(x^2 + \omega \cdot t) + \varepsilon) \right] - \dots \\
& + -SD \cdot \left[u_e \omega \cdot (\sin(x^2 + \omega \cdot t) + \varepsilon) - u_l \omega \cdot (\sin(x^2 + \omega \cdot t) + \varepsilon) \right] \dots \\
& + \tau_{wl} - \tau_{ivl}
\end{aligned} \tag{20}$$

Entrainment liquid momentum conservation:

$$\begin{aligned}
\alpha_e \rho_l \frac{\partial u_e}{\partial t} = & -\alpha_e \rho_l u_e \frac{\partial u_e}{\partial x} - \alpha_e \rho_l g - \alpha_e \frac{\partial P}{\partial x} - \\
& + \tau_{i,ve} + \eta \Gamma_C''(u_v - u_l) - S_E''(u_e - u_l) + S_{emo}'''
\end{aligned} \tag{21}$$

$$\begin{aligned}
\text{Semo} := & (.3 \cdot \sin(x^2 + \omega \cdot t) + .5) \cdot \rho_1 \cdot u_{e0} \cos(x^2 + \omega \cdot t) \cdot \omega \dots \\
& + 2 \cdot (.3 \cdot \sin(x^2 + \omega \cdot t) + .5) \cdot \rho_1 \cdot u_{e0}^2 \cdot (\sin(x^2 + \omega \cdot t) + \varepsilon) \cdot \dots \\
& + \dots \cdot (\cos(x^2 + \omega \cdot t)) \cdot x + (.3 \cdot \sin(x^2 + \omega \cdot t) + .5) \cdot \rho_1 \cdot g + \dots \\
& + 2 \cdot (.3 \cdot \sin(x^2 + \omega \cdot t) + .5) \cdot P_0 \cos(x^2 + \omega \cdot t) \cdot x - \dots \\
& + -\eta \cdot \Gamma C \cdot [u_{v0} \cdot (\sin(x^2 + \omega \cdot t) + \varepsilon) - u_{l0} \cdot (\sin(x^2 + \omega \cdot t) + \varepsilon)] \dots \\
& + SE \cdot [u_{e0} \cdot (\sin(x^2 + \omega \cdot t) + \varepsilon) - u_{l0} \cdot (\sin(x^2 + \omega \cdot t) + \varepsilon)] - \tau_{ive}
\end{aligned} \tag{22}$$

Installation of these special source terms into the 1-D two-phase code proved to be invaluable in detecting and isolating coding errors.

References

1. Roache, P. J., "Code Verification by the Method of Manufactured Solutions," Trans. ASME, J. Fluids Engineering, 124, pp. 4-10, 2002,
2. Salari, K., Knupp, P.: Code Verification by the Method of Manufactured Solutions, *Sandia Report*, Sandia National Laboratories, 2000.